PROCEEDINGS OF THE 16TH ANNUAL CONFERENCE ON RESEARCH IN UNDERGRADUATE MATHEMATICS EDUCATION

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FOREWARD

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its sixteenth annual Conference on Research in Undergraduate Mathematics Education in Denver, Colorado from February 21 - 23, 2013.

The conference is a forum for researchers in collegiate mathematics education to share results of research addressing issues pertinent to the learning and teaching of undergraduate mathematics. The conference is organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The program included plenary addresses by Dr. Patrick Thompson, Dr. Koene Gravemeijer, Dr. Loretta Jones, and Dr. Keith Weber and the presentation of over 117 contributed, preliminary, and theoretical research reports and posters. In addition to these activities, faculty, students and artists contributed to displays on Art and Undergraduate Mathematics Education.

The Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education are our record of the presentations given and it is our hope that they will serve both as a resource for future research, as our field continues to expand in its areas of interest, methodological approaches, theoretical frameworks, and analytical paradigms, and as a resource for faculty in mathematics departments, who wish to use research to inform mathematics instruction in the university classroom.

Volume 1, *RUME Conference Papers*, includes conference papers that underwent a rigorous review by two or more reviewers. These papers represent current work in the field of undergraduate mathematics education and are elaborations of selected RUME Conference Reports. Volume 1 begins with the winner of the best paper award and the papers receiving honorable mention. These awards are bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or providing significant or unique insights into existing research programs.

Volume 2, *RUME Conference Reports*, includes the Poster Abstracts and the Contributed, Preliminary and Theoretical Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. Theoretical Research Reports describe new theoretical perspectives and frameworks for research on undergraduate mathematics education.

Last but not least, we wish to acknowledge the conference program committee and reviewers, for their substantial contributions to RUME and our institutions, for their support.

Sincerely,
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VOLUME 2

CONFERENCE REPORTS

TABLE OF CONTENTS

Contributed Research Reports

Theoretical Reports

Preliminary Research Reports

Posters

IN-SERVICE SECONDARY TEACHERS' CONCEPTUALIZATION OF COMPLEX NUMBERS

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This study explores in-service high school mathematics teachers' conception of various forms of a complex number and the ways that they transition between different representations' (algebraic and geometric) of these forms. Data were collected from three high school mathematics teachers via a ninety- minute interview after they completed professional development on complex numbers. Results indicate that these teachers do not necessarily objectify exponential form of complex numbers and only conceptualized it at the operational level. On the other hand, two teachers were very comfortable with Cartesian form and showed process/object duality by translating between different representations of this form. It appeared that our participants' ability to develop a dual conception of complex numbers was bound by their conceptualization of the various forms, which in turn was hindered by their representations of each form.

Key words: Complex numbers, In-service secondary teachers, Operational/structural conceptualization, Representations

Introduction

Understanding the complex number system, including performing arithmetic operations with complex numbers and representing complex numbers and their operations on the complex plane, is one of the Mathematics standards for high school highlighted in the Common Core State Standards Initiative (CCSSI, 2010 Appendix A, p. 60). The document emphasizes the need for students to work with multiple representations of complex numbers (e.g., algebraic and geometric), and also recommends that students know how to represent complex numbers using rectangular and polar forms. In order for students to develop these notions, it is necessary for teachers to have deep content knowledge as well as knowledge about teaching the field of complex numbers. Deep content knowledge of the field of complex numbers entails knowing the multiple representations and forms, understanding the connections among them, translating between forms flexibly, and recognizing which representations and forms would be more suitable to use in a given task. However, teachers' understanding of complex numbers as well as the required pedagogical content knowledge has been understudied in mathematics education. Investigating this phenomenon is natural given that representations have played a significant role in the field of mathematics education (e.g., Eisner 2004; Janvier 1987; NCTM 2000) and the fact that multiple representations are an integral characteristic of complex numbers.

The purpose of this report is to share our findings from a research study conducted with secondary mathematics teachers that investigates teachers' content knowledge of complex numbers. In this paper we address the following research question: How do secondary mathematics teachers conceptualize complex numbers and their arithmetic operations? More specifically we explore teachers' conception of different representations and forms of complex numbers.

Literature Review

Mathematics educators typically insist that learners should have access to multiple representations in order to reveal information or to illustrate solutions to problems or mathematical ideas. One reason for the emphasis on multiple representations is that "Different representations often illuminate different aspects of a complex concept or relationship" (NCTM, 2000, p.69). Research studies stress the importance of using multiple representations and indicate that learners develop mathematical reasoning and better conceptual understanding when they use various forms of representations (Cuoco & Curcio, 2001). In other words, learners improve their ability to think mathematically by engaging with multiple representations.

Although there is a vast amount of literature related to the role that representations play in the arithmetic of real numbers (Kilpatrick, Swafford, & Findell, 2001; Sowder, 1992), the same is not true for complex numbers. A handful of researchers (Conner et al., 2007; Danenhower, 2000; Nemirovsky et al., 2012; Panaoura, et al., 2006) have begun to explore students' as well as experts' (Soto-Johnson et al., 2011; Soto-Johnson et al., 2012) geometric interpretations of complex numbers and complex valued functions. In general, findings indicate that the experts effortlessly regarded complex numbers and their operations and complex valued functions as dynamic objects. However, similar results were not observed with students.

Conner et al. (2007), found that prospective secondary mathematics teachers perceived multiplication of a real number by -1 as a reflection rather than a rotation of $180°$. This perception may be a result of focusing on the real number line rather than the entire complex plane and may have contributed to their inability to illustrate how multiplication by the complex number $x + yi$ results in a rotation and a dilation of the other factor. The preservice teachers also described complex numbers as a pair of real numbers rather than as a single number. This perspective may have facilitated their ability to provide a geometric interpretation of the addition of complex numbers using vectors or by decomposing the numbers into the real and imaginary components. Unfortunately, this perception about complex numbers does not lend itself to the geometric interpretation of complex number multiplication. Nemirovsky et al. (2012), however, offered methods for fostering a dynamic view of multiplication by i. As part of a teaching experiment with preservice secondary teachers, where the classroom floor served as the complex plane, the participants physically engaged in exploring the behavior of multiplying $2 + \frac{1}{2}i$ by i. Such perceptuo-motor activity provided a setting where the participants discovered and conceptualized the structural components behind complex number addition and multiplication.

In another related study, Danenhower (2006) asked undergraduates to convert instantiations of the fraction $\frac{a+bi}{c+di}$ into Cartesian $(x + yi)$ or exponential $(re^{i\theta})$ form. The participants tended to avoid the exponential form, especially if it required converting between polar $(r(\cos\theta + i\sin\theta))$ and exponential forms due to their perceived weakness with trigonometric functions. Although the students easily worked with the Cartesian form, it was generally not the most efficient method to simplify the fraction. In many instances a geometric interpretation would have alleviated much of the computational effort, but the participants did not appear to draw on such an interpretation. Similar results were found by Panaoura et al. (2006), who investigated Greek high school students' ($N = 95$) proficiency moving between algebraic and geometric representations of complex-valued equations and inequalities of the form $|z - z_0| \le k$. In general, the participants were more successful when explicitly asked to provide an algebraic representation of a given geometric figure, but this was not evident when the students were asked to solve a similar problem-solving task. This may suggest "a lack of flexibility in using the geometric approach effectively with different representations of complex numbers" (p. 700).

While the research suggests that high school students and undergraduates tend to view i as a static object, struggle to provide geometric meaning to complex number arithmetic, and fail to recognize which form is more appropriate for a given situation, Soto-Johnson et al. (2011) and Soto-Johnson et al. (2012) indicates that experts do not have such difficulty. In attending to given tasks, the experts easily recognized which representation was better suited for each task. It was "natural" for them to view the complex numbers as vectors for addition, to represent them in Cartesian form for addition and polar form for multiplication, and to express z as an operator and w as the operand for multiplication. The experts easily connected algebraic and geometric representations and navigated between representations and forms. Furthermore, they recognized which form and which representation were most appropriate for more advanced tasks. This flexibility allowed them to provide responses involving metaphors, which highlight the dynamic aspects of complex numbers.

Theoretical Perspective

In an effort to explore teachers' conception of complex numbers, we incorporated Sfard's (1991) duality principle of conception for the different representations and forms of complex numbers. Sfard defines *conception* as "the whole cluster of internal representations and associations evoked by the concept [or notion]" (1991 p.3), and describes two types of conception: *operational* and *structural*. *Structural conception* refers to treating or seeing mathematical notions as abstract objects. An example includes perceiving a complex number as a number- a fully-fledged mathematical object on which processes can be performed. *Operational conception* focuses on the "processes, algorithms and actions" (p.4) performed on mathematical notions. For example, recognizing i as the square root of negative one. When there is no evidence of conception, Sfard (1991) classifies this as the pre-conceptual stage. The two conceptions, *operational* and *structural*, of the same notion complement each other and foster a dual conception.

In the development of a mathematical notion, operational conception precedes structural and three stages of development, *interiorization*, *condensation* and *reification*, illustrate the transition from process to object. During the stage of *interiorization* the learner skillfully performs processes on developed mathematical notions. At the stage of *condensation* the learner can perform many processes and is capable of viewing them as a whole without going into details of each step. As learners progress in this stage, they begin to manifest more flexibility in translating between different representations of the same notion. The progression continues until the learner starts to recognize the object as a new entity or is able to distinguish the object from the processes. *Reification* is the stage when the learner can extract the object from processes. In contrast to the previous stages, the shift to reification can be instantaneous. At this stage different representations of the same notion merge together.

In our study, our participants were introduced to the Cartesian, polar, and exponential forms of a complex number with algebraic and geometric representations for each form. We incorporated Sfard's (1991) duality principle as part of the professional development highlighting the connection between different representations to provide an opportunity for the teachers to condense and possibly reify the different forms of complex numbers. Sfard highlights the importance of such practices and warns "As long as the computational processes have been presented in the purely operational way, they could not be squeezed into static abstract entities, thus were not susceptible of being treated as objects." (p.24) We examined the three teachers' dual conception of each form to capture their overall conception of complex numbers. In our analysis, we explored participants' use of multiple representations with various forms in order to distinguish between their operational and structural conception of a form.

Methods

As part of this study, in-service high school mathematics teachers engaged in a three-day professional development (PD) program intended to strengthen their content knowledge of complex numbers. Besides introducing the participants to the three forms of a complex number, we also illustrated various representations for each form. As part of the PD the teachers engaged in discussions emphasizing the connection between these various forms, shared their perceptions of complex number arithmetic, discovered dynamic representations of the arithmetic of complex numbers using *GeoGebra*, compared real and complex number arithmetic, and provided algebraic and geometric explanations for "complex sentences."

Three teachers, Melissa, Aaron, and Troy (all pseudonyms), participated in an individual 90-minute task-based interview after the completion of the PD. The goal of the interview was to gain understanding of the ways in which the teachers used different representations in their mathematical reasoning of complex numbers with novel tasks presented in various forms. At the time of our study Melissa was in her first year as a full-time teacher and taught algebra II and geometry; Aaron taught geometry and was in his second year of teaching; and Troy, who was in his 21st year of teaching, taught IB mathematics. Both Aaron and Troy had Masters degrees in mathematics education. These interviews served as our primary source of data; other data included video-recording of each of the PD days and teachers' in-class work, which was used for triangulation purposes. All three interviews were fully transcribed and each member of the research team used deductive analysis techniques (Erickson, 2006) to code the teacher's responses. This entailed cataloging how and when the participants used various representations for each form. This allowed us to provide evidence regarding the dual conceptualization of a given form. We refined our results after sharing and discussing our individual analysis, which was followed with a cross-case analysis.

Results

Our analysis suggests that none of the interview participants (Melissa, Aaron, and Troy) had a dual conceptualization of a complex number, although each teacher articulated reasoning that conveyed structural conceptualization for some forms of complex numbers. In other words, our participants' conceptualization of a complex number tended to be bound by their conceptualization of each form.

Overall, Melissa had an operational conception of i, while evidence suggests both Aaron and Troy had a dual conception of i. During the interview Melissa referred to i as the square root of -1 multiple times and utilized the fact that i^2 is equal to -1 in her explanations of her solutions. Even though she recognized and used different representations of i, she did not flexibly connect the various representations. For example, when she wanted to represent i as a point on the Argand plane, she was hesitant whether it was the point $(0,1)$ and asked the interviewer if her point was correct. After receiving confirmation she was not hesitant anymore. Throughout the remainder of the interview she translated back and forth between a point representation and algebraic one as she performed manipulations with i. We interpreted Melissa's such actions as her trying to condense the form i by moving between representations. However, we did not find any evidence where she reified this particular form. For these reasons we believe that Melissa had an operational conception of i and appeared to be at the condensation stage. On the other hand, while Aaron and Troy both stated and used the fact that i was the square root of -1, they both also recognized and utilized different representations of i flexibly in their explanations and solutions during the interview.

Similarly, for the Cartesian form Melissa had an operational conception, while Aaron and Troy had a dual conception. When asked to describe how she thinks of a complex number, Melissa replied with "*Well I guess just the letter* i *and anything that correlates with having* i, *so like* $i + 1$ *and multiples of* i *and all that…*" She used this description of a complex number throughout the interview. Melissa relayed a complex number in Cartesian form as an algebraic process performed on i, which is evidence of an operational conception of this form. In contrast, the evidence suggests that a complex number in the form $a + bi$ is an object for both Troy and Aaron. At one point in the interview, Aaron stated "*So if you're telling me* z *is complex figure,* z *is going to be in the form* $a + bi$*,*" and a similar instance occurred with Troy. Such instances were coded as a structural conception of this form, since the participants used this form as an object. The reason for such coding decisions were from

Sfard's framework in which she suggests that "when tackling a genuinely complex problem, we do not always get far if we start with concrete operations; more often than not it would be better to turn first to the structural version of our concepts." (p.27) Moreover, both Aaron and Troy were able to consider multiple representations of the Cartesian form simultaneously. For example, while working on an interview task, they each declared that multiplying a complex number by i took the point (a, b) to the point $(-b, a)$.

During the interview, Melissa was not able to work effectively with the exponential form. For example, when asked to explain why $\frac{r_1 e^{i\phi}}{r_2 e^{i\theta}} = \frac{r_1}{r_2} e^{i(\phi-\theta)}$ was true, Melissa responded that her solution method was "*comparing both sides of the equals [sign]*". She continued to articulate how she would simply compare the symbols and use the "law of exponents", which she had just covered in class. Such a response led us to believe that her conceptualization of the exponential form was at the pre-conceptual level. On the other hand, both Troy and Aaron appeared to possess an operational conceptualization with the algebraic representation of the exponential form. This was evidenced with their quick response that the statement was true due to the law of exponents. Furthermore, Troy demonstrated a pre-conceptual level of the geometric representation for exponential form (a polar vector representation of exponential form was considered to be a geometric representation of this form), while Aaron provided evidence of a pseudostructural conceptualization for the geometric representation of the exponential form. These conclusions are based on the fact that Troy struggled to recall the meaning of the exponential form and his attempt to divide vectors was problematic, which seemed to suggest he possessed a preconceptual understanding of the exponential form. Similarly, Aaron viewed the complex numbers as vectors as evidenced in his statement, "*Or you can think of it as vector 1 being divided by vector 2. So vector 2 is acting on vector 1.*" The fact that Aaron perceived the task as division of vectors, which does not make mathematical sense, appeared to hinder his ability to provide a viable geometric representation illustrating his algebraic explanation. Sfard (1992) provides a special term for such a case, pseudostructural conception, meaning that a person has both operational and structural conceptualization at certain instances and neither at other cases, and states "such tendency may indicate a semantically debased conception." (p.75) This led us to believe that Aaron had a pseudostructural conception of the geometric representation of the exponential form of complex numbers.

Conclusion

Our results indicate that the participants did not have dual conception of complex numbers, however developed duality of some forms of complex numbers. Even though teachers were provided opportunities during PD to condense and reify complex numbers structurally by practicing using various forms and translating between them using various representations, such practices were only observed in certain forms of complex numbers. As Sfard (1991) states "The reification, which brings relational understanding, is difficult to achieve, it requires much effort […]." (p.33). It is quite possible our participants needed more time to reify the complex numbers.

Our experiences make us believe that universities need to examine how they train prospective teachers and offer PD for inservice teachers regarding complex numbers. We are not proposing that preservice teachers complete a complex variables course, but room must be made in the curriculum for prospective teachers to develop a dual conceptualization of complex numbers. For example, such exposure could exist in methods and technology courses designed specifically for prospective secondary mathematics teachers. In the methods course, opportunities could be provided for the prospective teachers to review high school texts in order to obtain a better idea of where and how complex numbers emerge in the curriculum. A technology course is another excellent venue where both preservice and inservice teachers can learn about complex numbers using software such as *Geometer's Sketchpad* or *GeoGebra* to explore the behavior of complex-valued functions. Such practices may reinforce the progress towards a dual conceptualization of complex numbers.

The recommendations put forth by the Common Core State Standards for high school students to understand the structure and properties of complex numbers as well as their arithmetic operations will hopefully transform how complex numbers are taught in high school level. But such a transformation will also require assistance from schools, universities, and assessment agencies. More research investigating both students and teachers' conceptualization of complex numbers and complex valued functions will help us to develop better teaching practices in this content domain.

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COMMONLY IDENTIFIED STUDENTS' MISCONCEPTIONS ABOUT VECTORS AND VECTOR OPERATIONS

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Abstract

In this report we present the commonly identified error patterns and students' misconceptions about vectors, vector operations, orthogonality, and linear combinations. Twenty three freshmen students participated in this study. The participants were non-mathematics majors pursuing liberal arts degrees. The main research question was: What misconceptions about vector algebra were still prevalent after the students completed a freshmen-level linear algebra course? We used qualitative data in the form of artifacts and students' work samples to identify, classify, and describe students' mathematical errors. Seventy four percent of students in this study were unable to correctly solve a task involving vectors and vector operations. Two types of errors were commonly identified across the sample: a lack of students' understanding about vector operations and projections, and a lack of understanding (or distinction) between vectors and scalars. Final results and conclusions include research suggestions and practitioner-based implications for teaching linear algebra in high school and college.

Key words: Linear algebra, vectors, and students' misconceptions

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Background

Research suggests that students transitioning from computation-heavy courses to more proof-oriented mathematics courses have a lot of difficulties, especially in topics of linear algebra (Rasmussen et.al, 2010). Linear algebra serves as the bridge between many mathematics domains due to its content and significant connections between lower and upper level mathematics. Common Core Standards (CCSS) has placed a strong emphasis on students' learning of linear algebra topics such as vectors and matrices during their high school years to help students to better transition into business algebra, linear algebra, and calculus in college (CCSS, 2010).

However, the issue of conceptualizing abstract ways of reasoning is becoming increasingly problematic for most of the students (Rasmussen et. al. 2010; Hillel & Dreyfus 2005; Stewart & Thomas, 2009; Tabaghi 2010). Research studies have strongly suggested that students are able to grasp and perform the computational aspects; however, they have trouble understanding the conceptual notions and the mathematical ideas behind their computations (Stewart & Thomas, 2009, p. 951; Gueudet-Chartier, 2004; Tall, 2004). With linear algebra becoming a strong emphasis of high school mathematics curriculum (CCSS, 2010), consequently affecting many high school students and freshmen entering college, more research is needed to better understand students' difficulties and mathematical misconceptions in linear algebra.

Theoretical Perspectives

Whether it is due to a lack of visual representation or the task of generalizing familiar concepts, dealing with forms of abstraction and proof appears to be very difficult for students (Rasmussen et. al p. 1577, Hillel & Dreyfus p.181). Stewart and Thomas (2009) studied a group of undergraduate students, who struggled with vector addition and spanning, as well as not being able to distinguish between linear combinations and linear equations. The authors concluded that perhaps the teaching methods of linear algebra need to be re-focused to emphasize the "embodied world, symbolic world, and formal world" of mathematics (Stewart and Thomas, 2009, p. 956). These three worlds were initially introduced by David Tall (2008), who defined them as: the *conceptual-embodied* world - based on perception, action, and thought, the *proceptual-symbolic* world – based on calculation and algebraic manipulation, and the *axiomatic-formal* world – based on set-theoretic concept definitions and mathematical proof (Tall, 2008).

Tabaghi (2010) suggests that students' transition between operational thinking to structural thinking is critical. However, teaching this transition is difficult considering most mathematics instruction up to that point focuses on procedures and algorithms, leading many students to mainly develop *operational thinking.* Tabaghi defined operational thinking as "conceiving a mathematical entity as a product of a certain process" while *structural thinking* involves the conception of mathematical entity as an object (p. 1507). Tabaghi (2010) found that most of her students described linear transformation only as vectors (operational thinking). She explained that "typically, students are unable to visually represent the concept or do not adequately picture all the possibilities" (Tabaghi, 2010, p. 1507).

In contrast, Harel (1989) found that his students had difficulties in proof because they held a limited point of view about what constitutes justification and evidence. Harel (1997) claimed that students do not develop adequate concept images for the definitions provided by teachers or textbooks. Harel explained that, most textbooks use "algebraic embodiments rather than geometric ones", which are in part problematic due to their puzzling notion of "unfamiliar algebraic systems" (Harel, 1997, p. 56)

In this study we focused on students' error patterns and mathematical misconceptions related to vectors, vector operations and dilations, orthogonality, and linear combinations.

The main research question we sought to investigate was: *What misconceptions about vector algebra are still prevalent after the students complete a linear algebra course?* We collected students' written work samples from summative assessment at the end of the course to be able to classify, describe, and document students' error patterns and mathematical misconceptions related to these topics. Based on the findings of this research, we provided research suggestions and practitioner-based implications for high school teachers and educators, who are working to help students improve their experiences learning linear algebra.

Research Methodology

We used *phenomenography* as the research methodology for this study, which investigates the qualitatively different ways in which people think about something (Marton, 1981; 1986). Qualitative data collection methods were used to capture students' thinking and understanding of concepts of vector algebra. We collected hand-written student work samples and artifacts focusing on constructive response questions that required students to provide answers as well as explanations to their answers. The goal was to collect data in such a way that the researchers imposed a minimal amount of mathematical influence or instructional bias on the participants and their data. The researchers were not the instructors of this course nor were they familiar with the students enrolled in the course.

Methods & Procedures

To ensure better-quality instruction, careful consideration was given to the selection of the linear algebra course and its instructor. The instructor of the selected course was a senior professor of mathematics with expertise in linear algebra, who had taught the course for many years and was the author of the course textbook.

Sample & Context

The participants of this study were twenty three first-year college students enrolled in a Linear Methods course in a small urban private liberal arts college. Linear Methods is a variation of a traditional Linear Algebra course. This course was developed in response to a new curriculum for incoming college freshmen. The student participants were liberal arts majors (e.g., business, social sciences); mathematics and science majors do not take this course. As a requirement, students must take one mathematics "beauty" course, that is not computation-heavy (such as calculus), introducing students to a more elegant aspect of mathematics. All the "beauty" courses are required to include explorations of basic concepts of logic and methods of proof. These types of courses are considered tamer versions of their advanced counterparts, such as: Number Theory, Differential Geometry, Topology, and Abstract Algebra. As part of the course, our participants touched upon some essential topics of linear algebra without delving far into the theory. The class met for a total of fifty minutes three times per week for eleven weeks. It was taught by a senior professor of mathematics in a lecture format with very little small group student interactions. The instructor encouraged students to attend supplementary review sessions organized and taught by the teaching assistant two times per week.

Linear Methods Course

In Linear Methods students studied several linear algebra topics, such as: vector manipulations, scalar multiplication and vector addition, systems of linear equations, inverse matrices, and linear transformations (e.g., rotations and reflections over lines). The instructor taught these topics mostly from a deductive perspective by concentrating students' attention and learning on the definitions and formal structures. For example, the curriculum is organized to sequence matrix addition before introducing vectors, and later in the course defines a vector as a special type of matrix. The curriculum includes very few opportunities for students to explore the reasons for why two matrices (and ultimately vectors) can be added or multiplied together, while focusing largely on the mathematical procedures and examples for students to practice mathematical computations of matrix (and vector) manipulations. Over the past few years, however, the instructor had expressed concerns that the students in this course are fundamentally struggling with graphing exercises, especially the principles of vector addition and scalar multiplication. He also indicated that the students lack understanding and do not see the connections between matrices and linear systems.

Mathematical Task

The following task was the main question of our students' data and analyses:

Express the vector $\vec{x} = [x_1, x_2, x_3]$ as a sum of two vectors, one of which is parallel to $\vec{y} = [y_1, y_2, y_3]$, and second is orthogonal to $\vec{y}$. Use fractions instead of decimals.

This task required students to present their work and demonstrate an understanding of six overarching mathematical concepts related to vectors:

- Finding a parallel vector to a given vector (i.e. shift of a vector)
- Finding a perpendicular vector to a given vector (i.e. dilation of a vector, the projection of a vector onto another vector)
- Vector multiplication by a scalar;
- Finding the length of a vector;
- Vector addition and subtraction;
- Representing vectors as a sum of two vectors;

We selected students' work from this task because it included multiple parts and necessitated students to illustrate their answers, which allowed us to thoroughly analyze their work and identify mathematical error patterns. This task was given to the students at the end of the course, as part of their final exam. Thus, students had a period of one semester to confront their misunderstandings of linear algebra by asking either the instructor or the teaching assistant assigned to the course.

Data Analyses

Our goal was to investigate student's responses, specifically their errors and misconceptions related to vector algebra. We coded students work for error patterns within the six abovementioned categories. We then analyzed the coded data for common themes of error patterns and misconceptions. We focused on the mathematical errors behind the solutions, rather than students' processes of obtaining the solution. We generally noted the arithmetic and computational errors in our analyses as well (i.e. a student multiplied two fractions by finding a common denominator), however we didn't focus our analyses on these types of errors. We also did not analyze (nor did we collect) the course grades of these students.

Findings

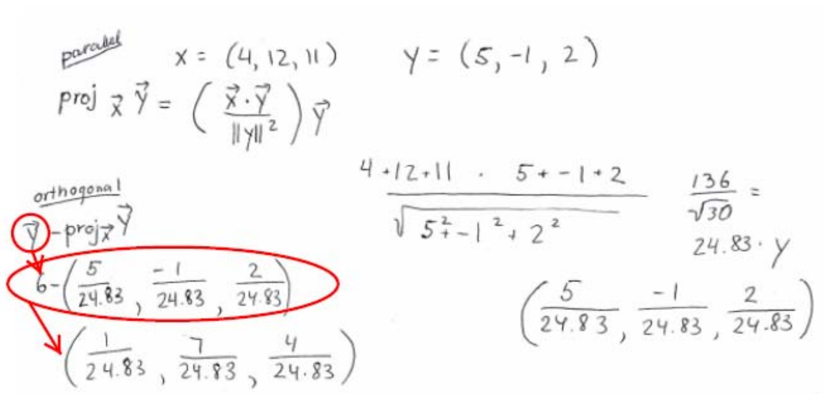
Only five students (out of 23 total) were able to correctly answer the question. One student did not provide a response to the question. The remaining seventeen students answered the question exhibiting two types of common misconceptions related to: the reasoning and spatial sense about vector operations and projections, and the understanding (or distinction) between vectors and scalars.

Lack of Understanding (or Distinction) Between Vectors and Scalars

Thirty five percent of students (8 out of 23) demonstrated fundamental misunderstandings of the meaning of vectors and scalars, and failed to differentiate between vectors and scalars. These errors were especially evident in students' operations with vectors - students confused vectors with scalars and performed arithmetic operations, often treating them as numbers. Many students in this category also have mistaken the vectors for scalars and used algebraic operations with them to obtain either vectors or scalars as a result.

For example, in Figure 1, Bobby computed a difference between a scalar and a vector and got a vector as a result.

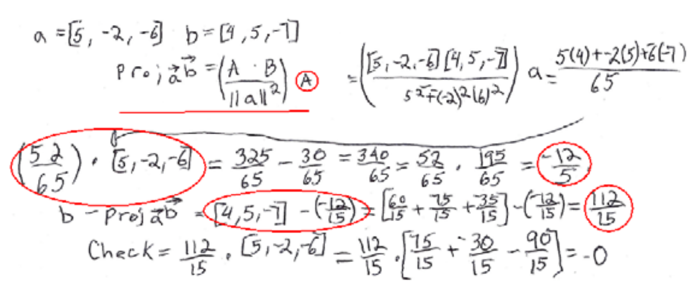
***Figure 1*. Bobby's Misconception: Scalar – Vector = Vector**



Other errors were also evident in Bobby's work, including: $\vec{y} = 6$; assumption that vector = scalar; incorrect reasoning about the projection of vectors; subtraction of fractions from whole numbers; calculation errors (136 in the numerator); and the use of square roots.

In contrast, Figure 2 illustrates the work of Casey, who subtracted a scalar from a vector and got a scalar as the result.

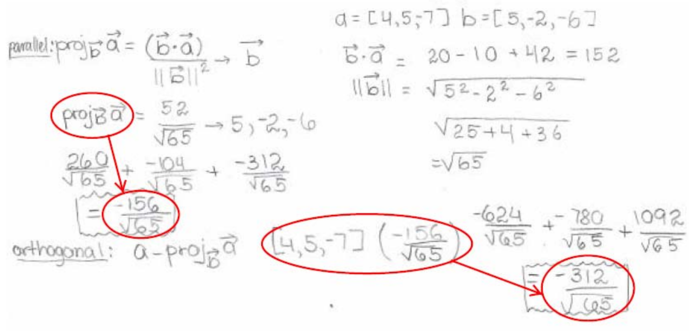
***Figure 2*. Casey's Misconception: Vector – Scalar = Scalar**



Similarly, Casey also used incorrect reasoning about: projection of vectors; assumption that vector = scalar; and a false interpretation for the dot product of a vector and a scalar.

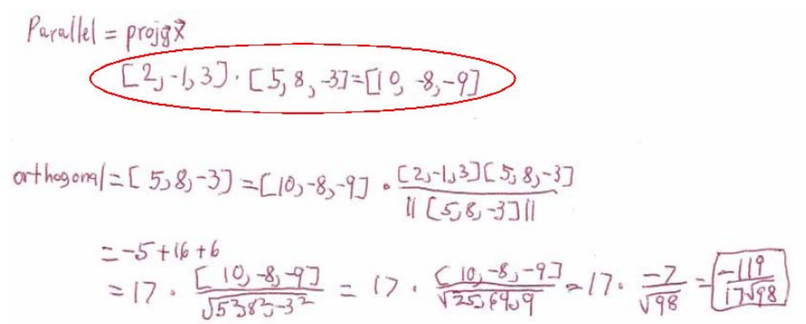
Indeed, the dot products of vectors and scalars have been common error patterns for most of our students' work samples. For example, Harper interpreted the dot product of a scalar and a vector as a vector. Harper, another student, also incorrectly assumed that vector = scalar, and used the square roots in the solution of this problem (see Figure 3).

***Figure 3*. Harper's Misconception: Scalar • Vector = Scalar**



The last work sample (Figure 4) in this category that we chose was the work of Hayden. This student demonstrated many difficulties and misconceptions.

***Figure 4*. Hayden's Misconception: Vector • Vector = Vector**

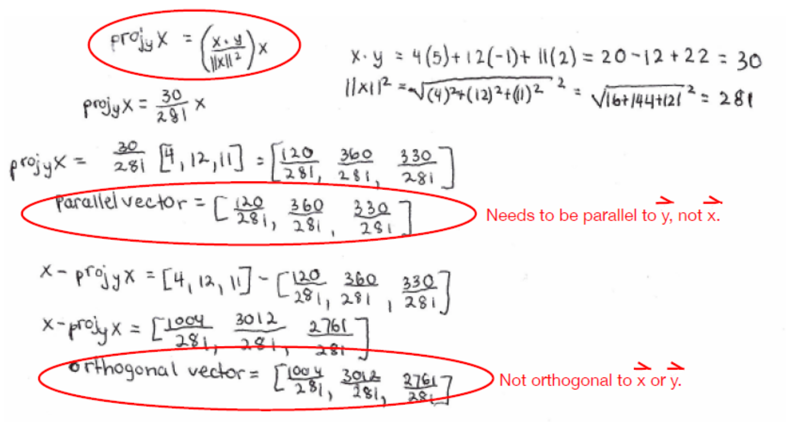


It was evident that Hayden struggled with simple arithmetic computations (multiplication of fractions by whole numbers), in addition to the false assumption that vector = scalar, incorrect reasoning about parallel and perpendicular vectors, and calculating the result of a dot product of two vectors as a vector (Figure 4).

Lack of Reasoning and Spatial Sense about Vector Orthogonality & Projections

Forty percent (9 out of 23) of students demonstrated limited understanding about the concept of parallel and perpendicular vectors. The common error was the fact that the students interpreted the vector projection as a scalar. These misconceptions were evident from the students' work samples above; however, we present one more example to strongly emphasize these misconceptions (see Figure 5,where $\vec{x}$ [4, 12, 11] and $\vec{y}$ [5, -1, 2])

***Figure 5*. Jordan's Misconception: Parallel and Perpendicular Vectors**



The results of these findings, also suggest that perhaps students used incorrect interpretation for the vector projections due to the lack of their understanding of the notation for vector projections. Another hypothesis is that students may have possibly confused the spatial orientation of the vector that they needed to project. Nonetheless, these errors demonstrate students' fundamental misconceptions about vectors, orthogonality, and the meaning of vector projections.

Discussion & Conclusion

We wanted to stress the fact that the students in our sample were freshmen college students completing a "beauty" mathematics course in linear algebra designed to meet the general education program course requirement for non-mathematics majors. One of the aspects that stood out in the students' work across the entire sample, however, was the fact that none of them used pictures to represent (or reason through) the solution to this task. As part of the recommendations for high school and entry-level college teaching of linear algebra, we would like to propose additional approaches that might be helpful for students' learning, especially approaches emphasizing pictorial representation.

Recent research strongly emphasizes the use of geometric approaches and representations in linear algebra. Gueudet-Chartier (2002) investigated students' geometric intuition, "use of geometrical or figural models", and its effect on students' ability to find mathematical models and develop conceptual understanding. The author suggested that mathematics instruction that focused on the use of drawings in general vector spaces was critical for his students; otherwise, the students were unable to find models and correct intuition to develop conceptual understanding of these topics (Gueudet-Chartier 2002; 2004).

Similarly, Harel (1989) found that, in comparison to strictly algebra-taught students, the students, who engaged in geometric interpretations outside of just algebraic ones, were able to answer more questions correctly and had an easier time visualizing and understanding the concepts using concept images.

Geometric representations also help to develop structural thinking (Tabaghi, 2010). Tabaghi argued that, typically, students are unable to visually represent the concept or do not adequately picture all the possibilities (Tabaghi, 2010, p. 1507). Therefore, incorporating opportunities for the students to explore abstract concepts through visual representations is critical and necessary to help students overcome the difficulties and misconceptions in linear algebra (Tabaghi, 2010).

One of the possible solutions of utilizing pictorial approach is included as a sample solution in Appendix A. This solution takes on an analytical (geometric) rather than procedural (computation-based) approach. First, this solution provides a geometric meaning that the projection of $\vec{x}$ onto $\vec{y}$ is parallel to $\vec{y}$ by shrinking/expanding $\vec{y}$ (thus the result is $c\vec{y}$, where c is a non-zero constant). Second, this solution emphasizes a geometric meaning that $\vec{x}$ minus the projection of $\vec{x}$ onto $\vec{y}$ is orthogonal/perpendicular to $\vec{y}$ (thus the dot product is zero). To help the students to "see" the vector (in blue), basic reasoning about geometric addition of vectors is needed. Additional prerequisite knowledge required for this task is: the distributive property of dot products and factoring out a constant.

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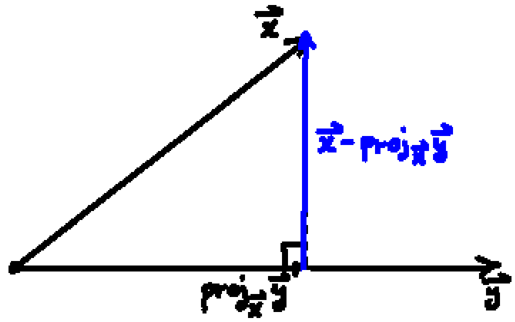
APPENDIX A



$$\underbrace{\text{proj}_{\vec{x}}\vec{y}}_{\text{PARALLEL TO } \vec{y}} + \underbrace{\left(\vec{x} - \text{proj}_{\vec{x}}\vec{y}\right)}_{\text{ORTHOGONAL TO } \vec{y}} = \vec{x}$$

SINCE $\vec{x} - \text{proj}_{\vec{x}}\vec{y}$ IS ORTHOGONAL TO $\vec{y}$, $(\vec{x} - \text{proj}_{\vec{x}}\vec{y}) \cdot \vec{y} = 0$.
ALSO, SINCE $\text{proj}_{\vec{x}}\vec{y}$ IS PARALLEL TO $\vec{y}$, $\text{proj}_{\vec{x}}\vec{y} = c\vec{y}$. NOW,

$$(\vec{x} - c\vec{y}) \cdot \vec{y} = 0$$
$$\vec{x} \cdot \vec{y} - c\vec{y} \cdot \vec{y} = 0$$
$$\vec{x} \cdot \vec{y} = c\vec{y} \cdot \vec{y}$$
$$\frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} = c$$

SO, $$\text{proj}_{\vec{x}}\vec{y} = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}\vec{y} = \frac{(4,5,-7) \cdot (5,-2,-6)}{(5,-2,-6) \cdot (5,-2,-6)}(5,-2,-6) = \frac{52}{65}(5,-2,-6).$$

FINALLY, $\text{proj}_{\vec{x}}\vec{y} = (4, \frac{-8}{5}, \frac{-24}{5})$ AND $$\vec{x} - \text{proj}_{\vec{x}}\vec{y} = (4,5,-7) - (4, \frac{-8}{5}, \frac{-24}{5})$$
$$= (0, \frac{33}{5}, \frac{-11}{5}).$$

PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING AND CONCEPTIONS OF TEACHING EFFECTIVENESS: ARE THEY RELATED?

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Salisbury University

Mathematical knowledge for teaching (MKT) is essential for effective teaching of elementary mathematics. Given the importance of MKT, MKT and conceptions of teaching effectiveness should not develop independently. The purpose of this study was to examine whether and how K-8 pre-service teachers' MKT and personal mathematics teacher efficacy beliefs are related. Results indicated overconfidence in teaching ability was prevalent, with the majority of participants exhibiting a strong sense of personal mathematics teacher efficacy but low levels of MKT. Pre-service teachers with high levels of MKT, however, reported a more accurate assessment of their teaching effectiveness. Results also indicated that examining pre-service teachers' self-evaluations of MKT is helpful for understanding pre-service teachers' personal mathematics teacher efficacy beliefs. Moreover, the results of this study point to the inadequacies of existing measures of teacher efficacy beliefs that do not parse out differences in efficacy beliefs according to a number of contextual factors.

Key words: Mathematical Knowledge for Teaching, Pre-service Teacher Education, Efficacy Beliefs

Introduction and Theoretical Background

Personal mathematics teacher efficacy beliefs (PMTE beliefs) are a teacher's beliefs about her abilities to teach mathematics effectively (see, e.g., Tschannen-Moran & Hoy, 2001). How effective a teacher will be for student learning depends on the level of mathematical knowledge that the teacher has. In particular, previous research has indicated that teachers with higher levels of *mathematical knowledge for teaching* (MKT) are more effective teachers (Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball 2005). Therefore, if a teacher is to develop accurate views of her teaching effectiveness, these views should be connected to the level of MKT that the teacher has.

Potential relationships between teacher efficacy beliefs and content knowledge for teaching have been examined in previous research. Such studies have produced inconsistent results, with efficacy beliefs and content knowledge for teaching found to be positively correlated (Swars, Smith, Smith, & Hart, 2009), weakly positively correlated (McCoy, 2011), negatively correlated (Wenner, 1993), or uncorrelated (Swars, Hart, Smith, & Tolar, 2007; Wenner, 1995).

These inconsistent results might be due to a mismatch between assessments of mathematical knowledge for teaching and assessments of personal mathematics teacher efficacy beliefs. Measures of mathematical knowledge for teaching are designed to reflect what teachers actually do in mathematics classrooms, but measures of personal mathematics teacher efficacy beliefs are typically not situated in classroom tasks. Additionally, typical efficacy-beliefs measures do not contain any actual mathematics. For example, both Swars et al. (2009) and McCoy (2011) measured personal mathematics teacher efficacy beliefs with a subscale of the Mathematics Teaching Efficacy Beliefs (MTEBI) Instrument. MTEBI items do not situate teacher efficacy beliefs in any classroom situations and do not contain any mathematics. Rather, items refer to mathematics teaching in a general way, such as in the items "I know how to teach mathematics concepts effectively" and "I wonder if I have the skills necessary to teach mathematics" (Enochs, Smith, & Huinker, 2000, p. 200-201).

Thus, measures of mathematical knowledge for teaching and measures of personal mathematics teacher efficacy beliefs are apparently disconnected. Measures of mathematical knowledge for teaching assess mathematical knowledge for teaching specific mathematical content, but measures of personal mathematics teacher efficacy beliefs measure beliefs not tied to specific content. Efficacy beliefs are task-specific constructs (Bandura, 1986), so personal mathematics teacher efficacy beliefs are likely to vary based on the content to be taught. One alternative is to measure both personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching in the context of specific mathematical teaching tasks. This is the approach taken in the current study.

The overall purpose of this study was to examine whether and how pre-service-teachers' personal teacher efficacy beliefs were related to their mathematical knowledge for teaching. No empirical studies identify whether there is an ideal relationship between these two constructs. However, given the importance of mathematical knowledge for teaching, having personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching that are aligned is perhaps preferable. In this article, personal mathematics teacher efficacy and mathematical knowledge for teaching will be considered *aligned* when higher levels of personal mathematics teacher efficacy accompany higher levels of mathematical knowledge for teaching, or lower levels of personal mathematics teacher efficacy accompany lower levels of mathematical knowledge for teaching. Pre-service teachers who have personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching that are misaligned might overestimate their teaching abilities. Such overestimation is a potential barrier to improving one's teaching, as dissatisfaction with one's performance can be a catalyst for change (Guest, Regehr, & Tiberius, 2001; Wheatley, 2002).

The study aimed to address the following research questions: (1) How prevalent is alignment of personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching? (2) What differences are evident between pre-service teachers with low mathematical knowledge for teaching and those with high mathematical knowledge for teaching with respect to alignment of personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching? (3) How do pre-service teachers' self-evaluations of their mathematical knowledge for teaching relate to their actual mathematical knowledge for the aligned and misaligned groups?

Methods

Forty-two K-8 pre-service teachers participated in the study. The study was conducted at a medium-sized university in the Northeastern United States. All participants were enrolled in the second course of a three-course series of mathematics courses required for the teacher education program at this university. Participants were randomly selected for participation from the pool of 209 students enrolled in this second course.

Pre-service teachers first participated in a 90-minute semi-structured interview in which they were asked to respond to four *Teaching Scenario Tasks*. For each task, pre-service teachers first gave a written response and then were asked to explain their answers orally. All interviews were audio-recorded; recordings were used to supplement written responses. Two to four weeks after the semi-structured interview, pre-service teachers participated in an individual 60-minute session to complete four *MKT tasks*.

Each of the Teaching Scenario Tasks presented a scenario that required the pre-service teachers to give a conceptual explanation to a student's "why" question about a problem involving fractions. A sample Teaching Scenario Task is displayed in Figure 1. Pre-service teachers' responses to the prompt "I am confident that my explanation would be effective in helping the students understand the relevant concepts" measured personal mathematics teacher efficacy for the given task. Responses of strongly disagree or disagree were considered *low PMTE* and responses of strongly agree or agree were considered *high PMTE*. Each of the Teaching Scenario Tasks had the same format.

Each MKT Task was designed to measure participants' MKT for the mathematics involved in the corresponding Teaching Scenario Task. Figure 2 contains the MKT Task that corresponds to the Teaching Scenario Task in Figure 1. For each task, a list of subcomponents involved in giving a complete mathematical explanation was constructed. Pre-service teachers could obtain a score of 0, 1, or 2 for each subcomponent. The total MKT score for a particular task was the sum of these scores across all subcomponents. Inter-rater reliability scores were obtained for MKT-coding on each task. The ratings for Tasks 1, 2, 3 and 4 were 81%, 82%, 82%, and 92% respectively.

For each task, a participant was considered *high* with respect to her exhibited MKT if her exhibited MKT score was at least 70% of the total possible score. Pre-service teachers with scores less than 70% of the total possible score were considered *low* with respect to her exhibited MKT. A cut-off of 70% was used because of the high standard for each subcomponent; that is, obtaining a score of 2 was difficult.

Results

On each MKT Task, pre-service teachers were asked to respond to the prompt "I am confident that I understand the mathematical concepts in this task." Responses to this prompt are displayed in Table 1. One notices from the table that pre-service teachers rated Task 2 most understandable mathematically and Task 4 least understandable mathematically. This result will be helpful in understanding the overall results for each of the three research questions.

To address research question 1, percentages of pre-service teachers for whom the two constructs were aligned or misaligned were calculated, as shown in Table 2. Misalignment was evident on Tasks 1 and 3 with a majority of pre-service teachers exhibiting high PMTE beliefs but low MKT on these tasks. On Tasks 2 and 4, the tasks rated most and least understandable respectively, higher frequencies of pre-service teachers had aligned PMTE beliefs and MKT. Overall, misalignment was prevalent, with 90 of the total 168 cases (42 participants on each of 4 tasks) falling into the High PMTE beliefs/Low MKT category.

To address research question 2, percentages of pre-service teachers with aligned PMTE beliefs and exhibited MKT by task and by level of MKT for that task were calculated, as shown in Table 3. One notices that high-MKT pre-service teachers tended to fall into the aligned category with greater frequency than low-MKT pre-service teachers, except on Task 4. Task 4 was, again, the task rated least understandable by the sample of pre-service teachers as a whole.

To address research question 3, the self-ratings of mathematical understanding that pre-service teachers gave on a particular task were compared to pre-service teachers' exhibited MKT for that task, as shown in Table 4. For the misaligned groups on each task, self-ratings of MKT and exhibited MKT were uncorrelated. On Tasks 2 and 4, pre-service teachers in the aligned groups who tended to rate their MKT higher also tended to exhibit higher levels of MKT.

Implications

The findings of this study indicate that pre-service teachers' personal mathematics teacher efficacy beliefs are more nuanced than previous research has suggested. The fact that results from the four tasks did not look identical suggests that personal mathematics teacher efficacy beliefs are highly contextual and, as such, should be measured in varying contexts. In particular, pre-service teachers' self-evaluations of their mathematical knowledge for teaching, with respect to how understandable the mathematics in a task was, seemed to influence pre-service teachers' personal mathematics teacher efficacy beliefs. Moreover, personal mathematics teacher efficacy beliefs were better aligned with pre-service teachers' evaluations of their mathematical knowledge than with pre-service teachers' actual mathematical knowledge.

Misalignment of personal mathematics teacher efficacy beliefs and mathematical knowledge for teaching was, overall, prevalent. In particular, the overall frequency of the "High PMTE beliefs/Low MKT" category (54%), arguably the most problematic category, is noteworthy. Many pre-service teachers likely need help in assessing their teaching effectiveness accurately, help that teacher educators might need to provide during teacher education programs. Pre-service teachers in the "High PMTE beliefs/Low MKT" category are likely those whose sense of personal mathematics teacher efficacy is inaccurate and whose MKT needs development. This problem is compounded by the fact that such pre-service teachers are less likely to recognize that they have low MKT (e.g., see Kruger and Dunning, 1999).

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Two students in your fifth-grade class, Joe and Amy, are trying to convert a mixed number 2 ¾ to an improper fraction. Each of their solutions is shown below:

Joe	Amy
$2\frac{3}{4}$	$2^3/4$
$\frac{2\cdot4+3}{4} = \frac{8+3}{4} = \frac{11}{4}$	$\frac{2\times3+4}{4} = \frac{6+4}{4} = \frac{10}{4} = \frac{5}{2}$

1. Joe and Amy both say that they are using a rule that they know for rewriting mixed numbers as improper fractions. They are not sure which rule gives the right answer. If you were Joe and Amy's teacher, how would you explain this to your class so that they would understand?

2. Thinking of yourself as Joe and Amy's teacher, respond to the following:

I am confident that my explanation would be effective in helping the students understand the relevant concepts.

Strongly Disagree Disagree Agree Strongly Agree

(circle one)

Why did you choose this rating?

Figure 1. Sample Teaching Scenario Task.

Joe	Amy
$2\frac{3}{4}$	$2^3/4$
$\frac{2\cdot4+3}{4} = \frac{8+3}{4} = \frac{11}{4}$	$\frac{2\times3+4}{4} = \frac{6+4}{4} = \frac{10}{4} = \frac{5}{2}$

1. Show with a picture which student is correct. Use the picture to show why this procedure works. Give a detailed conceptual explanation that explains your reasoning for each step of the procedure and use your picture to explain your reasoning.

2. Please respond to the following:

I am confident that I understand the mathematical concepts in this task.

Strongly Disagree Disagree Agree Strongly Agree

(circle one)

Figure 2. Sample Participants' MKT Task.

Table 1
Percentages of pre-service teachers' MKT self-evaluation ratings by task (n = 42)

	MKT Self-evaluation rating: "I am confident that I understand the mathematical concepts in this task"			
Task	Strongly Disagree	Disagree	Agree	Strongly Agree
Task 1	0	14	38	48
Task 2	0	0	43	57
Task 3	0	2	57	40
Task 4	2	17	62	19

Table 2
Percentages of pre-service teachers with aligned or misaligned PMTE beliefs and total MKT score by task (n = 42)

	Relationship between PMTE beliefs / Total MKT Score			
Task	High/High	High/Low	Low/High	Low/Low
Task 1	21	71	0	7
Task 2	43	36	5	17
Task 3	21	64	0	14
Task 4	14	43	10	33

Table 3
Percentages of pre-service teachers with aligned PMTE beliefs and total MKT score by task and by level of MKT shown on the task (n = 42)

	Percentage of High MKT and Low MKT pre-service teachers with aligned PMTE beliefs and total MKT score	
Task	High MKT	Low MKT
Task 1	100	9
Task 2	90	32
Task 3	100	18
Task 4	60	44

Table 4
Results from Spearman's rho tests for examining the relationship between MKT self-evaluation and total MKT score

	Aligned Group			Misaligned Group			Entire Group		
Task	n	value of ρ	p-value	n	value of ρ	p-value	n	value of ρ	p-value
Task 1	12	.331	.293	30	.226	.229	42	.335*	.030
Task 2	25	.437*	.029	17	.120	.645	42	.340*	.028
Task 3	15	.283	.304	27	-.043	.835	42	.128	.423
Task 4	20	.616**	.004	22	.052	.820	42	.334*	.031

*$p \leq 0.05$; ** $p \leq 0.01$

Understanding Mathematical Conjecturing

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In this study, we open up discussions regarding one of the unexplored aspects of mathematical sophistication, the inductive work of conjecturing. We consider the following questions: What does conjecturing entail? How do the conjectures of experts and novices differ? What characteristics, behaviors, practices, and viewpoints distinguish novice from expert conjecturers? and What activities enable individuals to make conjectures? To answer these questions, we conducted a qualitative research study of eight participants at various levels of mathematical maturity. Answers to our research questions will begin to provide an understanding about what helps students develop the ability to make mathematical conjectures and what characteristics of tasks and topics may effectively elicit such behaviors, informing curriculum development, assessment, and instruction.

Key words: Mathematical sophistication, Enculturation, Mathematical behaviors

Background

Much of contemporary research is concerned with helping students become more adept at problem solving and learning mathematical ideas. In fact, an ongoing concern is empowering students to develop deep understanding of mathematical concepts, instead of only developing shallow procedural proficiency; that is, we want students to be able to apply their knowledge to solve new problems.

From a sociocultural perspective (Bauersfeld, 1995), we argue that this challenge can be in part met by understanding the practices of the mathematical community. As we identify those practices that enable mathematicians to *do* mathematics and find ways to instill these in our students, we will empower them mathematically.

Many researchers have argued this point. Citing Cobb, Bowers, Lave, and Wenger, Rasmussen et al. (2005) argued that learning mathematics is synonymous with participation in mathematical practices; in other words, many of the activities used by the mathematical profession to build new mathematical artifacts are needed by learners to acquire those same artifacts. Carlson and Bloom (2005) argued that successful problem solving involves more than content knowledge; it requires cognitive control skills, methods, and heuristics. It is these mathematically sophisticated behaviors (such as conjecturing, testing, and modeling) that empower problem solvers to correct their own models and arrive at solutions (Moore et al., 2009).

Researchers have also observed and evidenced this. Seaman and Szydlik (2007) noted that even given ample time and resources, preservice teachers failed to relearn forgotten, common, elementary school mathematics concepts and skills because they lacked *Mathematical Sophistication*, that is habits of mind and practices of the mathematics community that *would have* empowered them to acquire mathematical knowledge; these practices include: making sense of definitions, seeking to understand patterns and structure, making analogies, making and testing conjectures, creating mental and physical models (examples and nonexamples of things), and seeking to understand why relationships make sense. Schoenfeld (1992) noted something similar, namely that novices lacked the skills and behaviors characteristic of expert mathematicians, skills which go beyond simple content knowledge, such as: attending carefully to language, building models and examples, making and testing conjectures, and making arguments based on the structure of a problem. Thus, mathematical sophistication is not only critical for prospective mathematicians, but for *anyone* who must engage in mathematical learning and problem solving.

In a recent study, Szydlik, Kuennen, Belnap, Parrott, and Seaman (2012) developed a measure of basic levels of mathematical sophistication and in doing so, found evidence that mathematical sophistication can be developed during the course of a class. So, in order to empower students to more effectively learn mathematics, we must understand the practices of the mathematical community and find ways for them to acquire these practices.

The body of mathematical knowledge develops as professional mathematicians engage in a variety of activities, most of which could be classified as either inductive or deductive work. The inductive work of mathematics involves activities that generate new mathematical ideas; through investigation, exploration, or study, mathematicians create conjectures (i.e. new and unproven hypotheses). Through deductive work, mathematicians take assumed mathematical ideas (axioms) and proven mathematical facts (theorems) and either prove or disprove those conjectures. Proven theorems then become new artifacts in the mathematical knowledge base.

In this study, we have chosen to focus our efforts on the mathematical activity of conjecturing for three main reasons. First, conjecturing is critical to the field of mathematics; it represents the potential mathematical knowledge of the field. Second, conjecturing tends to be a neglected aspect of mathematics classes and enculturation. Most lower-level mathematics courses focus on understanding and applying known theorems. At the upper-level, most attention is devoted to proofs and logic because of their complex and problematic nature, thus focusing predominantly on understanding content, testing conjectures, developing logical arguments, developing counterexamples, and writing deductive arguments--a focus mirrored by the research literature (Weber, 2001; Selden & Selden, 2003; Alcock & Weber, 2005). Third, (in theory) conjecturing is more accessible to undergraduates and novices because it is the generation of ideas which need not be certain; it is hypothesis-making and does not require all the intricacies that accompany deductive work.

This paper focuses on answering the following question: What does mathematical sophistication look like for conjecturing? To answer this question, we consider the following questions: What does conjecturing entail? How do the conjectures of experts and novices differ?

What characteristics, behaviors, practices, and viewpoints distinguish novice from expert conjecturers? and What activities enable individuals to make conjectures?

Methodology

To explore conjecturing, we conducted a qualitative research study, during winter semester 2012, in a mathematics department at a public university. We purposively selected eight participants at various levels of mathematical maturity: two undergraduate students (novice mathematicians), Scott and Laura; three graduate students (apprentice mathematicians), Ann, Noah, and Charlie; and three research mathematicians (i.e. expert mathematicians) Sam, Josh, and Lisa. Student participants were selected from a list of volunteers, who were enrolled in one of two specific courses; undergraduate students were taking a 200-level introduction to proofs course, while graduate students were enrolled in a graduate course in advanced Euclidean geometry, which incorporated a conjecturing component.

Participants were purposively selected to represent diversity in gender, background in Euclidean geometry, and area of expertise. Student participants were selected for diversity in their ability (high/medium/low) to do authentic mathematical work, as judged by their instructors and (in the case of graduate students) conjectures produced during conjecturing tasks. Mathematicians were selected to represent diverse areas of mathematical expertise, which included: probability theory, graph theory, and dynamical systems.

Data Collection

Data collection revolved around each participant's involvement in one individual conjecturing task in the area of Euclidean geometry. Participants were presented with a hardcopy of the task and given ample time and resources to work on it, then (after a break) participated in an interview regarding their experience, approach, and conjectures.

The task was intended to create a context where participants could do inductive work leading to the generation of conjectures. The task provided mathematical definitions for three new types of quadrilaterals; one faculty participant had previously encountered one of these quadrilaterals, otherwise all three definitions were novel for each participant. Participants were allowed to explore these definitions, with the goal of writing as many conjectures about them as possible; they were given as much time as they desired, with the exception of Josh, whom we cut off after about two hours.

During the task, participants had open access to the following resources: a ruler and compass; various colored and regular writing instruments; paper; a list of Euclidean postulates, common notions, definitions, and propositions; a glossary of common geometry terms; and GeoGebra, a free dynamic software program for constructing, measuring, and manipulating dynamic geometric objects. Researchers were also available throughout to answer questions on the task and definitions or to help with software usage.

After the task, each participant took a break, during which we prepared for the interview by discussing our observations and the participant's conjectures and making adjustments to the interview questions. The subsequent interview focused on: a) clarifying any behaviors and thinking that were not discernible by outward observation, b) understanding the participant's experience and perspective, c) understanding the written conjectures, and d) gaining details about the conjecturing process.

Data consisted of video recordings, written work, and observation notes. We took three video recordings during each task and subsequent interview: a) a video taken from the side, of them working on the task; b) a top-down recording of their written work; and c) an internal recording of their computer work. A synchronized compilation of these three video recordings served as the primary data source. Secondary data sources included each participant's written work and conjectures along with our observation notes taken during each task and interview.

Data Analysis

We analyzed the data using grounded theory techniques (Strauss & Corbin, 1998). Beginning with Scott (low-level novice), we independently reviewed the coordinated video feeds and made time-stamped annotations describing his behaviors throughout the task. Then we compared our annotations and negotiated differences, using the video and secondary sources of data to triangulate our observations. We began clustering our annotations around common themes to form initial categories of behaviors. We utilized our results to inform our next interview and observations (of Laura).

We repeated this same process working upward (by experience and level) through our participants, by next meeting with Laura (medium-level novice) and then Ann (low-level apprentice). After meeting with each participant, we coded independently by writing time-stamped annotations and applying our emerging categories; doing so, we modified our categories as appropriate to accurately describe the data; each time we made changes, we back-coded prior participants to examine how the framework reflected the data.

By our meeting with Noah (medium-apprentice), we noticed some broad, over-arching themes, so we independently synthesized our analyses of each individual into a vignette or synopsis, describing for each case (participant) the characteristic of each theme. We continued this process, independently writing, then collaboratively negotiating a final synopsis for each participant. As we compiled these synopses, we did a cross-case analysis, examining differences and similarities across the different levels of expertise. We herein present some of these differences as dimensions of the mathematical activity *conjecturing*.

Results

The distinction between novice, apprentice, and expert conjecturer is not a clear linear one, but varies by each of the five dimensions listed in table 1. These include: a) *overall process*, the process and problem-solving approach used during the task; b) *objects created for* *investigation*, the characteristics of and individual's view of objects created during the investigation; c) *nature of observations*, describing the things they noticed, paid attention to, or looked-for; d) *qualities of written conjectures*, mathematical and verbal qualities of a conjecture; and e) *qualifications of written conjectures*, the threshold of conviction required to consider an idea worthy to be considered a conjecture.

The behaviors listed for each dimension in table 1 represent the extremes observed. Each of a dimension's behaviors and characteristics range from unsophisticated to sophisticated by how they did or did not empower the individual to make conjectures. Other behaviors were noted that could be considered along a continuum, representing different degrees of sophistication. Furthermore, individuals exhibited combinations of these characteristics.

Discussion

Learning to conjecture appears to entail a variety of skills, knowledge, and values. It is affected by logic, content knowledge, practices of observation, experience with mathematical language, and persistence. Indeed, conjecturing appears to rely or draw upon many of the characteristics Seaman and Szydlik (2007) and Schoenfeld (1992) identified. Because of this

Table 1: Sophisticated and Unsophisticated Characteristics and Behaviors by Conjecturing Dimension

Dimension	Unsophisticated	Sophisticated
Overall process	Clear linear approach: make sense of task, explore, write conjectures. Random manipulation to stumble upon something.	Complex non-linear approach. Incorporation of other mathematical knowledge. Systematic analysis to discover or scrutinize. Scaling the problem in or out.
Objects created for investigation	Single *static* "prototype" assumed representative. Errors in constructions.	Multiple examples considered and required. Dynamic view of situation and examples.
Nature of observations	Reliance upon appearance *"looks like..."* Naive acceptance of definitions. Focus on superficial properties (e.g. orientation, location, etc.).	Consideration of degenerate (undefined) cases. Measuring or observing precise criteria and properties. Consideration of causal or covariational relationships.
Qualities of written conjectures	Perspective-dependent statements that do not present clear, testable, mathematical criteria.	Precise, testable statements. Conventional use of math terminology.

	Unconventional use of math terms. Reliance on common vernacular.	Inclusion of both general and special cases.
Qualification of conjectures	If you think it or see it once, it is a conjecture.	Level of conviction/scrutiny to be considered a conjecture. Must be an interesting/non-trivial result. Evidenced by multiple cases or a specially-designed dynamic construction.

and its accessibility, it may serve as a type of mathematical activity in which some of these characteristics can be discussed and developed. For example, the articulation of conjectures provides an opportunity to discuss logic and mathematical language; thus perhaps conjecturing could provide an earlier forum to deal with these topics and acclimate students earlier than a transition to proofs course.

One question at the front of our mind is the context. Because of its constructive nature and the many physical and virtual tools available, Euclidean geometry was a natural context for exploration and conjecturing. How could it be incorporated into other areas of mathematics?

In the end, we are left with several unanswered questions: How can conjecturing be fostered and developed? To what extent is conjecturing tied to content area? How can conjecturing be integrated into other branches of mathematics? and How does experience with conjecturing affect learning and success in mathematics?

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ON THE ROLE OF PEDAGOGICAL CONTENT KNOWLEDGE IN TEACHERS' UNDERSTANDING OF COMMUTATIVITY AND ASSOCIATIVITY

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The purpose of this study is to investigate a relationship between mathematical content knowledge and pedagogical knowledge of content and students (Hill, Ball, & Shilling, 2008), in the context of algebra. As participants in a paired teaching experiment, mathematics education doctoral students revealed their understandings of commutativity and associativity (cf. Larsen, 2010). Although the participants' knowledge of children's initial understandings of algebra and familiarity with mathematics education literature influenced their own mathematics reasoning, the difficulties they encountered were similar to those of undergraduates without such pedagogical content knowledge.

Keywords: Abstract Algebra, Teaching Experiment, Pedagogical Content Knowledge

Introduction

The purpose of this study is to investigate the ways in which doctoral students' learning of introductory abstract algebra differs from that of undergraduate students. Though the mathematics course background of undergraduates and mathematics education doctoral students may be similar, doctoral students generally have greater study of pedagogy. Thus, implicit in this goal is an exploration of how pedagogical content knowledge (Shulman, 1986) on the part of the student affects the learning of mathematics content.

Background

Larsen (2009) articulated a curriculum for an inquiry-based undergraduate abstract algebra course that includes students' reconstruction of the group axioms, using the design constructs of guided reinvention (Gravemeijer & Doorman, 1999) and emergent models (Gravemeijer, 1999). Guided reinvention is an instructional technique by which students gain a sense of ownership of their understandings, in contrast with the more authoritarian lecture format that is more typical of undergraduate teaching. Whereas a traditional abstract algebra course may begin with an expressed definition of a group and use S_3 as an example, the first unit in Larsen's (2012) curriculum includes students' identification of the symmetries of an equilateral triangle and subsequent creation of their own symbol systems, notation, and organization structures. Thus, the instructional design begins with students' creation of models of their own mathematics experiences. As students reflect on these emergent models, their conjectures and attention become less focused on the specific actions that resulted in the models' creation, and, via teacher prompting, their abstractions can be guided toward a formalization of the group axioms (Larsen, 2009).

Larsen (2010) reported on the results of two single-session teaching experiments with pairs of undergraduate mathematics students, the purpose of which was to inform the instructional design for use in a whole-class setting. A main finding of the paired teaching experiments was undergraduates' difficulties with the associative and commutative properties. Upon consideration of the research literature showing similar findings involving children and teachers, Larsen (2010) concluded that such difficulties "may stem from 1) a tendency to think about expressions involving binary operations in terms of a sequential procedure and 2) a lack of preciseness in the informal language used in association with these properties" (p. 42).

Theoretical Framework

Hill, Ball, and Shilling's (2008) elaboration of Shulman's (1986) seminal work distinguishing the types of knowledge of content and pedagogy required of K-12 teachers is used to frame the discussion of the role that specialized knowledge for teaching plays in the learning of mathematics (see Figure 1). Hill et al. (2008) describe specialized content knowledge (SCK) as the knowledge of the mathematics that is particularly relevant for teaching, beyond the common content knowledge (CCK) that might be used by a more general population. They describe knowledge of content and students (KCS) to be the understanding of *how* students learn the content, while knowledge of content and teaching (KCT) is understanding of the effect of teacher actions on said learning. Knowledge at the mathematical horizon (KMH) is both an "advanced perspective on elementary knowledge" and an application of "advanced mathematical knowledge" to lower-level curricula (Mamolo & Zazkis, 2011, p. 4).

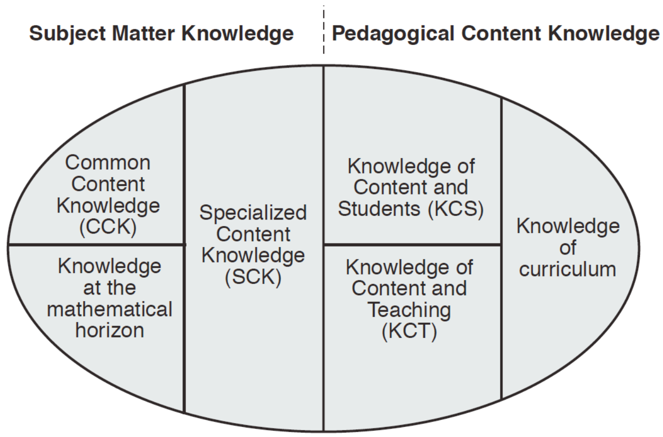

Figure 1. Hill et al. (2008) Domain Map of Knowledge for Mathematics Teaching

Hill et al. (2008) acknowledge that in their conceptualization, "KCS is distinct from teachers' subject matter knowledge. A teacher might have strong knowledge of the content itself but weak knowledge about how students learn the content or vice versa" (p. 7). The model has been extended to university level mathematics in studies of instructors' KCS (e.g., Alcock & Simpson, 2009; Johnson & Larsen, 2012; Speer & Wagner, 2009). The current study considers a reverse relationship – the effect of a *student's* pedagogical content knowledge on her *own* learning of university level mathematics content.

Participants

Both of the participants in this study, Zowie and Mary, were female second-year doctoral students in mathematics education at a large Southeastern University. Zowie and Mary each had over ten years of mathematics teaching experience; Zowie's was primarily with middle school geometry in China, while Mary's was with the elementary grades in the U.S. Though they each had some exposure to group theory from prior coursework, neither Zowie nor Mary recalled completing a university course devoted to abstract algebra. Both participants (and the researcher) were enrolled in a doctoral seminar that examined research in undergraduate mathematics education, and they were recruited because of their expressed desire to learn the mathematics content before digesting educational research about its teaching and learning.

Method

Larsen's (2012) instructional materials were used [with permission] to partially replicate his 2010 study; major differences included the participant population (mathematics education doctoral students instead of undergraduate math majors) and the number of sessions (four rather than one). The general trajectory of Larsen's (2012) group unit begins with students manipulating an equilateral triangle and developing notation for the compositions of its symmetries, writing the symmetries in terms of compositions of a single reflection and a single rotation, organizing them in a Cayley table, and constructing a minimal set of properties that could be used to perform the compositions. In the current study, the participants did not fully develop the notion of group by the end of their last session, so the researcher disclosed the intent of the instructional sequence and the relationship between the activities and the formal definition of a group at the end of the final session.

The researcher videotaped each session, and he watched the videos between sessions and read the online instructional material provided by Larsen (2012) to prepare materials for the following session. The role of the researcher throughout the first three hour-long sessions was that of a facilitator, as the participants generally questioned each other's thinking and asked for clarification without prompting. The researcher encouraged the participants to verbalize not only their own mathematics, but also their hypotheses for why they (or their partner) might reason a particular way. For example, before developing an initial notation system for the symmetries of an equilateral triangle, they were given definitions of isometry, symmetry, and equivalent symmetries. The participants negotiated the meanings for these definitions, and they subsequently verbalized their perceptions of the sources of their concept images (Vinner, 1991).

Results

Zowie and Mary developed their own (shared) notation for the six symmetries of the equilateral triangle (see Figure 2). They were tasked with writing each of the six symmetries in terms of the following: the 120-degree clockwise rotation (R) and the reflection about the vertical axis (F). Zowie and Mary developed additive notation, e.g., R + R + R = 3R, which was used throughout the sessions. Protocol 1 begins as they begin to complete a table of pair-wise combinations of symmetries and are deciding how to name the identity [S: Researcher, M: Mary, Z: Zowie].

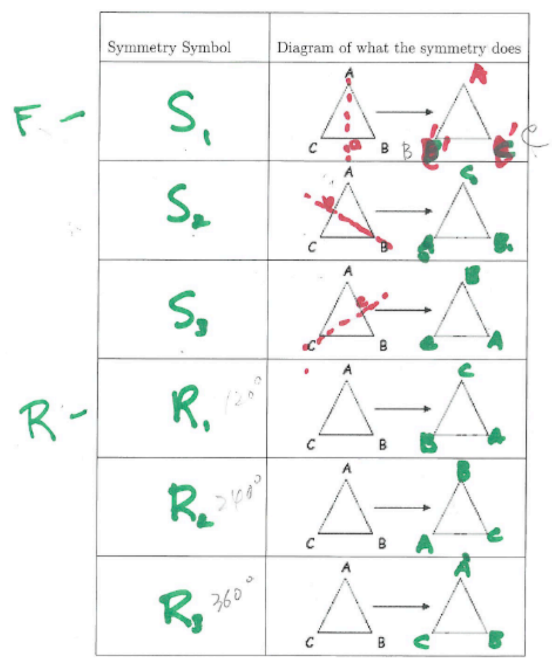

Figure 2. Zowie and Mary's Six Symmetries

Protocol 1:Session Two Discussion about commutativity

S: What is your identity the same as, in terms of Rs and Fs?

M: R_3.
Z: [at the same time] 3R.
M: 3R. [Writes 3R in the box corresponding to F+F].
Z: Okay, then we go, this [F] this [R + F]?
M: Um hmm. So two flips and one rotation.
Z: No. One flip, one ro[tation], and one flip [pointing at F, and then R, and then F]. I will go in this way.
M: You want to go in that order? Let me go in my order, and we'll see if we get the same thing. [Both participants complete the operations]
Z: 2R.
M: I got BCA. You got CBA? [Mary demonstrates what she did while Zowie watches, and then vice versa.]
M: Okay, so order matters.
Z: Yes, that's right. [Mary writes the rule "order matters" at the bottom of her paper.]
Z: It's the same as last time when you present…the equal sign is not truly equal.
M: Laughs. I have to know this – so we know that commutativity does not play a role.

Zowie's comment about the equal sign indicates that she was associating the use of equivalence classes of symmetries with a lack of commutativity. In the first session, she had expressed concern over whether three 120-degree rotations would be considered equivalent to one 360-degree rotation. When Zowie found that F + R + F did not result in the same symmetry as F + F + R, she returned to a conception that equivalent symmetries were not "truly equal." Her objection to Mary's suggestion to compute F + F was not because it would require changing the sequence of operations; she resisted replacing two actions with their composition.

After the participants completed the chart, they were asked what other rules or patterns they had noticed. The intent was for them to identify a minimal list of rules that would be necessary to complete the table. The rules the participants had formulated at this point in the teaching episode were: "3R = identity; 2F = identity; R + F = F + 2R; F + R = 2R + F; Order matters (commutativity does not work)". The next activity was to re-complete their Cayley table using only these rules – the original chart and triangles were removed from view. Protocol 2 describes how Zowie found some of the more complicated relationships on scratch paper using an associative property without initially acknowledging its use.

Protocol 2: Session Two Discussion about associativity
S: How did you get the identity here [2R + F + 2R + F = I]?
Z: I did it in two ways. I switch this [circles the first 2R + F] to make F + R + 2R + F. This [circles R + 2R] is 3R, which is the identity. So [we have] F + F, which is the identity. [Goes on to demonstrate that another way began with substituting for middle, F + 2R].
S: So in all these ways, it didn't matter if you substituted for this[underlined first 2R + F], this [underlined F + 2R] or this [underlined second 2R + F].
Z: Yes, because it's not order. I didn't switch the order. I just used a different combination.
S: And that's okay. Is that important? Should we write that down, is it obvious?
M: I think it's important.
S: How can we write that down?
M: That's in a way, saying that left to right is not important. How would you describe that? Order of operations, in a way? I'm trying to think of how to say that.
S: [3 second pause] So you didn't change the order that you worked. What you changed was, which operation you did first.

M: [Quietly, as if questioning] Like associative property?
S: Why do you say it's like associative property?
M: Like if you had parentheses around it, you could change the parentheses, which would indicate where you started first. What operation should begin, first.
Z: Wouldn't that be switching the order?
M: We're switching the order of the operation, not the order of the addends.
Z: Um hmm.
S: So you're still maintaining the pair, [pointing to 2R + F] in that order, you're not switching this to be F + 2R.
M: We're just deciding which operation to begin with, which is more associative.
Z: I didn't switch the order of addends, and I [stopped mid-sentence]
S: Right here, [pointing to paper] you did R + 2F first, even though F was written first.
M: It was like you had parentheses that grouped them, and instead of doing this, I'm going to put my parentheses around here and start here first. So that's how I look at it. The associative property focuses on which operation you're going to begin with, by using parentheses. Switching those parentheses is indicative of the associative property.
Z: I'm thinking, theoretically I only look at the operation – here [gestures to paper] it's right. But then you think back to actions – you're actually changing the order of the actions.
M: That's what an operation is, isn't it? The order of actions, isn't that the order of the operations? What's the difference?
Z: I should say it in this way. For this [indicates paper] I'm just playing a mathematics game. In here. But not the action.

Zowie's last comment is strikingly similar to a statement by Erika in Larsen (2010), who also referred to the use of the associative property as having " nothing to do with the actual order you're flipping the triangle in. Like it's all a paper game kinda" (p. 40). Mary's verbalization that "switching those parentheses is indicative of the associative property" is evidence of her thinking of how students understand associativity –as recognition of a change in grouping symbols' placement rather than a property of a binary operation. In the subsequent session, both Zowie and Mary declined to give up the rule that "order matters," a finding also reported by the undergraduates in Larsen (2010). Zowie and Mary each gave powerful statements demonstrating how their KCS influenced their reasoning.

Protocol 3: Session Three Discussion about the necessity of the "order matters" rule.
S: Why do we need to know that order does matter in order to fill out the chart?
M: Otherwise you're going to start to group your similar letters.
Z: Um hmm.
S: Why?
M: It's based on what we know about…
Z: You want to do association [makes swapping motion with hands].
M: That's something different, the associative. The commutative is what you're thinking of.
S: Why do we want to do that though?
Z: Because we want to find 3R + 2F.
M: Because our brains work in such a way that…
Z: We always want to simplify.
M: Because we've computed so many times, we've based our ideas on those basic properties of addition and multiplication, and those basic properties are what we rely upon to make our adding and our multiplying easier. So, you have to state that so you know not to depend on those.

Z: I have a different view. According to our stereotype of math, you need to make it simple – your result must be simplified. Like if I have 3R is the identity, then I can use 1 to represent it. Or I add two or subtract two and they cancel, so I can get 0, which means I can simplify.

M: We need to know when to simplify and when not to simplify.

Mary's suggestion that without being told that order matters that one would "start to group similar letters" is based on her understanding of how students assimilate experiences into available ways of operating. She suggests that without the warning, combining like terms is a cognitive necessity, and she justifies her rationale by appealing to her knowledge of how students come to know operations with real numbers. Zowie suggests that since mathematics practitioners are accustomed to a *need* to simplify, the rules must include an explicit exemption from that requirement. In their justification for the retention of the logically superfluous but pedagogically necessary "order matters" rule in a minimal list, the participants demonstrated how their KCS affected their own learning of mathematics.

Discussion

Both the doctoral students and the undergraduates in Larsen's study (2010) questioned the degree to which the symbols they used retained a structure, e.g., whether using a '+' sign would mean that an operation was additive. None of the pairs immediately resolved the issue of the two types of order, and they each created and retained the rule 'order matters.' Larsen (2010) argued that the difficulties that undergraduates had with commutativity and associativity were quite similar to the difficulties found in the literature pertaining to middle-school students and teachers. The current study provides additional evidence of the persistence of this difficulty. Furthermore, the doctoral students' KCS appeared to play a large role in their explanations for their mathematical thinking; they did not sunder their knowledge of how students learn mathematics from their own learning of mathematics.

In the final session the doctoral students expressed that they had been actively looking for ways to connect the abstract algebra they were learning to other content areas (e.g., functions, matrix theory) within the domain of teaching and learning mathematics. Thus, the connectedness of the participants' mathematics knowledge and pedagogical content knowledge may have enabled their construction of the advanced mathematical knowledge they were learning as KMH. One might consider that the mathematics coursework preparation of the participants in this study is not unlike the mathematics preparation of secondary teachers when they first learn about abstract algebra, and, in some cases, nearly identical to the mathematics preparation of in-service elementary or middle school teachers. Therefore, the results suggest that even if teachers do not take an abstract algebra course, they may build KMH from an opportunity to engage in similar mathematical activities that engender reflection on the properties of binary operations.

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PREPARING STUDENTS FOR CALCULUS

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This quantitative study compared the implementation of a problem-based curriculum in precalculus and a modular-style implementation of traditional curriculum in precalculus to the historical instructional methods at a western Tier 2 public university. The goal of the study was to determine if either alternative approach improved student performance in precalculus and better prepared students for success in a calculus sequence. The study used quantitative data collection and analysis. Results indicate students who experienced the problem-based curriculum should be better prepared to learn calculus but mixed results in terms of retention and success in calculus.

Key words: Precalculus, calculus, problem-based learning

If Calculus is the gateway to higher-level mathematics, then Precalculus is the course that should prepare students to be students of calculus. Students in first-semester mathematics courses continue to receive passing grades at low rates. In a report on factors effecting student success in first-year courses in business, mathematics, and science at a western Tier 2 public university, Benford and Gess-Newsome (2006) identify student academic under-preparedness and ineffective and inequitable instructional techniques as factors that contribute to the situation. The department of mathematics and statistics has been particularly concerned about the success rate of students enrolled in Calculus. Anecdotal data indicated faculty felt students entering the calculus sequence were under-prepared. Students did not have a deep understanding of the concept of function, a "central underlying concept in calculus" (Vinner, 1992), and were not able to solve problems at the level expected in the calculus sequence. Upon examining their preparation of students for first-semester calculus, the department discovered students in Precalculus also experienced a low rate of passing grades (grades of C or higher).

Thus, as part of a university-wide initiative to improve student success in first-year courses with a high rate of non-passing grades (grades of D, F, W), the department of mathematics and statistics chose to examine two alternatives to the traditional curriculum in precalculus. The goals of this initiative were to increase the rate of passing grades in precalculus *and* calculus and improve retention rates for students in higher-level mathematics. Historically, students participating in a precalculus course experience lecture-based instruction, using a traditional textbook, with little opportunity to practice problems and engage with the content during class. In light of the report and faculty concerns, the department chose two alternative methods for teaching precalculus that focused on offering students greater opportunity to master the precalculus content, gain a deeper understanding of the concept of function, and improve their problem-solving skills.

For the first option, the department adapted a modular approach used at the University of Texas in El Paso. In this model, the precalculus curriculum is split into three time periods, Modules 1, 2 and 3. Each module is 5 weeks in length. Students must pass an exam at the end of each module to continue to the next. If a student does not pass the exam at the end of a module, they may retake the current module over the next 5 weeks. If a student does not finish all three modules by the end of the 16-week semester, they may continue the sequence the following semester (including summer semesters). The advantage of this approach is that students are able to repeat material they have not mastered without the fear of earning a non-passing grade at the end of a traditional 16-week semester. That is, this approach gives students more time to remediate, if needed. The disadvantages are (1) instruction is not changed (i.e., students continue to experience traditional, lecture-based instruction) and (2) students must pay for an additional semester of precalculus if they are not able to finish all three modules in a single semester.

The second option offered by the department was a reform-based curriculum focused on a quantitative approach to learning concepts in precalculus (need to look up this reference) and a problem-based classroom environment. This curriculum was specifically designed to develop students' conceptual understanding of function (including trigonometric functions), problem solving abilities and skills that are foundational to calculus. Students engaged in problem-based learning in groups on a daily basis. Lecture became the exception, rather than the rule, and students were expected to learn mathematics through investigating problem situations. The advantages to this curriculum are students (1) engage in solving problems every class period; (2) learn by "doing mathematics," and (3) use a research-based curriculum that reflects what students need to know to achieve success in calculus. The disadvantage to this curriculum is that instructors and students are often unfamiliar with teaching and learning in a problem-based environment using group learning. Thus, establishing classroom norms may take longer than in a traditional college course.

The research questions for this study were as follows:

1. Does implementation of a problem-based curriculum or the adaptation of the modular approach improve student success in *Precalculus Mathematics* compared to traditional instructional methods?
2. Does implementation of a problem-based curriculum or the adaptation of the modular approach improve student preparation for *Calculus I* compared to traditional instructional methods?

Theoretical Framework.

The theoretical framework for this study combined ideas from work on the reasoning abilities and understandings students need to be successful in calculus (e.g., Selden & Selden, 1999, Jensen, 2010), Social Cognitive Theory (Bandura, 2001), and research on the relationship between students' attitudes toward mathematics and mathematical achievement (e.g., Alkhateeb & Mji, 2005). It is well documented that a complete notion of function, covariation, function composition, function inverse, quantity, exponential growth, and trigonometry are essential to learning in precalculus and calculus (Dubinsky & Harel, 1992; Rasmussen, 2000; Carlson et al., 2002; Engelke, Oehrtman & Carlson, 2005; Oehrtman, Carlson & Thompson, 2008; Carlson, Oehrtman & Engelke, 2010). In addition, Stanley (2002) found that students who experience problem-based learning in precalculus increased their ability to solve real world problems, identify and use appropriate resources, and take a more active role in their learning. Using these results, the research team chose a research-based curriculum for experimental group A that included a problem-based approach to learning and emphasized development of the function concept, covariational reasoning, and trigonometry. These results also informed the selection of the tool used to assess student preparation for calculus (see Methodology).

Social Cognitive Theory (SCT) holds that human behavior is often predicted by what students believe they are capable of rather than the realization of their capabilities (Bandura, 2001). In other words, students determine what to do with specific mathematical knowledge and skills by their self-efficacy rather than what they might actually understand mathematically. Their behavior is part of a three-way reciprocal interaction between personal factors (e.g., cognition and affect), behavior and the environment. The design of this study assumed that a student's affect about mathematics will impact their desire to continue their mathematical learning and success in the subsequent calculus sequence, an assumption supported by several studies (Lester, Garofalo, & Kroll, 1989; House, 1995; Randhawa, Beamer & Lundberig, 1993). Hence, assessment of student success included a survey of student efficacy around learning in mathematics.

Methodology.

This project used a quantitative approach of program evaluation across three types of course offerings available at a western Tier 2 public university during the 2010/2011 and 2011/2012 academic years. Quantitative methods were used to measure student preparation for first semester calculus and retention in precalculus and calculus. In addition, qualitative methods were used to describe differences in instructor teaching strategies that might interact with the data collected through quantitative methods. This inclusion of qualitative description helped the investigators identify any mediating variables attributed to instructional styles.

To answer research question 1, we measured overall student success in Precalculus using end-of-semester grades. To answer research question 2, we analyzed scores from the Precalculus Concept Assessment Tool (PCA; Carlson, Oehrtman & Engelke, 2010) and pass/fail rates among students who completed Calculus I the semester following completion of Precalculus. The 25-item PCA multiple-choice test is a valid and reliable instrument that measures "the reasoning abilities and understandings central to precalculus and foundational for beginning calculus." Eighteen items assess student understanding of the concept of function; five items assess student understanding of trigonometric functions; and four items assess student understanding of exponential functions. In addition, ten items require students to solve novel problem situations using quantitative reasoning and ideas of function, function composition, or function inverse. However, we recognize that instructional methods in Calculus I at this particular university might not align with research-based instructional practices in teaching and learning Calculus. Hence, we also compared student grades in Calculus I among students who completed the course the semester immediately following completion of Precalculus.

All students enrolled in Precalculus were required to complete the PCA instrument. However students were able to choose whether their PCA score was included in the study, and students' class grades were not based on their performance on the PCA. In the control group (traditional curriculum, primarily lecture-based instruction) and the experimental group A (the reform-based curriculum), the PCA was administered during the last week of classes for each semester. In experimental group B (the modular approach using a traditional curriculum), the PCA was administered during the last week of Module Three. Student efficacy around learning in mathematics was measured through the Mathematics Confidence and Attitude Survey (Piper, 2008). This survey was administered via email using the Google Education Suite in the final week of each term.

Results.

In order to determine if a problem-based curriculum or the adaptation of the modular approach improved student success in Precalculus compared to traditional instructional methods offered at this university, we compared end-of-semester grades for the 2010/2011 and 2011/2012 academic years using a *t*-test with the type of curriculum (tradition, modular or problem-based) used as the independent variable. At this university, student success is defined as completing a course with a letter grade of A, B or C. A letter grade of D or F is considered failure since it does not earn a student credit toward their degree. Hence, we compared the mean pass/fail rate for each type of curriculum. Over these two academic years, descriptive statistics indicate that students who experienced the modular approach or the problem-based curriculum were more successful in Precalculus, with students experiencing the modular approach enjoying slightly higher success rates.

	Curriculum	N	Mean	Std. Deviation	Std. Error Mean
Pass/Fail	Traditional	882	.71	.452	.015
	Modular	804	.83	.372	.013
	Problem-based	376	.79	.406	.021

Table 1. Mean pass rates for 2010/2011 and 2011/2012 academic years

The differences in mean pass/fail rates were statistically significant between the traditional and modular approach and between the traditional and problem-based approach with *p*-values of .000 and .004, respectively. There was not a statistically significant difference between the mean pass/fail rates for the modular approach and problem-based curriculum.

Curriculum	**Difference**	**Standard Error**	***t*-ratio**	***p*-value**
Traditional/Modular	-0.120	0.20	-5.935	.000
Traditional/Problem-based	-0.078	0.027	-.2.896	.004
Modular/Problem-based	0.042	0.024	1.756	.079

Table 2. t-test statistics for Mean Pass rate

Student scores on the PCA from the 2010/2011 and 2011/2012 academic years were compared using a *t*-test with the type of curriculum (tradition, modular or problem-based) used as the independent variable.

	Curriculum	N	Mean	Std. Deviation	Std. Error Mean
Pass/Fail	Traditional	311	7.91	3.289	.187
	Modular	96	8.17	3.361	.343
	Problem-based	111	10.78	3.502	.332

Table 3. PCA mean scores for the 2010/2011 academic year

	Curriculum	N	Mean	Std. Deviation	Std. Error Mean
Pass/Fail	Traditional	69	7.19	3.112	.375
	Modular	347	8.76	3.505	.188
	Problem-based	171	10.17	4.219	.323

Table 4. PCA mean scores for the 2011/2012 academic year

It should be noted that this university transitioned out of the traditional, lecture-based curriculum after the Fall 2011 semester. Only the modular approach and the problem-based approach were offered in the Spring 2012 semester. Thus the n=69 for the traditional curriculum is much lower than one might expect. This was accounted for in subsequent t-tests for independent samples by using a t-test for unequal variances between the traditional curriculum and the modular approach.

Curriculum	**Difference**	**Standard Error**	***t*-ratio**	***p*-value**
Traditional/Modular	-1.567	.419	-3.737	.000
Traditional/Problem-based	-2.981	.561	-5.312	.000
Modular/Problem-based	-1.415	.351	-4.032	.000

Table 5. t-test statistics for mean PCA scores 2010/2011 and 2011/2012

Descriptive statistics show that the mean score of students who experienced the problem-based curriculum was greater than the mean score of students who experienced the traditional curriculum or the modular approach in both academic years. Furthermore, the difference in mean scores was statistically significant between all three curricula with *p*-values less than 0.0001.

Student semester grades in Calculus I were compared for students who completed Calculus I the immediate semester after completing Precalculus. Grades were taken from the Spring 2011, Fall 2011, Spring 2012, and Fall 2012 semesters. We were only interested in whether experiencing a specific curriculum in Precalculus helped students pass Calculus I. Hence, we analyzed semester grades in terms of passing score (i.e., A, B or C) and failing scores (i.e., D or F). Scores were analyzed across the population of students satisfying the above requirement. We used independent sample t-tests to compare the pass/fail rate in Calculus I between students who experienced each type of curriculum in Precalculus.

	Curriculum*	N	Mean	Std. Deviation	Std. Error Mean
Pass/Fail	Traditional	169	.61	.489	.038
	Modular	205	.69	.465	.032
	Problem-based	62	.65	.482	.061

Table 6. Mean pass rates for Calculus

Descriptive statistics show that the mean pass/fail rate in Calculus I for students who experienced the modular and problem-based curriculum were slightly higher than the pass/fail rate for students who experienced the traditional, lecture-based curriculum in Precalculus. However, the differences are not statistically significant at the $\alpha = 0.05$ level.

Curriculum	**Difference**	**Standard Error**	***t*-ratio**	***p*-value**
Traditional/Modular	-0.078	0.50	-1.576	.116
Traditional/Problem-based	-0.036	0.072	-.496	.621
Modular/Problem-based	0.043	0.069	.615	.540

Table 7. t-test statistics for mean pass rates in Calculus

Since the population sizes were so different for the control group and both experimental groups, we also took a simple random sample of 60 scores from each population (i.e., students who completed the traditional curriculum in Precalculus, the modular approach, or the reform-based curriculum) to verify the results above.

	Curriculum	N	Mean	Std. Deviation	Std. Error Mean
Pass/Fail	1	60	.57	.500	.065
	2	60	.70	.462	.060
	Problem-based	60	.63	.486	.063

Table 8. Mean pass rates for Calculus with simple random sample

Curriculum	**Difference**	**Standard Error**	***t*-ratio**	***p*-value**
Traditional/Modular	-0.133	0.088	-1.517	.132
Traditional/Problem-based	-0.067	0.090	-0.741	.460
Modular/Problem-based	0.067	0.087	0.770	.443

Table 9. t-test statistics for mean pass rates in Calculus with simple random sample

Descriptive and t-test statistics for the simple random sample of 60 students in each group show similar results. The mean pass/fail rate in Calculus I for students who experienced the modular and problem-based curriculum were slightly higher than the pass/fail rate for students who experienced the traditional, lecture-based curriculum in Precalculus. However, the differences are not statistically significant at the $\alpha = 0.05$ level.

Connection to Theory and Practice.

Precalculus and Calculus I are staples of the curriculum of STEM degrees across the country. For many students, these courses are hurdles or barriers that delay or impede their degree progress. Furthermore, Calculus I instructors may often be disappointed in their students' knowledge of precalculus concepts. While many colleges and universities deliver these courses in the traditional lecture format, others are experimenting with other methods, including problem-based and modular curricula. In theory, curricular decisions should be based on which curriculum is most likely to promote student success. In practice, other factors are also part of the curriculum decision-making process, such as the availability of financial, human, and physical resources that are needed to implement the curriculum.

The popularity of the traditional lecture format may be historical, but it probably requires the least resources. Generally, all that is needed is a chalkboard and a piece of chalk, or a PowerPoint presentation and a projector. On the other hand, modular-based curriculum can be logistically more difficult to schedule and staff. In addition, faculty probably need additional time to prepare for classes that use a problem-based curriculum than those that use a lecture format. How to balance providing the most effective curriculum and pedagogy with the reality of available resources will continue to be an issue that colleges and universities must face.

At our university, we have moved from a traditional lecture-based format to a modular-based curriculum. It is uncertain whether this is a permanent change—only time will tell. What we do believe is that, for us, the traditional lecture format is the least effective of the three formats discussed here. This is supported by the data presented above that suggest that students from precalculus sections taught in the traditional lecture format are not as successful as those taught in the modular or problem-based format as measured by their precalculus grade or subsequent success in calculus.

The results of this study contribute to the knowledge base of best practices that are associated with the teaching and learning of precalculus and calculus. Although further research is needed, these results suggest that the traditional lecture format found in most university and college classrooms may not be the most effective method of instruction. Rather, students may learn best by being exposed to problem-based curricula that allow them to explore mathematical content in a way that develops their conceptual understanding of the mathematics instead of only their algorithmic knowledge of the procedures. We hope that these results will prompt precalculus teachers (including those at our own university) to reexamine their instructional strategies and practices.

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PARTIAL UNPACKING AND INDIRECT PROOFS: A STUDY OF STUDENTS' PRODUCTIVE USE OF THE SYMBOLIC PROOF SCHEME

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This paper examines mathematics majors' evaluations of indirect proofs and of the compound statements used in this form of proof. Responses to survey items with a cohort of 23 students and six 1-hour clinical interviews, indicate that the students who could successfully evaluate indirect arguments and who could successfully recognize logically equivalent statements, tended to use partially unpacked (Selden & Selden, 1995) versions of the statements and the proofs and, in so doing, demonstrated a productive use of the symbolic proof scheme, whereas both successful and unsuccessful students tended to use a proof framework (Selden & Selden, 1995) for indirect proofs. Moreover, successful students' approaches are suggestive of activities, which are rarely found in introductory proof texts, yet may benefit novice proof writers.

Key words: Indirect Proof, Symbolic Proof Scheme, Proof Frameworks

Introduction

Research on students' production, understanding, and evaluation of indirect proofs suggests a general lack of preference for this form of proof. Harel and Sowder (1998) reported that students in their teaching experiments disliked indirect proofs and argued that this is due to students' preference for constructive proofs – proofs that construct mathematical objects and relations – over proofs that solely establish the logical necessity of a mathematical relation or object, as is the case with indirect proofs. Prior to Harel and Sowder's work, Leron (1985) also proposed that students' difficulties deriving a sense of conviction from indirect proofs may be rooted in the non-constructive nature of such proofs. Specifically, he argued that "most non-trivial proofs pivot around an act of construction – a construction of a new mathematical object," whereas with indirect proofs, we engage in acts of mathematical "destruction, not construction" (p. 323). Additionally, Harel and Sowder (2007), in their discussion of Aristotilian causality, have explored how students' tendency to view implications as causal statements can act as barrier to students' acceptance of indirect proofs. Indeed, if $P \rightarrow Q$ means P causes Q then how can $\sim Q \rightarrow \sim P$ show P causes Q?

Antonini and Mariotti (2008) explored students' proof preferences related to indirect proof and provided a rationale for these difficulties that is not solely rooted in issues of constructiveness (Leron, 1985; Harel & Sowder, 1998) and causality (Harel & Sowder, 2007). Working within the Cognitive Unity theoretical framework, Antonini and Mariotti showed that indirect proofs call on students to move from a principal statement (*e.g.*, $P \rightarrow Q$) to a secondary statement (*e.g.,* in a proof by contraposition, $\sim Q \rightarrow \sim P$) and to interpret or produce the proof of the secondary statement. They argue that it is this jump between principal statements (S) and secondary statements (S*) that is the source of students' difficulties and refer to such difficulties as *metatheoretical*. "Referring to their meta-theoretical status, we call the statement $S^* \rightarrow S$ meta-statement, the proof of $S^* \rightarrow S$ meta-proof, and the logical theory, in which the meta-proof makes sense, meta-theory" (p. 405). To illustrate students' metatheoretical difficulties, Antonini and Mariotti asked students to evaluate indirect proofs, including a proof by contraposition of the statement, "If n^2 is even then n is even." They showed that students struggle to accept the validity of the principal statement (*i.e.*, If n^2 is even then n is even), given the proof of the secondary statement (i.e., If n is odd, then n^2 is odd). For instance, Fabio, a university student remarked, "The problem is that in this way we proved that n is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one." (p. 407). We see here that Fabio has accepted that a claim has been made and a proof given of the secondary statement, but is not "satisfied" with regard to the principal statement. Thus, it is the "jump," S*→S, which he finds problematic. Interestingly, if one considers the ideas proposed by Leron (1985), and Harel and Sowder (1998, 2007), then a plausible rationale for the meta-theoretical difficulties discussed by Antonini and Mariotti (2008) emerges. Indeed, it may be that students look to such arguments to understand how S* *causes* S (*i.e.*, they seek causality in the argument) or how negating a claim of nonexistence (assuming a negation) enables one to *know*, in an epistemological sense, existence, that is, S* *means* S.

Reframing the Problem

According to Antonini and Mariotti, the Italian university students in their case studies recognized the transition to secondary statements (S*) and the proof of S* but experienced metatheoretical difficulties related to the "jump" between statements (*i.e.*, S*→S). It is unclear, however, if the majority of students would be as successful at recognizing and understanding this jump. Certainly, if one does not recognize and understand the jump between a principal and secondary statement, then one cannot accurately evaluate the logical structure of an indirect proof. Moreover, it may be the case that students' difficulties *accepting* the jump do not arise until and unless they are able to *recognize* a jump.

Drawing on series of clinical interview, Brown (2012) explored the extent to which advanced mathematics students (students enrolled in their 3rd and 4th university year, $n = 6$) were able to explain the logical structure of four indirect proofs involving basic number theory statements. Findings from this study indicated that only two of the advanced students could successfully explain the indirect proofs that were proofs by contradiction, while all but one student could successfully explain the indirect proofs that were proofs by contraposition. Additionally, the students who were successful at explaining the proofs by contradiction were also able to successfully evaluate a series of theorem statements, in terms of their logical equivalence, whereas the other four students struggled to do so. Specifically, among the other four students, after several unsuccessful attempts and repeatedly expressing doubts regarding the validity of their own evaluations of the argument, two were eventually able to explain the logical structure of the proofs by contradiction. The other two of the four students were not able to successfully explain the arguments even after repeated attempts and both misinterpreted key aspects of the logical structure of the arguments, including but not limited to the logical form of the secondary statement. Findings from these clinical interviews point to the possibility that many students' difficulties may be at the level of *recognizing* and *understanding* the jump between statements (S*→S). Moreover, they also suggest that the various forms of indirect proof (proof by contradiction, proof by contraposition) are not uniformly difficult for students.

As illustrated thus far, much of the research on indirect proof has focused on either students' lack of preference for or their difficulties with indirect proof (Leron, 1985, Harel & Sowder, 1998, Antonini & Mariotti, 2008). While understanding the nature of students' difficulties and the junctures at which these difficulties may manifest themselves is important, it is also possible that progress may be made by studying students who can (1) successfully interpret and evaluate indirect proofs, and (2) determine the logical equivalence of principal and secondary statements. The purpose of this paper is to share findings related to students' successful approaches related to interpreting indirect proofs. In particular, building on Selden and Selden's (1995) description of *unpacking* and their construct of a *proof framework* and Harel and Sowder's (1998) construct of a *symbolic proof scheme*, we will demonstrate that the students' successful attempts represent instances of students' use of a partial unpacking of the theorems and are examples of students' productive use of the symbolic proof scheme, while use of a proof framework for indirect proof was a characteristic of both successful and unsuccessful attempts. Interpreted in terms of the work of Antonini and Mariotti (2008), these findings shed light on the approaches used by students who can successfully engage in metatheoretical modes of thought.

The Study

The data presented in the paper are drawn from the *Bridges to Advanced Mathematics* study, which aims to identify content specific barriers to students' transition to advanced mathematics. One component of this study was a small-scale exploration of students' proof preferences, as they related to indirect proof. This exploration involved developing and administering an 8-item proof preference survey, which was administered to 15 students enrolled in courses typically taken by 3rd and 4th year mathematics majors (*e.g.*, Topology, Analysis) and 8 students during the last week of an introduction to proof course in which more than half of students were in either their 3rd or 4th year. The survey instrument included three types of proof comparison tasks and two 'proof-related' tasks. Proof comparison tasks provided students with two proofs and asked the students to rate the extent to which they were confident in their understanding of each proof and to indicate which proof "is the most convincing" and which "is the best proof?" Three forms of proof comparisons items were used in the survey. The items ask the participants to compare: (1) a direct proof to an indirect proof (Type I); (2) a *Constructive* to a *Non-constructive Existence Proof* (Type II); and (3) a proof by contraposition to a proof by contradiction (Type III). Type III items were included to gather data on the question of whether or not there might be psychological distinctions to be made between these two forms of indirect proof. Type IV survey items were 'proof-related' comparison tasks, which asked participants to select a statement to prove out of three statements. Choices for the three statements include a principal statement ($\forall n, P \Rightarrow Q$) and two secondary statements. Secondary statements were either of the logical form $\forall n, \sim Q \Rightarrow \sim P$, which is the Contra-P form, or of the form "there exists no *n* such that, $P \wedge \sim Q$," which is the Contra-D form.

Following the administration of the surveys, 6 video-recorded, one-hour clinical interviews were conducted. During the interviews, participants were asked to discuss their responses to three of the proof comparison tasks and one statement selection task. The interviews were semi-structured to allow for clarification questions. The questions used in the interviews included asking the student to explain: (1) each proof to the interviewer (Can you explain this argument to me?); (2) any similarities or differences between the two arguments (Do you see any similarities or differences in the two arguments?) (3) his or her selection of the most convincing argument and of the best proof (Can you share with me how you thought about the two proofs as you decided which was more convincing? Can you share with me how you thought about the two proofs as you decided which was the best proof?). Furthermore, if the student did not comment on the proof type, students were asked at the end of the discussion of a comparison task, "would you describe one or either of the proofs as being a particular type of proof?" Thought of in terms of the work of Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012), the questions asked of participants were primarily *local* comprehension questions, which they describe as questions focused on: "students' understanding of key terms and statements in the proof;" "students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven;" and, "students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements" (p. 15). Additionally, some of the questions could be considered *holistic* - focused on students' understanding of the "proof as a whole" (p. 15).

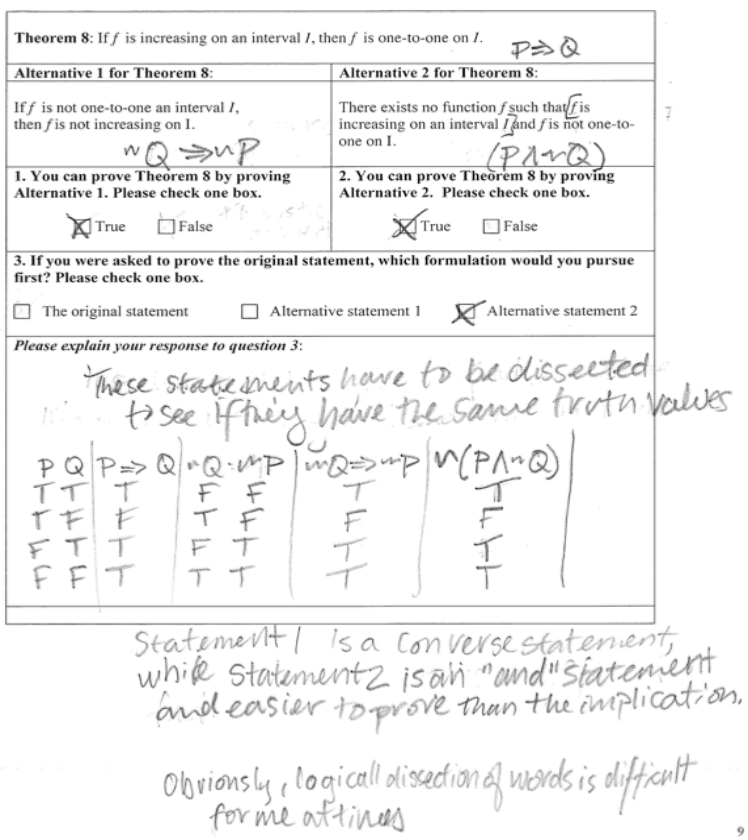
It should be noted that in Mejia-Ramos et al., the taxonomy of comprehension questions is geared towards evaluating students' comprehension of a single proof rather than a pairing of two proofs. Thus, this study's comparative questions (i.e., those focused on similarities and differences) do not fit within their taxonomy of comprehension questions. However, it can be argued that asking students to engage in comparative acts may provide additional insights into their comprehension of a given pair of arguments. Specifically, features that students may feel are not noteworthy may be important to making distinctions between two arguments. Indeed, comparative tasks may provide a context for eliciting a richer model of students' understanding of a given argument. Nevertheless, if such activities were non-normative then one could argue that it would be unlikely to provide deeper insight since student may be unprepared to engage in comparative work. Yet, it seems unlikely that this is the case. Students often engage in such activities. For instance, having constructed a proof for a theorem a student may compare their proof to an alternative proof provided by a teacher, a classmate, or in a text. One final issue to consider is that comparative questions may draw attention to specific features while diminishing others, a potentially problematic aspect of such questions. For instance, a comparison between two indirect proofs, such as in Tall (1979), might draw attention to specific details of the arguments, whereas comparison between an indirect and direct proof may draw attention to the indirect nature of an argument. In the study reported, however, understanding students' interpretations of the structure of indirect arguments was a primary research goal. Thus, the inclusion of comparative questions was warranted due to their potential focusing effect.

Analytic Approach

The analysis of the video-recorded interview data was informed by two constructs developed by Selden and Selden (1995): *unpacking* and *proof framework*. *Unpacking* refers to unpacking the logical structure of a statement; that is, the development of a symbolic, set-theoretic statement from a statement written in words. For example, the statement, "A function f is increasing on an interval I means that for any numbers x_1 and x_2 in I, if $x_1 < x_2$ then $f(x_1) < f(x_2)$" could be unpacked as, "$(\forall f \in \mathbf{F})(\forall I \in \mathbf{I})[(\text{f increasing on } I) \Leftrightarrow (\forall\, x_1 \in I)(\forall\, x_2 \in I)\ \{((x_1 < x_2) \Rightarrow f(x_1) < f(x_2))\}]$" (Selden & Selden, 1995, p. 138). The notion of *unpacking* was incorporated into the analyses due to observations of students' use of symbolic statements during the clinical interviews and in written survey work. A *proof framework* is a "representation of the 'top-level' logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts" (p. 129). The construct of a proof framework was relevant to the analysis since students were asked both to explain the proofs and to describe the proof type, questions which might provoke a students' proof framework. Lastly, since some interview questions focused on the extent to which the students' found a particular argument convincing, the construct of a *proof scheme* – "what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244) – informed the analysis.

Findings

The findings reported in this paper focus on students' successful attempts interpreting and evaluating indirect proofs and the logical equivalence of secondary statements. Attention was drawn to this aspect of the data for two reasons. First, when asked to explain the indirect arguments many students experienced difficulties with the proofs by contradiction, which drew attention to the few successful students. Second, roughly 1/3 of the students did not successfully identify the contra-D form statements as equivalent in the Type IV survey items. Among the successful attempts was Anna's response, which is shown in Figure 1. This response pointed to the possibility that successful students may use a partial unpacking of the theorems and their proofs when determining logical equivalence and/or structure. *Partial unpacking* refers to the use of symbolic statements that are not fully quantified and would not be considered an *unpacking* as defined by Selden and Selden (1995) but do represent a movement from written words towards a symbolic form, hence the name *partial unpacking*. Observe that in Figure 1 the student has noted an implication ($P \Rightarrow Q$), then symbolically identified Alternative 1 as the contrapositive ($\sim Q \Rightarrow \sim P$) and employed a truth table in an effort to determine the logical equivalence, $(P \Rightarrow Q) \equiv \sim (P \wedge \sim Q)$. The student also noted, "Obviously, logical dissection of words is difficult for me at times." This remark points to the student's need to move away from a word-based language to a symbolic language in order to analyze the underlying logical structure. Thus, through a partial unpacking and a series of symbolic manipulations the student appears to have ascertained the equivalence of the statements in question; in other words, it appears that the student was "thinking of the symbols as though they possess a life of their own without reference to their possible functional or quantitative reference" (p. 250), in other words, a symbolic proof scheme was enacted.



Theorem 8: If f is increasing on an interval I, then f is one-to-one on I. $P \Rightarrow Q$

Alternative 1 for Theorem 8:	**Alternative 2 for Theorem 8**:
If f is not one-to-one an interval I, then f is not increasing on I. $\sim Q \Rightarrow \sim P$	There exists no function f such that f is increasing on an interval I and f is not one-to-one on I. $(P \wedge \sim Q)$
1. You can prove Theorem 8 by proving Alternative 1. Please check one box. ☒ True ☐ False	**2. You can prove Theorem 8 by proving Alternative 2. Please check one box.** ☒ True ☐ False

3. If you were asked to prove the original statement, which formulation would you pursue first? Please check one box.

☐ The original statement ☐ Alternative statement 1 ☒ Alternative statement 2

Please explain your response to question 3:

These statements have to be dissected to see if they have the same truth values

P	Q	$P \Rightarrow Q$	$\sim Q \cdot \sim P$		$\sim Q \Rightarrow \sim P$	$\sim(P \wedge \sim Q)$
T	T	T	F	F	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Statement 1 is a converse statement, while Statement 2 is an "and" statement and easier to prove than the implication.

Obviously, logical dissection of words is difficult for me at times

9

Figure 1. Anna's Survey Response

Anna's response was an anomaly among the written survey responses. It was not until the clinical interviews that reasoning similar to that in Figure 1 was observed. Due to space limitation, in this paper we will describe only one of the two successful cases.

Lillian was a sophomore mathematics major at the time of the interview. She was observed reasoning symbolically during a proof comparison task and a statement equivalence task. The proof comparison task is shown in Appendix A. When asked to describe the two arguments, Lillian immediately described Argument A as a proof of the contrapositive and then, after a period of hesitation, described Argument B as a direct proof. She was then asked to explain how Argument B was a direct proof. Proceeding, Lillian was observed rereading Argument B at least four times, repeatedly returning to the first sentence, and underlining the two assumptions stated. After approximately 90 seconds she had not yet responded to the interviewer's question. The interviewer proceeded by asking Lillian why she had repeatedly underlined "*x* and *y* have opposite parity." She responded by saying "it's not correct" and then "I need to read it again." After another 45 seconds, Lillian sat back from the paper and remarked, "Oh, I see … it's correct" and then noted, "the conclusion is contrapositive, umm, contradiction to what they assumed." Analysis of her scratch paper, showed two statements, $P \rightarrow Q$ and $\sim Q \rightarrow \sim P$, both of which she had gestured towards during her reading of the arguments, most often crossing over the latter statement – a gesture one could interpret as a "crossing out." At this point in the interview it was unclear how Lillian had arrived at the conclusion that the proof was valid. However, what was clear from her remarks, gestures, and markings was that she had struggled interpreting Argument B, had recognized the use of two assumptions, one of which was a negation of the conclusion of Theorem 6, and that she knew the basic structure of the arguments was not of the form $\sim Q \rightarrow \sim P$. With that said, deeper insight into Lillian's reasoning was obtained from her response to the Theorem 7 statement equivalence task (see Figure 2). As was the case with the previous task, Lillian immediately identified Alternative 1 as a statement of the contrapositive of the original theorem. When asked to explain, she labeled "*n* is not a perfect square" as Q and "$n(\text{mod } 3) \equiv 2$" as P and explained that Alternative 1 was of the form $\sim Q \rightarrow \sim P$.

Theorem 7: For all positive integers n, if n mod(3) = 2, then n is not a perfect square.	
Alternative 1 for Theorem 7:	**Alternative 2 for Theorem 7**:
For all positive integers n, if n is a perfect square, then n mod(3) $\neq$ 2.	There exists no positive integer n such that n mod(3) = 2 and n is a perfect square.

Figure 2. Theorem Statement Equivalence Task

She then proceeded to examine Alternative 2 by rereading the statement multiple times and making a series of markings on her scratch paper, which are shown in Figure 3. She then proceeded to explain that the original statement was "for all" and that you could prove for all statements by, "showing that there exists no n such that it's … it's statement is false," she then gestured to her written work (the vertical arrow, phase 3) and noted that "these are equivalent," in reference to the statements $P \rightarrow Q$ and $\sim P \vee Q$, "so, this (points to $P \wedge \sim Q$) is the negation."

$P \rightarrow Q$	$P \rightarrow Q$ $\updownarrow$ $\sim P \vee Q$	$P \rightarrow Q$ $\updownarrow$ $\sim(\sim P \vee Q) \equiv (P \wedge \sim Q)$	$P \rightarrow Q$ $\updownarrow$ $\sim(\sim P \vee Q) \equiv (P \wedge \sim Q)$ no $n \mid P \wedge \sim Q$
Phase 1	Phase 2	Phase 3	Phase 4

Figure 3. Phases of Lillian's Written Work

Lillian's response resembles that shown in Figure 1 in that she produced a partial unpacking and then worked within the symbolic statements to determine equivalence, without any observable reference to the meaning of the statements, other than to confirm their logical status (*e.g.*, that one statement was the negation of another). Thus, ascertainment of the validity of an equivalence occurred at a symbolic level through the students' production of a partial unpacking and series of symbolic manipulations. Returning to Lillian's response to Theorem 6, her work with Theorem 7 offers a plausible rationale for her sudden realization of the structure of Argument B. Early in her work on Theorem 6, Lillian produced two symbolic statements $P \rightarrow Q$ and $\sim Q \rightarrow \sim P$, and reread Argument B repeatedly. Her crossing-out gestures indicate that she repeatedly rejected the idea Argument B was of either form. Thus, though it is unclear how she was able to move past her rejection of the argument and come to recognize it as disproving rather than proving $P \wedge \sim Q$, it is clear that a set of partially unpacked statements were used as tools for examining the logical structure of the argument. Moreover, with both tasks, Lillian constructed a series of partially unpacked statements and reasoned with those symbols as though "they have a life of their own" – in other words, she used a symbolic proof scheme.

It is also the case that Lillian and many other students recognized that within Argument B (and the other indirect arguments) a contradiction occurred and then used this realization to discuss the structure of the proof by contradiction. When asked to explain how the argument related to the theorem, however, many students became confused and were unable to make connections between the theorem statement and the basic assumptions in the initial sentence of the argument. Thus, it appears that a rudimentary proof framework for proof by contradiction was invoked among all of the students but that this was not a characteristic that distinguished successful from unsuccessful students.

Concluding Remarks

Researchers, such as Harel and Sowder (1998) and Healy and Hoyles (2000), have provided many examples of students' unproductive use of a symbolic proof scheme in their research. Few if any researchers have provided evidence of productive uses – of what Harel et al. alluded to when they remarked, " symbolic reasoning can either be superficial and mathematically vacuous, or a very powerful technique" (p.250). The findings in this paper, however, highlight the possibility of a potentially productive use of the symbolic proof scheme – one that may aid students in their effort to navigate the logic complexities and metatheoretical issues of indirect proofs. Moreover, the successful students' approaches are suggestive of activities, such as partially unpacking, which are rarely found in proof texts, yet may benefit students in their efforts to understand the structure of indirect proofs.

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Appendix A

<table>
<tr><td colspan="2">Theorem 6: If x and y are two integers for which $x + y$ is even, then x and y have the same parity.[1]</td></tr>
<tr><td>Argument A:

Assume x and y have opposite parity. Since one of these integers is even and the other odd, there is no loss of generality to suppose x is even and y is odd. Thus, there are integers k and m for which $x = 2k$ and $y = 2m+1$. Now then, we compute the sum $x + y = 2k + 2m + 1 = 2(k+m) + 1$, which is an odd integer by definition.</td><td>Argument B:

Assume x and y are two integers for which $x + y$ is even and that x and y have opposite parity. Since one of these integers is even and the other odd, there is no loss of generality to suppose x is even and y is odd. Now then, we compute the sum $x + y = 2k + 2m + 1 = 2(k+m) + 1$, which is an odd integer by definition. However, by assumption x and y are two integers for which $x + y$ is even. Since $x + y$ cannot be odd and even, either x and y must have the same parity or $x + y$ is not even.</td></tr>
<tr><td>1. I am confident about my understanding of Argument A. (Please mark one)

☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree</td><td>2. I am confident about my understanding of Argument B. (Please mark one)

☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree</td></tr>
<tr><td colspan="2">3. Which argument, in your opinion, is the most convincing? ☐ Argument A ☐ Argument B

Please explain your selection. (If you need additional space please use the back of this page.)</td></tr>
<tr><td colspan="2">4. Which argument, in your opinion, is the best proof? ☐ Argument A ☐ Argument B</td></tr>
</table>

1. Two integers are said to have the same **parity** if they are both odd or both even.

PRE-SERVICE SECONDARY TEACHERS' MEANINGS FOR FRACTIONS AND DIVISION

Cameron Byerley, Arizona State University
Neil Hatfield, Arizona State University

In this study, seventeen math education majors completed a test on fractions and quotient. From this group, one above-average calculus student was selected to participate in a six-lesson teaching experiment. The major question investigated was "what constrains and affords the development of the productive meanings for division and fractions articulated by Thompson and Saldanha (2003)?" The student's thinking was described using Steffe and Olive's (2010) models of fractional knowledge. The report focuses on the student's part-whole meaning for fractions and her difficulty assimilating instruction on partitive meanings for quotient. Her part-whole meaning for fractions led to the resilient belief that any partition of a length of size m must result in m, unit size pieces. It was non-trivial to develop the basic meanings underlying the concept of rate of change, even with a future math teacher who passed calculus.

Key Words: Pre-service Secondary Teachers, Rate of Change, Fractions, Teaching Experiment, Division

Research indicates that many students leave the middle grades well trained to operate symbolically on fractions while having under-developed meanings for fraction symbols (Hiebert & Behr, 1991). It has been suggested that strong meanings for secondary topics such as rate of change and proportion are dependent on strong meanings for fractions (Norton & Hackenberg, 2010). Although there are not many studies connecting students' meanings of fractions to their meanings for rate of change, the potential relationship can be justified mathematically. Rate of change can be understood as a comparison of the relative sizes of associated changes in two quantities. The mature meaning of fractions as reciprocal relationships of relative size, described by Thompson and Saldanha (2003), coheres well with the above meaning of rate of change. Research makes it clear that secondary teachers can anticipate needing to help their students develop quantitative meanings for fractions. Furthermore, many studies report that rate of change is a challenging topic for high school and university students (Asiala, Dubinsky, Cottrill, & Schwingendorf, 1997; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Coe, 2007; Orton, 1983). With the acknowledgment that we have no evidence demonstrating that immature meanings for rate of change are related to immature meanings for fractions, we propose that it is important to study secondary teachers' meanings for fractions because of the mathematical importance of comparisons of relative size in secondary mathematics. Furthermore, secondary teachers will be more likely to address their student's weak meanings for fractions if they have a strong quantitative understanding of the topic.

Most, but not all, studies designed to investigate teachers' meanings for fractions or division describe elementary teachers. Numerous researchers have found that teachers find it difficult to model a situation using division (Ball, 1990; Simon, 1993).

Additionally, teachers' demonstrations of fractions and division primarily involve computations and do not focus on quantitative meanings for the operations (Harel & Behr, 1995). Finally, teachers struggle to draw models of fraction situations while teaching (Izsák, 2008). In prior attempts to influence teachers' fraction schemes it was found "challenging at best and impossible at worst" to encourage teachers to override automatized procedures for computation and think meaningfully about multiplying and dividing fractions (Armstrong & Bezuk, 1995).

Methodology and Research Question

To understand the challenges in teaching adults new meanings for fractions we conducted a six session teaching experiment with a pre-service secondary mathematics teacher named Jacqui (Steffe & Thompson, 2000). The instructional goal was to develop the reciprocal relationships of relative size meaning for fractions articulated by Thompson and Saldanha (2003) by engaging the pre-service teacher in a modified activity sequence originally designed by O'Bryan and Thompson (unpublished). During the teaching experiment the witness, second author, and the instructor, first author, assumed that Jacqui had different meanings for fractions and division than we did. Cobb and Steffe (2011) describe a student's "actual meaning" as the one that is created as the student interprets instruction and not necessarily the teacher's intended meaning (p. 86). We believed that by engaging her in specific tasks her thinking could be modeled in a way that could shed light on the challenges associated with teaching adults new meanings for fractions. The teaching experiment was videotaped, transcribed and analyzed using methods described by Corbin and Strauss (2007). The primary research question was "what meanings for fractions and division constrain and afford the development of a reciprocal relationships of relative size meaning for fractions?

Constructs Used to Model Meanings for Fractions

Descriptions of Jacqui's thinking are based on the constructs reciprocal relationships of relative size from Thompson and Saldanha (2003), part-whole and iterative fraction schemes from Steffe & Olive (2010), and partitive and quotitive division described by Simon (1993).

Thompson and Saldanha (2003) stress that a multiplicative understanding of fraction requires that students understand fractions as reciprocal relationships of relative size. They describe this relationship as "Amount A is $1/n$ the size of amount B means that amount B is n times as large as amount A. Amount A being n times as large as amount B means that amount B is $1/n$ as large as amount A" (2003, p. 31). Thompson and Saldanha contrast this mature multiplicative meaning with an additive part-whole fraction scheme.

In a part-whole fraction scheme the student understands seven-tenths as seven out of ten equal parts without an awareness that the one-tenth is an iterable unit or a length that stands in comparison to the whole (Steffe & Olive, 2010, p. 323). For example, Nik Pa (1987) found that 10 and 11 year old children could not find 1/5 of 10 items because "one-fifth" referred to one and five single items (cited in Steffe and Olive, 2010, p. 3). A part-whole meaning is often sufficient to correctly answer common fraction problems such as describing what fraction of marbles in a bag are red. Notice that in the statement "3/5 of the marbles are red" the denominator refers to the number of marbles in the bag.

In the more advanced iterative fraction scheme, the denominator refers to the number of times the original quantity is partitioned, and not necessarily the number of distinct objects in the physical situation. Students with an iterative fraction scheme can imagine fractions such as 7/5 as a length because they can imagine disembedding 1/5 of a whole and iterating it seven times. After a student partitions, disembeds, and iterates to find 7/5 the result stands in "multiplicative relationship to the whole" meaning that it is 1/5 seven times (Hackenberg, 2010, p. 394). Although Steffe and Olive (2010) and Thompson and Saldanha (2003) do not explicitly draw connections between their constructs, it seems mathematically logical that developing a reciprocal relationships of relative size meaning for fractions requires an iterative fraction scheme and an ability to reverse mental operations.

Understanding fractions as reciprocal relationships of relative size requires a quantitative meaning for the quotient. There are at least three quantitative meanings that allow one to interpret the quotient as providing information about real-world quantities. In the partitive meaning the quotient A/B refers to the size of each group when A is cut up into B equal-sized groups. In the quotitive meaning, the quotient A/B represents how many times something of length B would fit when laid end to end next to something of length A (Simon, 1993). Quotitive division could also be described as measuring A with a ruler of length B. In the third meaning, A/B quotient A/B tells us the relative size of quantity A and B. In the relative size meaning A is A/B times as large as B (Thompson & Saldanha, 2003). Speaking of division as a measure of relative size is particularly useful when trying to explain rate of change or the definition of the derivative.

Results of Initial Assessment and Teaching Experiment

Jacqui was selected as a typical representative of a group of 17 pre-service secondary mathematics teachers who completed an assessment on meanings for division and fractions. Assessment questions were inspired by Hackenberg's (2009) research on reversible multiplicative relationships, Simon's (1993) and Ball's (1990) studies on division, O'Bryan's and Thompson's activity sequence and Coe's study of teachers' meanings for rate of change (2007). Jacqui received A's and B's in all of her math courses up to and including Calculus I and plans to teach high school math. Although everyone who took the assessment had at least passed Calculus I, the majority struggled to answer questions related to fractions and division. Ten out of seventeen pre-service teachers' were able to give a scenario in which you would divide by a fraction. Six out of seventeen were able to use a picture to explain the meaning of a quotient in a problem involving division by a decimal. Only one teacher was able to explain why division is used to calculate slope. The acceptable response described division as a measure of relative size in changes in y and changes in x. Like many respondents, Jacqui was able to correctly answer some tasks, and drew visual representations of fractions, but struggled to explain or model division and tended to express fractions as parts out of wholes.

After a summary of the teaching experiment, two of Jacqui's resilient, problematic meanings will be described in more depth. Throughout the teaching experiment the witness and instructor attempted to help Jacqui develop an iterative fraction scheme by interpreting A/B as A copies of $1/B$ of one. This required developing a meaning for multiplication as making copies as well as understanding $1/B$ as the amount in one piece when we partition one into B equal pieces. The item in *Figure 1* was intended to help Jacqui interpret fractions as reciprocal relationships of relative size.

Some amount, call it B, is partitioned into n equal parts.

a) How large is B compared to the size of each part?

b) How large is each part in relation to B?

Figure 1. Teaching experiment item designed by O'Byran and Thompson.

Figure 2 shows an item added from Hackenberg's research in an attempt to necessitate the coordination of multiple levels of units, an ability thought to be related to advanced fraction schemes (Steffe & Olive, 2010).

The unmarked rectangle shown represents 3/5ths of a candy bar.

a) Draw a picture of the whole candy bar.

b) Suppose the entire bar is shared equally among 10 students. What fraction of the entire bar will they receive? Why?

c) What fraction of $1/5^{th}$ of the bar will they receive?

Figure 2. Teaching experiment item inspired by Hackenberg (2010).

Additionally, we wanted Jacqui to understand that an iterative fraction scheme could give a quantitative meaning to an "improper" fraction because in her initial assessment she interpreted improper fractions only after converting them to mixed numbers. Although Jacqui was often able to answer the questions correctly using procedures and her primarily part-whole meaning for fractions, she revealed many problematic assumptions when she explained her work. For example, she believed that fractions must be less than one, that multiplication makes bigger, and that the word "of" in "1/4 of 1/6" means division because in her diagram it looked like she was "pulling out" 1/6 from the whole 24 times. She did not think that dividing by a fraction had the same quantitative meaning as dividing by a whole number because dividing by a fraction was really multiplying. When asked how fractions and division were related she replied that it was possible to divide by a fraction but gave no evidence of a stronger connection between the two ideas. After these issues were noticed, it was typically possible to ask Jacqui a question that caused her to see her mistake. With practice, she learned to speak of A/B as A copies of $1/B$, the product $u*v$ as u copies of size v and to use division to determine how many times as large A is as B, where A and B were any real numbers.

Despite some success, there were two related issues that took a number of focused attempts to resolve that were strongly rooted in Jacqui's meanings for division and fractions. The first issue was that we wanted Jacqui to distinguish between partitive and quotitive division so that when she spoke about dividing while explaining her answers we had an image of what she meant. On the first day we gave an example of the difference between the partitive and quotitive division and asked her to summarize the two models at the end of the day. In both of her models the quotient A/B represented the number of groups of size B when A was measured with length B. She used slightly different language to explain each model. In one case she described seeing how many times B could be pulled out of A. In the other case she described how many copies of B fit into A. In these explanations, Jacqui used the word partition to describe a quotitive model of division. She said "I'm trying to figure out how many times one third can go into four thirds so I can partition four thirds into one third sections and then evaluate how many one third pieces make up a complete four thirds piece." We immediately made the first of many attempts to check that Jacqui understood our meaning for partition by asking her to interpret the quotient as the size of the group resulting from a partition. In the next session we asked her to summarize the two models of division and she again explained two quotitive models. We intervened with another explanation of the partitive model of division and asked her to explain what A/B meant in the partitive model. She replied, "so you are partitioning A into B pieces… So how ever many times, so each one is $1/B$ of A, so [pause] I don't know where to go from here. That's just it, it's just $1/B$ of A." Despite direct instruction moments before, she had not assimilated that the size of the piece resulting from the partition stands for the quotient. We attempted to resolve this problem on five separate occasions and after a 25-minute focused lesson on partitioning using manipulatives she still had a different meaning for partitioning that resembled the quotitive model.

The other problematic issue for Jacqui was her repeated insistence that one part out of a length A cut up into B equal pieces was size $1/A$. She also divided a length of four into two equal pieces and called the size of each piece $1/4$ instead of $1/2$. She confounded the total number of objects in a group (four) and the number of partitions of the group (two). Her behavior is consistent with Steffe's description of part-whole meanings for fractions that focus on the denominator as the total number of objects in a group. This meaning for fractions is problematic when attempting to understand the statement "Amount A is $1/n$ the size of amount B means that amount B is n times as large as amount A." In Jacqui's case she automatically assumed that after cutting up a quantity into groups that each group must be size one. Jacqui struggled with the question in *Figure 3* because of this assumption.

Imagine that a pack of bubble gum is split equally among a group of 11 friends. What fraction of the bubble gum will each friend receive?

Figure 3. Teaching experiment item from O'Bryan and Thompson.

Although she knew that each person received 1/11 of the pack she repeatedly insisted that there must be 11 wrapped pieces of gum in the pack and that if the pack was partitioned equally it was not possible for one person to receive 1/2 of a wrapped piece of gum. Even after a prolonged discussion about the bubble gum problem, her part-whole meaning fractions took over and she continued assume the denominator represented the total number of objects in the next problem. Another indication of her part-whole thinking was her tendency to describe the fraction 5/5 as "five pieces of five" because "you look at it as five pieces of length one."

Although Jacqui initially viewed fractions and division separately, she began to associate both fractions and division with the expression A / B. It appears that Jacqui knew a process for finding a quotient using the quotitive method, but when she imagined partitioning an amount into equal pieces she automatically assumed that he pieces were of size one. This is one explanation for why she would not have assimilated the instruction focused on interpreting the quotient as the size of a group resulting from a partition. Furthermore, her meaning for the word partition was associated with the hash marks she made when she measured A in terms of length B. When I referred to the partitioning model of division, she imagined quotitive division and learned a meaning for partition that was inconsistent with my intentions.

Jacqui's Constraints and Affordances

Speaking carefully, anticipating and checking for non-productive meanings and watching videotaped sessions all contributed to the resolution of a number of issues Jacqui faced. Many of the cases of miscommunication were subtle and only apparent in retrospect. It became clear that drawing a rectangle partitioned into pieces could mean partitioning to one person, copying to another and measuring to a third. A major constraint was that Jacqui believed she understood me, when in fact she had assimilated what I had said to unintended images. She openly admitted she believed we were just using different words but speaking about the same idea when we were in the midst of a major miscommunication about partitive and quotitive division. It is possible that when teachers are asked to learn a new language about fractions and they do not develop the associated quantitative meanings, they will view the request as arbitrary pickiness.

Another constraint to developing the intended meanings is that Jacqui often was able to provide descriptions such as "9/7 means nine copies of 1/7" without altering her problematic assumptions about the size of the part in a fraction. Using symbolic skills paired with part-whole meanings for fractions, Jacqui was able to answer questions that were designed to challenge students who had primarily part-whole meanings in Hackenberg's research (2010). Often she inserted correct numbers in the blanks in the activity sequence and needed personal feedback on the quality of her explanations to alert her to errors in her thinking. Even after substantial intervention, we did not reach our goal of developing a reciprocal relationships of relative size meaning with Jacqui. The part-whole fraction meaning Jacqui most likely developed in school constrained her ability to assimilate partitive division. This lack of shared meaning for the word partition was one major constraint that stalled our progress in developing a reciprocal relationships of relative size meaning for fractions. Although not formally investigated in this study, it seems possible that Jacqui's difficulty explaining the slope formula could be because she struggled to view division and fractions as a measure of relative size of two quantities.

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PERFORMANCE AND PERSISTENCE AMONG UNDERGRADUATE MATHEMATICS MAJORS

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There is little mixed methods research into the patterns of course taking, performance, and persistence among mathematics majors, in general, and among secondary mathematics majors, in particular. Drawing from a sample of 42,825 mathematics enrollment records at two universities over a six-year period, this study presents quantitative summaries of mathematics majors' performance and persistence in undergraduate mathematics courses alongside qualitative themes from interviews of nine secondary mathematics majors at one of the universities. Implications include potential strategies for mathematics programs and faculty to support the success of mathematics majors in undergraduate mathematics coursework, with special emphasis on prospective secondary mathematics teachers.

Keywords*:* mathematics performance, persistence, mathematics majors, preservice teachers

A recent U.S. federal advisory panel report recommended expanding mathematics instruction by faculty from other disciplines (e.g., physics or engineering) and building "a new pipeline for producing K-12 mathematics teachers from undergraduate and graduate programs in mathematics intensive fields other than mathematics." (PCAST, 2012, p. vii). Citing mathematics as a bottleneck in STEM education, the report suggests students in a range of STEM programs may be better served by learning the specialized mathematics needed for their fields from faculty in those fields, views held in direct contrast of many mathematicians (AMS, 2012). In the context of the mathematics preparation of teachers, the PCAST recommendations pose a serious challenge to the primary role of university mathematics faculty in the preparation of secondary mathematics teachers.

Nearly all undergraduate teacher education programs build prospective secondary mathematics teachers' content knowledge through courses in an undergraduate mathematics major, including courses such as single- and multi-variate calculus, differential equations, linear algebra, introduction to proof, statistics, and abstract algebra (Conference Board of Mathematical Sciences, 2012) . Though research is unclear on whether advanced undergraduate mathematics courses necessarily improve the specialized knowledge of mathematics needed in middle and high school classrooms (Hill et al., 2005; Kahan et al., 2003), U.S. secondary mathematics teachers typically complete at least four times as many courses in mathematics as they do in education (Monk, 1994).

Mathematics majors collectively consist of just 1% (12,363 of 1,119,579) of bachelor's degree earners in the U.S. (Lutzer, 2002), and there has been limited research on the patterns of academic outcomes among the even smaller number of students concentrating in secondary mathematics teaching. Existing research does, however, suggest earning a bachelor's degree in mathematics can be a difficult experience. Mathematics courses have the second lowest grade distribution among all majors (Rask, 2010), and more than 3 of 5 students who declare a mathematics major switch out of the major (Seymour & Hewitt, 1997). Moreover, students' early work toward earning mathematics degrees may be complicated by the role of mathematics courses as a filter for business and STEM programs (Moore & Shulock, 2009) and a variety of social factors, such as a lack of family support and the commonly held societal view of mathematics as a male domain (Ma & Kishor, 1997). Secondary mathematics majors may also face the challenge of navigating between grading norms in mathematics courses and those established in education courses, which have the highest grade distribution among all college majors (Rask, 2010).

Though learning undergraduate mathematics is central to learning to teach mathematics (Cooney & Wiegel, 2003), there has been limited research specifically addressing the performance of prospective secondary mathematics teachers in undergraduate mathematics courses. This may be due in part to small enrollment numbers. Students majoring in secondary mathematics are typically outnumbered in nearly all required courses by STEM majors outside of mathematics (Lutzer, Rodi, Kirkman, & Maxwell, 2007). In 2005, just 3,400 secondary mathematics majors joined about 15,000 applied or liberal arts mathematics majors among 699,000 students enrolled in undergraduate mathematics courses at the calculus level or beyond (Lutzer et al.). There have been investigations into the subject matter knowledge gained by secondary mathematics majors through undergraduate mathematics programs (e.g., Bryan, 1999; Even, 1993), there is need for more research on mathematics performance and persistence.

The research questions address (a) the patterns of course enrollment and letter grades earned by mathematics majors in mathematics courses, and (b) how secondary mathematics majors experience and respond to difficulties in undergraduate mathematics courses:

1. (Quantitative) What characterizes mathematics majors' academic performance and persistence in undergraduate mathematics courses at two universities?
2. (Qualitative) What describes prospective secondary mathematics teachers' perceived performance in undergraduate mathematics courses at one university?

Methods

The two universities included in the quantitative strand are similar in many ways – each is a minority serving public institution located in the same central U.S. state, enrolls about 7,500 full-time undergraduate students, admits about 90% of applicants (many of which are community-college transfers), and offers undergraduate and master's programs in mathematics. The main contrasts come from the facts that University A is a regional campus in a mid-sized city, while University B is in a large metropolitan area and enrolls mostly female students. Data collection consisted of institutional academic records from both universities and face-to-face interview and survey data from University B. Quantitative data included student, course, and performance variables for all students enrolled in one or more mathematics courses during the study period of six academic years (fall 2005 to spring 2011). For each enrolled mathematics student, the institutional records included unique student identifier, university, ethnicity, sex, ACT/SAT mathematics score, high school grade point average, age, undergraduate major, academic level, course name, course section, instructor, term, and final letter grade.

The qualitative strand of the study employed semi-structured interviews crafted after the protocol of Seymour and Hewitt's (1997, p. 401) large-scale study of switching- and non-switching undergraduate science, mathematics, and engineering majors. Interview questions invited study participants to discuss experiences surrounding choosing a college major, performance in high school and college mathematics, exam and course grades in mathematics, quality of mathematics instruction, sources of academic support, (self- and peer) experiences of major switching, and career plans. As part of the interview protocol, participants completed a modified version of the 30-item Mathematics Self-Efficacy and Anxiety Questionnaire (May, 2009) and self-reported their mathematics self-efficacy (Bandura, 1997) in the content of each of their undergraduate mathematics courses on a scale of 0 (not confident) to 10 (very confident).

The quantitative data sample initially included all N = 42, 825 enrollments in mathematics courses by undergraduate students at the two universities during the six-year study period. While this larger sample allowed for analysis of course enrollment patterns, measures of persistence in mathematics were focused on a subsample of n = 12,522 mathematics course enrollments by students who completed all their mathematics classes during the six-year study period. Demographics in the overall sample suggests the data set included more female (72%) than male (28%) students, and no majority student ethnicity (48% White, 31% Hispanic, 14% Black, 4% Asian, 2% Other). In addition, the sample included comparable enrollments by Freshmen (27%), Sophomores (17%), Juniors (20%), and Seniors (26%). Most students were either between the ages of 17 to 22 (48%) or 23 to 27 (35%), with just 16% older than 27 years old.

The nine qualitative interview participants were purposefully sampled at University B. All undergraduates from University B in the persistence subsample who had previously declared a secondary mathematics major (approximately 40 students) were invited to participate in the study, so the interview participants represent a self-selected group of secondary mathematics teachers interested in sharing their experiences in mathematics courses. Eight of the participants were still mathematics majors at the time of the interview, and six participants planned on working as a teacher after completing their degree. The numbers of mathematics enrollments were 4, 10, 10, 12, 12, 13, 18, 19, and 27, respectively.

Results

Performance of Mathematics Majors in Mathematics Courses

Mathematics majors represented 1.9% of the unique students who took one or more mathematics courses during the study period. However, enrollments by mathematics majors accounted for 5.8% of all the mathematics enrollments, and at least one mathematics major completed every one of the 28 mathematics course titles offered during the study period. Mathematics majors were found in the highest percentages in classes with the lowest enrollments. For example, mathematics majors formed just 0.8% of the enrollments in the 8 introductory courses (e.g., College Algebra) which account for 77% (33,177 of 42,825) of the combined mathematics enrollments. Mathematics majors formed less than 10% of enrollments in Trigonometry, Precalculus, and Calculus I, and less than a third of enrollments in Probability & Statistics, Calculus II, and Discrete Math I & II. Mathematics majors were the minority in 6 of the 10 most taken courses by mathematics majors, including Probability & Statistics, Calculus I & II, Discrete Math I, Linear Algebra, and Introductory Statistics I.

Though mathematics majors were among the least common major types enrolled in mathematics courses (only "Undeclared" had lower absolute enrollment numbers), mathematics majors earned the highest overall distribution of letter grades. Forty-eight percent of mathematics majors maintained an average letter grade of B or better in their mathematics courses; the combined mean mathematics GPA of mathematics majors was 2.5 (*Mdn* = 2.9, *SD* = 1.3, Range = 0 to 4). As indicated in Table 1, more than one-in-three enrolled mathematics majors earned a letter grade of A (highest among all major types), and the overall DFWI rate for mathematics majors was 20.8% (lowest among all major types). However, six course titles showed DFWI rates among mathematics majors of the courses of more than 25%, including Abstract Algebra (32%), Introduction to Proof (32%), Linear Algebra (30%), Probability & Statistics (29%), Real Analysis (28%), and Calculus I (26%).

Table 1. Major types among undergraduate mathematics courses at two universities.

Major Type	%A	%B	%C	%D	%F	%W	%I	Total
Health Sciences	32.6	24.3	18	6.6	11.5	5	2	*11,780*
Liberal Arts	20	23.8	21.4	7.5	14.5	8.7	4.1	*8,843*
Sci/Engineering	23	23	22.1	9.9	12.6	8.1	1.3	*7,248*

Business	25.6	24.9	21.5	7.4	10.4	8	2.2	*6,531*
Education	22.7	24.1	21.6	7.4	12.6	6.9	4.6	*5,102*
Mathematics	34.9	26.4	17.9	6.9	8.5	4.7	0.7	*2,249*
Undecided	22.8	29.3	22.8	10.1	11.7	3.4	0	*386*
Combined	26.1	24.2	20.4	7.6	12.1	7	2.6	*42,139*

Persistence among Mathematics Majors in Mathematics Courses

The subsample of mathematics course enrollments by students who completed all their mathematics courses during the study period (n = 12,522) included just 98 unique students (0.8%) who declared a mathematics major during the study period. The mean number of mathematics courses attempted by mathematics majors was 5.0 (*Mdn* = 3, SD = 4.5, Range = 1 to 19), and 69.4% of mathematics majors took 5 or fewer mathematics courses. However, just 29.4% of mathematics majors passed 6 or more mathematics classes with a letter grade of C or better (M = 4.3, *Mdn* = 3, SD = 4.4, Range = 1 to 19). More than half (54.1%) of mathematics majors attempted 3 or fewer mathematics classes, 15.3% attempted 4 or 5 classes, 12.3% attempted 6 to 9 classes, and 18.4% attempted 10 or more classes. Based on the enrollment numbers, we estimate only about one in four mathematics majors completed enough mathematics courses to fulfill undergraduate degree requirements.

Overall, 35.7% of mathematics majors failed at least 1 mathematics class, including 39.3% of mathematics majors who enrolled in 7 or more mathematics classes. There were 8 courses for which 10% or more of the enrolled mathematics majors had previously attempted the course, including Introduction to Proof (19.3%), Abstract Algebra (16.0%), Real Analysis (15.2%), Probability & Statistics (14.7%), Calculus I (12.2%), and Linear Algebra (10.1%). Collectively, 9.5% of mathematics majors enrolled in mathematics courses were retaking the course, with 1.3% attempting the course for a third, fourth, or fifth time.

Perceived Performance in Mathematics Courses among Secondary Mathematics Majors

The major themes in our qualitative interviews included struggles described by mathematics majors in mathematics courses and the diverse ways in which some of the students worked to overcome those obstacles. In all, we found eight areas identified by multiple students as sources of personal frustration in mathematics courses, including the course instructor (5 participants), family/parenting (4 participants), difficult content (4 participants), inadequate understanding of prerequisite content (3 participants), online course format (3 participants), difficulty in understanding the textbook (2 participants), and a perceived lack of real-world applications (2 participants). All of the study participants described substantial struggles in at least one of their undergraduate mathematics courses. Five participants failed to pass (with a C or better) at least one required mathematics course on their first attempt, and three of the participants took a specific mathematics course three or more times. Consistent with Champion (2010), these participants described performing well as elementary and secondary school mathematics students and recalled rarely struggling to learn new mathematical ideas prior to college. In all cases except Goldie, participants described first encountering substantial difficulties in a mathematics courses while in college.

Though each student's stories of struggles were unique, Kirsten's description of her undergraduate mathematics coursework exemplifies the extent to which self-perceived performance had deep impacts on mathematics majors' lives. Kirsten excelled in high school mathematics courses and entered college planning to become a high school mathematics teacher. After earning high grades in the required lower-division mathematics classes, Kirsten described first encountering major struggles when she earned a D in Real Analysis:

> I had a complete lack of understanding [of Real Analysis]…. I had to retake it because it was a major class, but I just, I couldn't go back to school. I didn't know what I wanted to do with my life. That D really stung me a lot. So…. for the fall, I substitute

> taught, and that was important in getting me back to school, because I did a lot of substitute teaching and I liked the school.

Kirsten's poor performance in Real Analysis led her to temporarily withdraw from the university and return home. She reenrolled a semester later and devoted a significant amount of time and effort to Abstract Algebra:

> I worked myself to death …. This is what I call my crazy time. I got so depressed after that class that I actually started seeing a psychologist…. I took Analysis again. I took Abstract Algebra again. I took a Geometry class…. I kept on trying, thinking I would maybe just be done, but I was emotionally shot at this point.

After failing Abstract Algebra, Real Analysis, and Geometry, Kirsten formally withdrew from the university and took two years to work and seek treatment for clinical depression. She attributed her mathematics hardships as major sources of her depression, recalling, "I'm struggling with math classes. How can I even graduate? How can I become a math teacher? Is this the right path? What am I supposed to do with my life? What do I do now?" Before returning to school, Kirsten described having to work through a fear of failure in mathematics with a therapist. She eventually came to believe, "I probably am capable of graduating college. I am capable of this… Just because I failed once or twice doesn't mean that I will continue to fail in every other thing." So, three years after leaving the university, Kristen moved back to her home state and enrolled part-time in one mathematics course at University B. The next semester she took two courses (including Abstract Algebra), and after earning an A in all her classes, described plans to return as a full-time student in the fall. She was unsure of whether or not she still wanted to teach high school mathematics, but confidently expected to complete her undergraduate mathematics degree in the near future.

Though seemingly extreme, it was unclear whether Kirsten's reports of intense and interconnected personal and academic struggles in mathematics courses were atypical. Three participants described intense feelings of hopelessness in calculus courses as they worked to overcome perceived deficiencies in understanding of pre-requisite content. Four participants recalled struggling to catch-up in classes following personal crises, including caring for children, grieving the death of a close relative, and even having a child mid-way through an Abstract Algebra course.

Discussion

Several aspects of the study findings (e.g., interview themes, grade distributions of mathematics majors) should be considered exploratory and tied to the context at the two research sites. Nonetheless, the findings suggest several insights into how mathematics programs can help secondary mathematics majors succeed in an undergraduate mathematics programs. The statistical results suggest (1) mathematics majors are among the best performing students in mathematics courses and (2) many mathematics majors choose not to persist in undergraduate mathematics. Fewer than half of mathematics majors took more than three mathematics courses, and nearly two in five mathematics majors who completed seven or more mathematics classes failed at least one course. The results suggest that mathematics majors, though typically successful in mathematics courses, often experience significant difficulties with at least one mathematics course which may lead them to reconsider their choice of the major.

The quantitative and qualitative data also support the common sense claim that students' struggle most in mathematics classes with unfamiliar content, especially abstract and proof-based courses. Our interviews suggested these struggles may be amplified when students struggled early in course, perceived instructors to have strict expectations, or faced assignments that deemphasized computational procedures. Some of these struggles may be unavoidable, and with proper support, challenging courses can provide excellent opportunities for learning. Well-designed courses like Introduction to Proof, Abstract Algebra, and Real Analysis can deeply extend mathematics majors' understanding, and can serve the especially important function of recalibrating the beliefs of secondary teaching majors who, prior to such courses, may think of mathematics in terms of memorization and assume the role of mathematics teachers is to lecture (Cooney, Shealy, & Arvold, 1998). Much of the gap between mathematics majors' history of success in mathematics courses and their struggles in challenging courses may well be bridged by clear communication of expectations along with formal and informal support structures.

The interview data also suggests approaches faculty might consider in order to help mathematics majors be successful in mathematics classes. For example, faculty might consider explicitly describing the qualitative meaning students can ascribe to exam scores (e.g., "a score above 70% on this test means you're on track"). Instructors might also use group quizzes or in-class activities to help mathematics majors make normative judgments of their understanding in relation to that of their peers. Fukawa-Connelly (2012) has recently outlined a way in which abstract algebra students can reach deep understanding of proofs by engaging in a student-centered classroom atmosphere in which students present and defend their proofs, and "high discourse communities" (Imm & Stylianou, 2012) can help students' understandings become the focus of class discussions and could help break the cycle of procedural-based instruction that makes it difficult for some secondary mathematics teachers to shift to conceptual-based teaching approaches (Cooney, Shealy, & Arvold, 1998).

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TEACHING UNDERGRADUATE CALCULUS FOR TRANSFER: A QUALITATIVE CASE STUDY OF THE CALCULUS SEQUENCE AT ONE LIBERAL ARTS COLLEGE

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At small liberal arts colleges, a single calculus sequence must successfully accommodate students from various majors, such as mathematics, biology, chemistry, and economics. This qualitative case study considers mathematics professors' perspectives about the required nature of calculus in various disciplines, attempts to identify how calculus instructors teach with the aim of preparing students to apply calculus knowledge in their future coursework, and how the disciplinary focus of their students affects professors' design and teaching of calculus courses. Framed using aspects of teaching and learning shown to promote transfer of knowledge, results suggest that the professors teach for understanding and allow in-class processing time, but could improve their emphasis on applying calculus in non-mathematics disciplines. This study contributes to the growing body of undergraduate mathematics education research intended to document undergraduate teaching practices.

Keywords: Undergraduate Calculus, Transfer, Qualitative Case Study

Objectives

Undergraduate courses like the calculus sequence serve students from various majors, such as mathematics, biology, chemistry, and economics. Small liberal arts colleges (SLACs) do not have the ability to offer multiple calculus courses specifically for students from these departments, but instead must offer one calculus sequence to successfully serve these students and departments. Newly added majors in Biochemistry and Molecular Biology at the SLAC in this study, both of which require significant calculus, are changing the composition of students taking the calculus sequence to include more science students. This change in student composition prompts the need to examine how calculus instructors design and teach courses in the calculus sequence, largely populated by non-mathematics students. Specifically, research is needed to determine how calculus instructors prepare students who are required to take calculus for their major to use calculus in their major disciplines.

This qualitative case study considers mathematics professors' perspectives about the required nature of calculus in various disciplines, and attempts to identify aspects of professors' design and teaching of calculus courses that might increase students' ability to use their calculus knowledge in subsequent courses. In particular, the study asks: (1) Why do mathematics professors believe specific departments on campus require their students to take all or part of the calculus sequence?; and (2) How do calculus instructors use teaching methods known to improve learning for transfer?

Background Literature and Theoretical Framing

While it is often acknowledged that calculus serves a wide audience (Bressoud, 1992), little research has formalized how calculus instruction might be designed to accommodate the different disciplines that make up a typical calculus course at a SLAC. A personal account of a successful restructuring of Calculus I that was recently undertaken at Macalester College, a small liberal arts institution, illustrates how knowledge about calculus use in non-mathematics departments can be used to inform calculus course design (Bressoud, 2008). There have also been calls for quantitative skills in disciplines such as biology (e.g., Bialek & Botstein, 2004) and surveys attempting to connect quantitative literacy with economic literacy (e.g., Schuhmann, McGoldrick, & Burrus, 2005). Despite these calls, surveys, and personal accounts of relevant calculus restructuring, research has not been conducted to formalize how, or if, calculus professors attempt to teach calculus for transfer of knowledge to non-mathematics disciplines. In fact, empirical research in undergraduate mathematics education is limited, and calls have been made for more research to illuminate college teaching practices (Speer, Smith, & Horvath, 2010).

In order to characterize how mathematics instructors design and teach calculus courses for transfer of knowledge into other disciplines, both knowledge about professors' perceptions of the required nature of calculus in other disciplines and knowledge of the ways in which professors currently teach calculus courses are needed. According to cognitive learning theory, learning can be considered as knowledge construction (Mayer, 1992). In this perspective, learners control their own cognitive processes during learning, and teachers are also participants in the learning process, actively constructing meaning during learning situations. As a result, finding ways to help students effectively process information becomes a central instructional issue. This study explored how calculus instructors teach calculus for transfer, which Bransford, Brown, and Cocking (2000) define as “the ability to extend what has been learned in one context to new contexts” (p. 51).

Initial learning is essential to transfer, and many types of learning experiences have been shown to promote transfer, such as learning with understanding as opposed to only memorizing facts or procedures without a connection to why those procedures or facts are used or work, learning with time to process complex material, and learning with frequent feedback focused on when, where, and how to use knowledge (Bransford et al., 2000). Bransford et al. also point out that research has shown that students' knowledge transfer is increased when potential ways in which the knowledge they are learning might be useful in the future are highlighted during initial learning. Beyond the types of initial learning experiences that have been shown to promote transfer, learning in multiple contexts can promote transfer of knowledge since students are able to create flexible knowledge of the topic by abstracting general features (Bransford et al., 2000).

Readily explorable in calculus teaching practices through interviews with professors, classroom observations, and examination of course materials, five characteristics of teaching and learning for transfer supported by the empirical literature framed this study: learning with understanding, giving processing time, giving frequent feedback focused on understanding, noting transfer implications during original study, and learning in multiple contexts. Although the professors were not familiar with the concept of knowledge transfer from a formal educational learning theory perspective (and I did not present these aspects of teaching as “ways to promote transfer” during interviews), they were aware that certain of these ideas, such as connecting concepts to future applications, could be of benefit to student learning.

Research Methodology

This study took the form of a qualitative case study because it provides an “in-depth description and analysis of a bounded system" (Merriam, 2009, p. 40), the calculus sequence at one liberal arts college. Data were collected in one Calculus I course and two Calculus II courses that were conducted during the Spring 2012 semester. Mathematics professors were the key informants in this study because they represent experts in calculus. Their perceptions of calculus, both as key in mathematics and as required for other disciplines, forms the basis of how the calculus sequence is designed and executed. Focal mathematics professors represented the mathematics department's general views as conveyed in an initial survey, and can therefore been seen as a typical (Patton, 2003) and purposeful sample (Merriam, 2009; Patton, 2003). Selecting focal professors who self-identified as having information to give about calculus is one way I ensured "information-rich" subjects to study (Merriam, 2009).

The data collection proceeded in four phases. An initial email survey provided basic information about departmental beliefs about calculus teaching and learning and was used to select the three focal calculus professors. Preliminary individual semi-structured interviews asked professors to provide perceptions of why calculus is required in other disciplines, the knowledge or skills they believe their calculus students obtain, and to discuss how (if at all) they believe they teach calculus so that their students are prepared to use calculus in their major disciplines. The third phase of data collection was a 50 minute class session observation, during which I was an "observer participant" (Merriam, 2009) generating knowledge of how calculus is used in the classroom for triangulation with the email survey results and interviews. Classroom observations informed the final phase of data collection, individual follow-up semi-structured interviews, where professors were asked to describe their calculus courses in more detail based on observation of their class and examination of their course materials.

Data, consisting of transcripts from interviews, field notes from observations, and course materials, were analyzed for the ways in which the calculus instructors teach for transfer, focusing on the five characteristics of learning and teaching that promote transfer. Collecting data from several different sources is one way I attempted to ensure internal validity, as it provided opportunities for triangulation of data (Merriam, 2009). My analysis began with an initial reading of the entire data set (Emerson, 1995), during which I asked questions of the data set (Emerson, 1995) (such as, "Where and how does the instructor emphasize understanding? And, where and how does the instructor emphasize computational procedures?"), as I initially open coded the data. Themes and patterns in the data were used to develop an analytic code list based on the five aspects of teaching that promote transfer, and focused coding of the data was undertaken (Emerson, 1995). In order to make assertions about answers to the research questions, this coded data was used to develop themes, make comparisons, and determine how the particulars of the data could be generalized (Miles & Huberman, 1994). Themes that appeared were often investigated further, and claims were generated. Then, data was searched for both confirming and disconfirming evidence of these claims (Miles & Huberman, 1994). Finally, these themes and general patterns, drawn from the interviews, observations, and course documents, were used to create hypotheses about how calculus professors teach for transfer, a few of which will now be described.

Results

When asked if they knew why other departments required calculus, all three calculus professors could quickly state applications of calculus in non-mathematics disciplines which require calculus of their students. All of the calculus professors in this study also articulated a justification for non-mathematics departments requiring calculus by drawing on their personal knowledge of these various disciplines, such as economics, chemistry, biology, and physics. One professor's explanation of this is typical of all three professors: “So, there's kind of the obvious idea that calculus tools are very good at describing and modeling change. So, any time things are changing, calculus may be useful.” The other two professors answered in a similar way, by giving examples from chemistry, biology, and physics, that were admittedly from their “personal contact” with others studying applied mathematics or simply from their personal knowledge of the discipline. Professors' conceptions of calculus, and ideas about what students should be learning in calculus, appear to come primarily from their own knowledge of calculus and personal experiences with non-mathematics disciplines.

Instructors of calculus at this SLAC are already teaching in some ways that might prepare students to transfer their calculus knowledge to future coursework. One such result will be highlighted here. During interviews, when outlining the knowledge and skills that they believe students should leave calculus courses knowing, each of the calculus professors mentioned both computational skills (e.g., being able to calculate the derivative of a polynomial) and some type of conceptual understanding, often described as deeper understanding, intuition, or fluency (e.g., knowing that the derivative is a rate of change, when/why that information might be useful). For example, one professor describes computational skill as follows: “...the mechanical skills are the symbol pushing. The, if I give you something, can you manipulate the symbols to get what I'm asking for?” While conceptual understanding gets at something deeper: “When I say intuition I mean a better sense about what is going on...it's trying to give them a framework, so that when they see something that can be expressed as a rate of change, that they understand that they can manipulate that to get more information.”

Each calculus professor articulated the desire for students to have conceptual understanding, noting something similar to, “I'm not particularly interested, as a college teacher, in training them to follow a memorized pattern. I want them to think about what's going on.” The professors also made attempts to foster such understanding during observed class sessions, by asking students to always justify their reasoning, suggesting that calculus students need more than computational skills to learn calculus successfully. Emphasis on conceptual understanding was particularly noted when professors gave students time to actively process the material being covered in class.

Despite the fact that conceptual understanding appears to be the ultimate goal for these calculus professors, each professor also acknowledged the foundational importance of computational skill. In fact, mechanical skill plays a large role in these calculus courses, and conceptual understanding may be just emerging for calculus students. Importantly, professors expressed difficulty in being able to assess whether students possess this conceptual understanding without having one-on-one conversations with students. One professor said, “I think that I give tests because I have to give grades, I don't think tests measure what people know. … So, I find out the most about what students know when they come in and sit down at that table right there and we talk about things that they are wrestling with.” It would be of benefit to further study how assessments might be adjusted to match what professors hope their students are taking away from calculus courses.

Areas of calculus teaching that could be improved to encourage promotion of transfer were also articulated during the study. Considering one example, calculus professors admit in interviews that they do not specifically focus on bringing in applications of calculus to non-mathematics disciplines, although course observations reveal that examples are widely used in these courses to illustrate calculus concepts and professors admit that applications could benefit student learning. One professor articulated this by saying, “I make small effort to bring in examples from more than just the areas that are in the textbook. I do know that seeing something applied in a discipline that they are actually interested in makes a difference for their understanding.” Additionally, one professor articulated how these examples are chosen so that students are able to understand both the specific variety of applications of a particular concept and the general abstract ideas underlying those concepts. Calculus textbooks include examples from other disciplines, and professors do make use of those built in applications at times, but do not go out of their way to find disciplinary applications outside of mathematics.

The professors articulated two reasons for not including more applications. Balancing the interests of students from a variety of majors: “So, what might be great for 5 of your 20 students, is worthless for the others.” And, a notable desire to maintain some type of a “traditional” calculus sequence: “I think, what are the primary things I want them to understand from the perspective of what mathematics *is*? And then, I certainly use examples to illustrate the utility of those concepts. But, to me, they are just examples.” Emphasizing calculus applications in other disciplines may prove to be a challenge worth further study in these foundational calculus courses.

Interview transcripts and field notes from classroom observations provide ample data to explore how the professors in this mathematics department currently teach calculus for transfer and how teaching might be altered to further promote such transfer, this is a small sample of the major findings.

Implications and Significance

Several opportunities for future research based on this study have already been noted. In particular, it could be beneficial to study calculus assessments and potential ways of improving assessment of students' conceptual calculus knowledge and to further explore whether calculus instructors might be able to increase student understanding by making more connections to disciplines outside of mathematics. This research also has direct benefits for the SLAC in this study as it can serve as the foundation for justifying restructure of the calculus sequence or emphasizing particular teaching practices. A limitation of this study is that it only focuses on mathematics professors' views of calculus teaching at the university in question. Justification for a restructuring of the calculus sequence to meet student needs could greatly benefit from both perspectives of professors in non-mathematics disciplines requiring calculus of their majors and student perspectives of the calculus sequence in relation to their studies.

SLACs also serving a variety of students with a single calculus sequence may benefit from gaining knowledge of some existing teaching practices that might promote the transfer of calculus knowledge. While personal accounts of teaching experiences exist, empirical research in undergraduate mathematics education is limited, and calls have been made for more research to illuminate college teaching practices (Speer et al., 2010). Making clear how calculus instructors design and teach calculus to ensure that students are acquiring the knowledge they need for success in their chosen majors will contribute to the growing field of undergraduate mathematics education research documenting undergraduate teaching practices.

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The Emergence of Algebraic Structure: Students Come to Understand Zero-Divisors
Contributed Research Report

Little is known about how students learn the basic ideas of ring theory. While the literature addressing student learning of group theory is certainly relevant, the concept of zero-divisor in particular is one for which group theory has no analog. In order to better understand how students come to understand zero-divisors, this talk will present results from a study that investigated how students can capitalize on their intuitive notions of solving equations to reinvent the definitions of ring, integral domain, and field. In particular, the emergence and progressive formalization of the concept of zero-divisor at various stages of the reinvention process will be detailed and discussed.

Keywords: abstract algebra, zero-divisors, guided reinvention, Realistic Mathematics Education

Introduction

Little is known about how students learn concepts that are specific to ring theory. In fact, nearly all of what is known about student learning of rings stems from conceptual ideas shared with groups (for example, binary operations and associativity). Ring-theoretic concepts for which group theory offers minimal insight include the distributive property, the allowance of elements without a multiplicative inverse, and zero-divisors. In particular, zero-divisors are likely to be completely foreign to students because, throughout school algebra and calculus, the real numbers (and, more generally, fields) are the realm within which most work is done. This concentration on fields, often considered "well-behaving" because of their commutative operations and closure under nonzero division, promotes a certain expectation or assumption that all structures should behave in this manner. How students come to understand a structure lacking one or several of the characteristic properties of the real numbers (or, in general, fields) remains a question the current literature is unable to definitely answer.

Literature. The literature provides two meaningful suggestions about how students might be able to identify zero-divisors on an informal level. First, Findell (2001) asked one of his student participants, Wendy, whether Z_6 is a group under multiplication modulo 6. Referring to the operation table she had constructed, Wendy noted that "I have tried every element, 0, 1, … 0 through 5, multiplied by 2 to see if I can get the identity, 1, and I can't get it. So therefore, Z_6 is not a group under multiplication" (p. 136). Indeed, this excerpt serves to explain why most studies of students learning group theory prove unable to provide much information about how they learn zero-divisors: a multiplicative structure with a zero-divisor can not be a group under multiplication. Nonetheless, Wendy's use of the operation table to identify 2 as a problematic element suggests that operation tables may be an effective means for detecting the presence and examining the consequences of zero-divisors in a finite ring.

Second, Simpson and Stehlikova's (2006) case study examined how a student, Molly, apprehended the commutative ring Z_{99}. In her guided explorations of the structure, Molly discovered zero-divisors by noticing that that, when solving multiplicative equations of the form $ax=b$, certain elements a caused the equation to have multiple solutions. These certain elements, she noticed, were those that shared common factors with 99. Molly's experience suggests that solving basic multiplicative equations could be an effective means of developing an initial concept image of zero-divisor.

In both cases, the student was able to identify these elements because of their peculiar and unfamiliar behavior in the context of familiar activities: examining operation tables and solving linear equations. These episodes suggest that operation tables and linear equations have the potential to be an effective means through which students can discover and address the consequences of zero-divisors in the greater ring structure. What remains to be documented in the literature, however, is how students might be able to capitalize on these intuitive notions in an effort to understand zero-divisors in a formal algebraic setting.

Purpose and scope. This paper refers to a study that investigated how students might be able to reinvent the definitions of ring, integral domain, and field (Cook, 2012). Specifically, the primary focus is to describe the emergence and evolution of the concept of zero-divisor throughout the reinvention process.

Theoretical Perspective

I adopted Realistic Mathematics Education (RME) as a theoretical perspective, which guided both the instructional design and the data analysis. Two RME heuristics were of critical significance to this study. First, the principle of *guided reinvention* (Freudenthal, 1991) served as the overarching objective. The reinvention principle seeks "to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999, p. 116). Secondly, the notion of an *emergent model* (Gravemeijer, 1998) was integral to the design of the instructional tasks and was used to identify milestones of the reinvention process. Emergent models are based upon the idea that students often develop informal procedures to solve contextual problems in a manner that anticipates and serves as a guide for more formal activity. In this way, the concept emerges as a *model-of* the students' informal mathematical activity, gradually transitioning into a *model-for* more formal mathematics. Gravemeijer (1998) delineated this transition into four phases:

1. The *situational* phase involves working to achieve mathematical goals in an experientially real context.
2. The *referential* phase includes models-of that refer to previous activity in the original task setting.
3. The *general* phase is characterized by models-for that support interpretations independent of the original task setting.
4. The *formal* phase entails student activity that reflects the emergence of a new mathematical reality.

Each phase simultaneously builds on previous activity while *anticipating* the more formal activity that follows. To accommodate the inherent complexity of the emergence of the model, I introduced three intermediate, anticipatory phases[1]:

- The *situational anticipating referential* phase involves activity still firmly rooted in the original situational setting that lays the groundwork for future referential activity.
- The *referential anticipating general* phase is characterized by models-of that provide an overview of previous work in preparation for abstract or general activity.
- The *general anticipating formal* phase includes models-for that encourage more efficient or concise use of the mathematics at hand in preparation for formal use.

Together, these seven phases served as a framework to identify and analyze the milestones of the reinvention process. For the purposes of instructional design, I viewed solving linear equations as an emergent model because of its potential to incite powerful informal understandings of zero-divisors and other intricacies of a ring structure.

[1] While it is not necessary to incorporate these additional phases, they provide a useful, organic means to more thoroughly describe the emergent model transition.

Methods

I employed a *developmental research* design (Gravemeijer, 1998), which was compatible with and followed from my theoretical perspective in that the primary goal is "the constitution of a domain specific instructional theory for realistic mathematics education" (p. 278). Following Gravemeijer's (1995) suggestion that the teaching experiment methodology is useful for such purposes, I adopted the guidelines of the *constructivist teaching experiment* (Cobb, 2000; Steffe & Thompson, 2000). In the constructivist teaching experiment, the researcher serves as the teacher and interacts with the students individually or in small groups (Cobb, 2000). This paper focuses on a teaching experiment I conducted with two students, consisting of six sessions of up to 120 minutes each. The participants, Haden (19) and Laura (18), were mathematics majors recruited from a discrete mathematics course at a large comprehensive research university. The students were both above-average (earning As in discrete mathematics) and had no direct prior exposure to abstract algebra. Each session was video-recorded. Additionally, the students' written work from each session was collected. In accordance with the techniques of multiple iterative analysis of video data (Lesh & Lehrer, 2000), I produced a detailed, analytical summary of each session by viewing each session multiple times, each time incorporating more detail and comprehensive identification of emerging themes.

In accordance with Zazkis' (1999) recommendation that "working with non-conventional structures helps students in constructing richer and more abstract schemas" (p. 651), the instructional tasks centered on solving equations and proving the cancellation laws on a variety of rings with different structural features: Z_{12}, Z_5, Z, Q, Z[X], and $M_2(Q)$. I then engaged the students in *proofs and refutations* (Larsen & Zandieh, 2007) as they attempted to prove versions of the additive and multiplicative cancellation laws for each of the six structures. Subsequent tasks encouraged the students to use their experience with equations and the cancellation laws to identify both common and distinguishing characteristics of each of these structures[2].

Results

In this section, I describe student activity with zero-divisors in each phase of the emergent model transition. Using the expanded version of Gravemeijer's emergent model transition as a theoretical lens, student activity with zero-divisors in each stage is detailed and explained. These results focus on the students' activity with Z_{12} because of its presence in most phases and prevalence in discussions (informal or otherwise) about zero-divisors.

Preliminary: Constructing and examining operation tables. Using clock arithmetic as an informal starting point for modular arithmetic, Haden and Laura constructed the multiplication table for Z_{12}:

•	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	16	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

[2] More information about the specific instructional tasks and the corresponding rationale for their inclusion is available in Cook (2012).

After noticing that 12 is the zero element in Z_{12}, a discussion ensued about the factors of 12:

> Inst.: I heard you guys say something about how factors of 12 are different from the other numbers. Which ones specifically are those and what is different about them?
> Laura: If there is a regularity, if it repeats itself, then it is a factor of 12? 2, 3, 4, 6.
> Haden: Or it contains factors of 12.

Thus, zero-divisors were evident in the repetition for certain elements in the multiplication table. In contrast, Haden and Laura correctly remarked about the lack of such repetition in the multiplication table for Z_5.

Situational: Solving specific linear equations on Z_{12} and Z_5. Just as Molly in Simpson and Stehlikova's (2006) case study, Haden and Laura quickly identified the elements with repetition in the table as elements that cause multiple solutions. Furthermore, when attempting to algebraically solve the equations $9x=6$ and $3x=12$ on Z_{12}, the students made precisely the same observation (and use nearly the same phrasing) as Findell's (2001) student:

> Laura: What about $3x=12$?
> Haden: $3x$ is not going to be as easy as … there is nothing that makes it be 1.
> Laura: Could we define some multiplication that makes it 1?
> Haden: What do you mean?
> Laura: [Now talking about $9x=6$.] Like nothing times 9 … I'm trying to figure out how to get it to be 1 somehow.

Thus, in this phase, the students demonstrated their initial conceptions of zero-divisors as elements that cause multiple solutions to $ax=b$ and, consequently, do not have an inverse.

Situational anticipating referential: Proving the cancellation laws. Now somewhat familiar with the innerworkings of Z_{12} as a result of solving equations, I prompted Haden and Laura to prove the multiplicative cancellation law [$ax=ab$ (a nonzero) $\Rightarrow$ $x=b$] on Z_{12}. They noticed immediately that it did not hold:

> Haden: If [the multiplicative inverse axiom] is true, it forces [cancellation] to be true.
> Laura: So that's false.
> Haden: Not only is it not provable, it's actually false.

I then engaged Haden and Laura in the process of *proofs and refutation* (Larsen & Zandieh, 2007) in which I asked them if the cancellation law held for certain, if not all, values of a. This sparked a discussion that culminated in the following proof:

$ax = ab$
$\frac{1}{a}\cdot(a\cdot x) = \frac{1}{a}(a\cdot b)$ multiplicative inverse
$(\frac{1}{a}\cdot a)\cdot x = (\frac{1}{a}\cdot a)\cdot b$ associative property
$1\cdot x = 1\cdot b$ definition of $\frac{1}{a}$
$\therefore x = b$ $\square$ identity property

this is false in general, but if it is true for $a \in S \subset \mathbb{Z}_{12}$, $S=\{1,5,7,11\}$

$a=1 \Leftrightarrow \frac{1}{a}=1$ $\quad a=7 \Leftrightarrow \frac{1}{a}=7$
$a=5 \Leftrightarrow \frac{1}{a}=5$ $\quad a=11 \Leftrightarrow \frac{1}{a}=11$

This proof highlights the students' concept image of zero-divisors as elements that create problems when attempting to cancel. Alternatively, it highlights the units of a ring as exactly the opposite: the elements that support cancellation. This parallel between zero-divisors and units continued throughout the study.

Referential and referential anticipating general: Summarizing the cancellation laws and sorting the structures. So that students would begin to both identify common and distinguishing features amongst the example structures, I encouraged Haden and Laura to summarize their different methods of proving the cancellation laws. Reflecting on their previous activity, they noted that the multiplicative cancellation law could, for certain values of *a*, be proved using multiplicative inverses or the zero-product property. Unsurprisingly, zero-divisors proved to be a characteristic of interest, as evidenced by Haden commenting that "determinant A not being zero is the analog of the coefficient *a* not being or containing a factor of 12 in Z_{12}." As a result, Haden and Laura "grouped" Z_{12} with $M_2(Q)$ on the basis that both contained elements that caused problems with cancellation (the term "zero-divisor" had not yet been introduced). The other groupings included Q with Z_5 (on the basis of multiplicative inverses) and Z with Z[x] (on the basis of a lack of both multiplicative inverses and zero-divisors).

General and general anticipating formal: The process of defining. Whereas the *presence* of zero-divisors served as a discriminating feature in the previous phases, the process of defining saw the *absence* of zero-divisors take a prominent role. I introduced the term "zero-product property" once they had identified that the lack of zero-divisors was a critical feature in the multiplicative structure of a ring. To begin the process of defining, I prompted the students to list all of the rules and properties they had deemed important while solving equations and use this to characterize each set of sorted structures. The students, in turn, having already recognized its value, included it in their definition of what would eventually become the definition of an integral domain, while excluding it from their preliminary definition of ring with identity[3].

A ring with identity is a set R...
with binary operations $+: R^2 \rightarrow R$ and $\cdot : R^2 \rightarrow R$
such that
1. ADDITIVE INVERSE $\forall a\ \exists(-a);\ a+(-a)=0$
2. ASSOCIATIVITY OF ADDITION $\forall a,b,c\quad (a+b)+c = a+(b+c)$
3. COMMUTATIVITY OF ADDITION $\forall a,b\quad a+b=b+a$
4. ADDITIVE IDENTITY $\exists 0 \in R; \forall a\quad a+0=a$
5. DISTRIBUTIVITY $\forall a,b,c\quad a\cdot b + a\cdot c = a\cdot(b+c)$, $b\cdot a + c\cdot a = (b+c)\cdot a$
6. MULTIPLICATIVE IDENTITY $\exists 1 \in R; \forall a\quad a\cdot 1 = a = 1\cdot a$
7. ASSOCIATIVITY OF MULTIPLICATION $\forall a,b,c\quad (a\cdot b)\cdot c = a\cdot(b\cdot c)$

An integral domain is a ring with identity R such that
8. COMMUTATIVITY OF MULTIPLICATION: $\forall a \forall b\quad a\cdot b = b\cdot a$
9. ZERO PRODUCT PROPERTY: $a\cdot b = 0 \Rightarrow a=0 \vee b=0$

This established that the students were able to capitalize on their informal understandings of zero-divisors for use in a more formal setting. What remained was using these definitions and ideas in a more formal setting.

Formal: Using the reinvented definitions to apprehend new structures. One of the first example structures I asked Haden and Laura to classify was the direct product $Z_3 \times Z_3$. Almost immediately, they identified that not every element had a multiplicative inverse and that zero-divisors existed:

[3] For conciseness, the preliminary definitions in the actual defining process have been omitted. For more details, see Cook (2012).

multiplicative inverses
(1,1) (2,1)
(1,2) (2,2)

$(0,1)\cdot(1,0)=(0,0)$

In this way, zero-divisors and units had come to the fore in the students' minds as a means of characterizing algebraic structure. After attending to the remaining axioms, they concluded that $Z_3 \times Z_3$ is, at most, a (commutative) ring with identity.

I then prompted Haden and Laura to generate their own examples of structures closed under addition and multiplication. Their list included Q[X], R, N, and Z_n for *n* prime and composite. The following conversation occurred when attempting to classify Z_n:

Haden: Z_n composite would be rings.
Laura: Yeah, not an integral domain.
Inst.: Why not an integral domain?
Laura: Because it doesn't have … what property was it? Commutativity or zero-product?
Haden: It's multiplicative inverse that's not defined on all of them. Zero-product property is what we proved from that.

Again, the contrasting relationship between units and zero-divisors (manifested in this episode as discerning the zero-product property) emerged as a tool by which the students could apprehend new structures.

Conclusions and Discussion

The results provide an empirically-established trajectory for the concept of zero-divisor throughout the reinvention process. Additionally, the concept of units often arose simultaneously and in contrast to that of zero-divisors, admitting a noticeable dichotomy in the development of these two intrinsically related, yet radically different, concepts:

Emergent Model Phase	Role of Zero-divisors	Role of Units
Preliminary: operation tables for finite rings	The elements whose rows and columns have repeating patterns	The elements whose rows and columns do not repeat
Situational: solving linear equations	The elements *a* for which $ax=b$ has multiple solutions; division to isolate x is not possible	The elements *a* for which $ax=b$ has a unique solution; division to isolate x is possible
Situational anticipating referential: proving the cancellation laws	The elements that disrupt multiplicative cancellation	The elements that support multiplicative cancellation
Referential: summarizing the methods used to prove the cancellation laws	Emerge as a means to differentiate structures for which the zero-product property holds always, sometimes, or never	Emerge as a means to differentiate structures on which division is possible always, sometimes, or never
Referential anticipating general: sorting structures		
General: process of defining	Zero-product property is	Division (multiplicative inverse) is

General anticipating formal: writing nested definitions	defining characteristic of integral domains	defining characteristic of fields
Formal: using reinvented definitions	Used to determine if new structure is an integral domain	Used to determine if new structure is a field

Originating at a level as intuitive as a repetitive pattern in a multiplication table and transitioning into a means by which structure can be characterized, the identifications set forth in the above tables provide a functional conceptual framework for the development of the concepts of zero-divisor and (as a result) multiplicative inverse. This framework adds to the knowledge about how students learn zero-divisors by building upon suggestions in the research literature in order to document the complete evolution of the concept from an informal, intuitive level to a more formal, rigorous level.

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Coherence from Calculus to Differential Equations

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Introduction Recent reform movements, including the Mathematical Association of America's Curriculum Foundations Project, have sought to make calculus courses more coherent with other fields (Hughes Hallet, 2000). Despite this and other interdisciplinary reform efforts, post-calculus instructors still express extreme dissatisfaction with their students' mathematical preparation (Ferguson, 2012). We do not downplay the importance of the lamentations of other disciplines, but instead suggest that there also exist coherence issues within mathematics. Our hypothesis is that there is a mismatch between how students are expected to know calculus at the end of a course and how students are required to use calculus in future courses. Specifically, we examine calculus knowledge for differential equations.

Background Learning differential equations (DEs) is difficult and that difficulty is compounded by the multitude of other areas also needed, such as science, lifelike situations, and complex mathematics. DEs classrooms have benefitted from successful school mathematics approaches applied in a college classroom context (e.g., O. Kwon, Allen, & Rasmussen, 2005; Rasmussen & O. N. Kwon, 2007; Czocher & Baker, 2011). Past research has shown that there are foundational ideas from pre-calculus and calculus that are vital to success in a DEs course. Such ideas include: equation and solution (Raychaudhuri, 2008), mathematical differences among quantity, rate and rate of change (Rowland & Jovanoski, 2004; Rowland, 2006). Despite the existence of such foundational ideas, their coherence from calculus to DEs has not been explored.

Beginning from the calculus reform movements of the 1980s, calculus has been studied in many different contexts from many different research angles. Calculus instruction has benefited from taking advantage of contemporary theories specific to mathematical thinking. Many aspects of the calculus reform the emphasis on conceptual understanding and multiple representations, together with increased use of technology since the early 1990's are so embedded in calculus practice that they are considered mainstream (Ferrini-Mundy & Gucler, 2009). We suspect that these instructional advances cannot be utilized to their fullest without examining how the calculus must be known in future courses and in our case, in DEs.

Our purpose in this research was to examine how expected understandings of calculus topics align with the expected understandings of these topics in later courses. fThat is we examine if and how calculus and DEs align epistemically.

Research Frameworks In this work, we adopted the Calculus Content Framework (CCF) to examine important calculus concepts and skills (Sofronas et al., 2011). The CCF was built to organize goals from the first year of calculus into four strands: the first strand is sufficient for our purposes as it covers mastery of fundamental concepts and skills. The concepts are: derivative, integral, limits, approximation, sequences and series, Riemann sums, parametric and polar equations, continuity, and optimization. The skills are: derivative computations, manipulating algebraic expressions, area and volume, parametric equations, polar coordinates, trigonometric manipulations, facility with logarithms and exponentials, epsilon-delta proofs, listening and reading comprehension, facility with definitions and notation.

To meet our theoretical needs, we draw from parts of APOS theory (Dubinsky & McDonald, 2001), conceptual analysis (Thompson, 2002; Thompson, 2008), and didactical phenomenology (Freudenthal, 1983). Though APOS theory decomposes a concept into four stages (Action-Process-Object-Schema), it does not account for multiple concepts being utilized within the same mathematical setting. Didactical phenomenology and conceptual analysis suggest how a learner might sequence a mathematical idea, but is not embedded in a mathematical setting that a student might encounter. We required a theoretical perspective that could account for connections among concepts and that could be situated within the kinds of mathematical settings that are described in syllabi.

Combining pieces of all these frameworks, we created a new technique of analysis we termed *mathematics-in-use* which builds on Freudenthal's 1983 position that mathematical objects are created as an organizational scheme for mathematical phenomena. Our technique examined, through reflective reading of the texts and paradigmatic exercises, conceptual mathematical prerequisites, how these multiple mathematical concepts come together, how they are used, and how understandings of them may shift in order to structure a mathematical setting (task, example). Mathematics-in-use is an in-depth examination of how mathematical objects (concepts, ideas) must be interpreted for the problem solver to use them within the context of an example or task. We explore applying mathematics-in-use to different aspects including calculus concepts as a whole and in worked exercises. We believe that the best way to share our intentions for mathematics-in-use as a method of analysis is through an example, however that is beyond the scope of this current paper. We will focus on the calculus concepts in this paper, but a detailed examination of a worked exercise is in press (Czocher, Tague, & Baker, in press).

Methods In working with the above frameworks, we needed to use a multistage approach. Ferguson (2012) used a similar approach to examine how other disciplines required their students to know calculus. First, identify the topics and skills vital to DEs. Second, clarify the calculus topics and skills needed for successful understanding of the DE topics and skills. Third, apply our technique of mathematics-in-use to reflect on the process of solving exercises.

Before we describe our methods, we note that the differential equations course we draw from in this study is specifically for scientist and engineers. Since approximately 95% of the students in first-year mathematics courses are not majoring in mathematics (Ganter & Barker, 2004), our choice is most certainly warranted.

In the first stage of our multistage approach, we examined two textbooks currently in use at The Ohio State University (OSU). The textbooks are described in greater detail in some of our previous work (Czocher & Baker, 2011), but we offer some brief descriptions here. *Elementary Differential Equations and Boundary Value Problems* is a common differential equations textbook within the United States (Boyce & DiPrima, 2009). The overall emphasis of the book is on analytic techniques and it is organized by solution type. *An Introduction to Differential Equations for Scientists and Engineers* was written specifically to meet the needs of scientists and engineering majors at OSU (Baker, 2012). The organization of the book centers around exemplary problems from science and engineering and approaches those problems through mathematical modeling. Linear algebra is not a prerequisite for DEs at OSU, and so neither book requires prior linear algebra knowledge. However, both books assume previous knowledge of multivariate calculus as well as sequences and series. We examined the textbook table of contents, homework assignments, course objectives, and syllabi. Lastly, we interviewed five engineering faculty members who were teaching courses that listed DEs as a prerequisite, two mathematics faculty members, and five teaching assistants assigned to a differential equations course.

In the second stage, we used the CCF (Sofronas et al., 2011) to identify the calculus topics and skills necessary to explain the DE topics and skills. Some additions to the framework were necessary, for example, the derivative computations topic needed to include partial derivative computations. We also added the fundamental theorem of calculus topic as it is vital to differential equations in explaining the relationships between derivative and integral.

Figure 1 offers a pictorial view of our decomposition results. It is a matrix with DE topics displayed horizontally across the top and calculus topics (via the CCF) vertically down the left side. The top panel of the left side includes calculus concepts and the bottom panel includes calculus skills. Where dots are present indicates that a particular calculus concept or skill is necessary for the above differential equation concept. For example, in the column labeled separable equations, the calculus concepts integral and fundamental theorem as well as the calculus skills integration techniques, algebraic expressions, trig manipulations, and logarithms and exponentials are necessary.

Using mathematics-in-use, our decomposition shows a set of epistemological mismatches between calculus and differential equations. These mismatches are examined more fully in the next section, summarized in Table 1, and an extended example is available in our submitted manuscript (see Czocher, Tague, & Baker, in press).

Content Analysis The matrix from Figure 1 provides a visual representation of the ways that differential equations content relies on calculus concepts and skills. Horizontally, across the top, are the DE topics and vertically down the left side are the calculus concepts and skills identified from the CCF (see Sofronas et al., 2011) with our necessary additions. If a dot appears in box i,j, then it signals that the calculus content in row i is necessary for the differential equations topic in column j.

Similar to Ferguson's 2012 study of calculus, the suggestions of faculty members for topics covered in differential equations were dependent on whether the individual was a teacher of mathematics or a user of the mathematics. Thus, only one engineering faculty member chose existence/uniqueness as crucial, whereas all mathematics faculty members listed it as vital. In cases such as these where there was consensus among an entire group, we included the topic. Some professors also suggested Laplace transforms as vital, however, this topic is not included in the introductory differential equations course for engineers and scientists, and as such it was not included in our content analysis.

It is evident from a quick glance at Figure 1 that there are far more marks in the bottom panel than in the top panel, leading to the conclusion that DEs is a skill-based collection of topics. Note also, that some calculus concepts and skills are not central to any DE topic. For example in calculus *concepts*, approximation, Riemann sums, continuity, and optimization are absent and in *skills*, polar coordinates and epsilon-delta proofs are also not used. We caution the assumption that because they are not explicitly listed as necessary that they are not utilized anywhere in DEs. Derivations of important equations in DEs provide exemplary reasons for this caution as they are supported through the concepts of approximation, continuity, and optimization.

Figure 1. Decomposition of differential equations topics relative to Calculus Content Framework (Sofronas et al., 2011)

	topics	common topics covered by both textbooks																							
		classification	FOL, variable coeffs	separable eqns	applications with FO	linear vs. nonlinear	autonomous eqns	popluation dynamics	exact eqns	integrating factor	existence / uniqueness	SO linear	fundamental solns	wronskian	complex/ repeated roots	undetermined coeffs	applications (vibrations)	higher order eqn	two-point BVP's	PDE's	Fourier series	separation of variables	heat, wave, laplace eqns	phase planes	systems of linear eqns
calculus concepts	derivative				●			●	●			●				●	●			●		●	●	●	●
	integral			●	●				●	●		●		●		●					●				●
	<u>fundamental thm</u>			●						●															
	limit																●				●		●	●	
	approximation																								
	sequences & series																			●	●		●		
	riemann sums																								
	paramteric & polar eqns																								●
	continuity																								
	optimization																								
calculus skills	derivative computations				●		●	●				●				●	●		●	●	●		●	●	
	integration techniques		●	●	●		●	●	●	●		●		●			●		●	●	●		●		
	limit calculations				●			●									●								
	sequences/series																				●	●	●		
	algebraic expressions		●	●	●				●			●	●	●	●	●	●		●	●	●	●	●	●	
	area/volume				●																				
	parametric eqns																								●
	polar coord.																								
	trig manipulations		●	●	●		●		●			●			●		●		●	●		●	●		●
	logarithms and exponentials		●	●	●				●	●		●		●	●	●			●	●			●		●
	epislon-delta																								
	listening & reading comprehension							●			●														
	definitions & notations	●				●					●							●							●

Epistemological Mismatches Through our mathematics-in-use analysis technique, we uncovered epistemological mismatches within these calculus concepts: derivative, integral, fundamental theorem of calculus, limit, sequences and series, parametric and polar equations, and continuity. Table 1 provides a summary of each of these mismatches and then elaboration on each concept follows. We draw attention to the fact that despite some concepts having no visible mismatches, vital concepts, such as derivative, are vastly mismatched.

Derivative Concept Throughout calculus, the concept of derivative is used to produce a numerical value. Students are taught computational methods for taking the derivative of a function and finding the derivative at a point for a given function. In most cases, solutions to derivative exercises are numerical values. In DEs, the derivative is used as an operator, specifically as the inverse to the integral operator.

Integral Concept During calculus courses, the concept of integral appears throughout the course. For example, it is embedded in: the area under a curve, the area between two curves; Riemann sums and summations; improper integrals definite and indefinite anti-derivatives; solids of revolution; work-energy relations; accumulation; measurement of attributes (e.g., arc length, surface area, volume, center of mass); fundamental theorem of calculus. Many times, computation is emphasized over application of the concept of integral. In contrast integrals in DEs are viewed and used mainly as the inverse operation of derivative to produce

Table 1: Comparison of uses of calculus ideas in calculus and differential equations

	Calculus	Differential Equations
derivative	computation; rate of change	rate of change; algebraic object; invertible operation
integral	anti-derivative; measurement of area, volume, accumulation	technique for solving differential equations
fundamental theorem of calculus	formal justification for using anti-derivatives instead of definite integrals; short cut for computing certain derivatives and definite integrals	creates free parameter for satisfying initial/boundary conditions
limit	algebraic computation; exposure to formal (i.e., ϵ-δ) definitions; basis for derivative definition	relates discrete and continuous models of a situation
sequences & series	using theorems to check for convergence; writing common functions in polynomial bases	means for conveniently representing functions
parametric & polar equations	alternative representations of functions	introducing auxiliary equations; tool for reducing order; tool for representing solutions graphically
continuity	property to be checked	mathematical tool for ensuring desirable mathematical properties

a family of functions. They are used also to compute coefficients of Fourier series. The overall epistemological mismatch is that while in calculus, students expect to obtain numerical values when using integrals while in DEs, it is vital as an invertible operator.

Fundamental Theorem of Calculus Concept The fundamental theorem of calculus provides the relationship between the definite integral and anti-differentiation in calculus courses. It also provides a simplification of computation of definite integrals. While the fundamental theorem of calculus is vital to DEs, it is not explicitly visible because differential equations themselves are collections of derivatives in combination with algebraic procedures that form conditions which are satisfied by a family of functions. It can also be hidden because there are very few cases when definite integrals are expressed in the form of the theorem statement. On the contrary, fixing initial or boundary conditions constrain the choice of a particular anti-derivative. Lastly, the fundamental theorem of calculus is hidden when using methods such as undetermined coefficients or separation of variables. When it is visible, it underpins the integral as the inverse operator of the derivative. Part of the reason why this topic is a mismatch might be due to the fact that derivatives and integrals are not seen as operators in calculus.

Limit Concept In calculus, limits are primarily algebraic computations that replace the brief introduction to epsilon-delta proofs. It forms a basis for the definition of derivative. In DEs the limit concept is used to derive differential equations and to discuss long-term behavior and stability. Whereas in calculus, the limit is treated as a computational exercise, in DEs, it is utilized in reasoning through behaviors of functions.

Sequences & Series Concept In calculus, sequences and series are given as topics in a vacuum. Most of the exercises required in calculus textbooks include reading and applying theorems to determine limits or checking for convergence. Occasionally, students are expected to use series to compute approximations of numbers, such as e or $\frac{pi^2}{6}$. MacLauren, Taylor, and power series are introduced to provide (possibly infinite) sets of monomials for approximating values of various functions, such as e^x, at or near a point of interest. However, rather than focusing on the new representation being a way of defining complex functions in terms of simpler functions, students in calculus are required only to compute higher order derivatives to construct the power series.

The emphasis is inverted in DEs. Power series are called upon regularly for their utility in representations of functions. Using a technique where one assumes the solution function can be represented as a convergent power series, the monomial coefficients can be determined through recursion relations. Fourier series construction is another place where series are used to transform ordinary and partial DEs. To summarize, in DEs, series provide new ways to construct and represent functions while in calculus, sequences provide an introduction to partial sums, which are then used to approximate specific values of functions.

Parametric & Polar Equations Concept Parametric and polar equations are used within calculus to provide an introduction to functions in alternate coordinate systems. Generally, exercises request translation between Cartesian, parametric, and polar systems or computing integrals in these systems. Because parametric equations are so closely related to vector-valued functions in DEs, they are used to convert a higher-order differential equation to a system of first order equations. This process allows the study of evolution of these systems numerically or graphically. Parameterizations are also used in dynamical systems approaches to represent solutions in graphs involving t as the input variable and graphs with the unknown function as the variable, requiring complex shifts in reasoning.

Continuity Concept During calculus courses, continuity is treated as a property to be checked using the vertical line test or limit checking. In DEs, continuity is used as a tool for ensuring desired mathematical properties.

Common Themes The previous subsections have provided a glimpse of the epistemological mismatches between calculus and DEs. One major theme across all examples is that in calculus, functions are treated as actions that produce output numbers given input numbers, whereas in DEs, functions must be viewed as objects. This distinction includes derivatives and integrals.

Interpretation and Concluding Remarks Using the CCF decomposition of DEs content into calculus topics, we reveal what calculus topics must be known, but not how they must be known. Our contributions are toward strengthening the CCF in its practical application of relating calculus-dependent topics to calculus coursework as well as utilizing the framework for exploring conceptual coherence across curricula.

Our results provide further evidence that a large amount of DEs requires proficiency in calculus concepts and skills. Adding to the decomposition with the mathematics-in-use technique showed several major concepts where epistemological mismatch occurs.

Ferguson's (2012) work revealed that there are epistemological mismatches between end-of-calculus knowledge and the following courses from other disciplines (Ferguson, 2012). For example, there were mismatches between what teachers of calculus wanted their students to know versus what users of calculus, such as calculus-based physics course instructors, wanted their students to know. Our results confirm her findings, and like Raman's (2002, 2004) findings of epistemological mismatches between pre-calculus, calculus, and analysis, they show conclusively that even within mathematics, instructors need a new approach to aid in mathematical coherence of their courses.

We chose to use DEs as an example setting, but there is a growing body of evidence in the literature that our expectations for calculus knowledge are out of line with our expectations for post-calculus courses, regardless of whether the students are following a mathematics major track or not. Thus, in order to address the epistemological incoherence we must explicitly focus on how we expect our students to know the content that the community decides is important. One way to do this through thorough evaluation of the phenomenology and the mathematics-in-use of the mathematical concepts, objects, tasks, examples, and exposition that we show our students. Our major suggestion would be to encourage coherence along phenomenological arcs from pre-calculus, to calculus, to post calculus, which would require cooperation among mathematics instructors and work to identify the "big ideas" and the concept-eliciting tasks that can codify them. Calculus is a multi-purpose course that is intended to serve many disciplines and to support many topics in mathematics. For it to be a functional course, we need to examine its content from many views, both post- and pre-calculus, and in our opinion, from the perspective of epistemological coherence.

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USING METAPHORS TO SUPPORT STUDENTS' ABILITY TO REASON ABOUT LOGIC

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In this paper, we describe an inquiry-oriented method of using metaphors to support students' development of conventional logical reasoning in advanced mathematics. Our model of instruction was developed to describe commonalities observed in the practice of two inquiry-oriented real analysis instructors. We present the model via a general thought experiment and one representative case study of a students' metaphorical reasoning. Part of the success of the instructional method relates to its ability to help students reason about, assess, and communicate about the logical structure of mathematical activity. In the case presented, this entailed a students' shift from using properties to describe examples to using examples to relate various properties. The metaphor thus imbued key example sequences with meta-theoretical significance. We introduce the term "wedge" to describe such examples that distinguish oft-conflated properties. We also present our analytical criteria for empirically verifying the specific influence of the metaphorical aspect of instruction.

Key words: Reasoning about logic, metaphor, real analysis, example use

Introduction

Philosophers classify logic as a form of meta-language because it describes the structure of more general forms of linguistic expression. It seems consistent then that reasoning specifically about logic is a form of metacognition. While the logical form of peoples' reasoning has been studied extensively, few studies carefully distinguish whether research subjects engage in linguistic analysis or metacognition, what we call *reasoning about logic*. Formalized logic, by definition, ignores the semantic content of statements or is at least generalizable beyond particular semantic content. Careful analysis of many previous studies on logical reasoning reveals that research subjects often reason about the semantic contents of statements or tasks, meaning the researcher may only observe the *logic of reasoning*. Since mathematics education studies show that students struggle with key logical structures in advanced mathematics like quantification (Epp, 2003, 2009; Roh & Lee, 2011), it seems important that students learn both standardized logical conventions and how to notice and assess the logical validity of their own mathematical reasoning. In other words, advanced mathematics students must develop tools for *reasoning about logic* and for transferring generalizable logical tools (ignorant of particular context) into their current context of reasoning (like sequence properties). This study addresses the following research questions:

1. How can mathematics instruction foster students' ability to reason about logical structure in mathematical activity and their adherence to logically valid forms?
2. How do metaphors for logical structure influence students' reasoning about particular examples in the context of real analysis?

The first question relates to our broader ongoing investigation of logic instruction in advanced mathematical courses. We present our findings from observations of two inquiry-oriented real analysis faculty via a thought experiment that models the common instructional method we observed in their classrooms. This method employed metaphors or stories to represent logical structure such that students could examine, assess, and communicate about such structure itself (*reason about logic*). The second question relates to the particular case we feature in this paper in which the professor used a metaphor to model and discuss the roles two key examples played in the class' theory of sequence properties.

Theoretical Backdrop

Our theoretical backdrop is inherently constructivist, more specifically aligned with the perspective called "radical" constructivism (von Glasersfeld, 1995). Learning should be characterized as students organizing their own mental and physical activity into schemes that support goal-oriented activity. The "transfer" or "application" of knowledge should not be viewed as use of an abstract tool for a specific situation, but rather treating a new situation as the "same" as a previously encountered one such that it is assimilated into the prior scheme of activity (ibid, 1995). Our characterization of "transfer" is consonant with Lobato and Siebert's (2002) notion of actor-oriented transfer and with Wagner's (2006) notion of transfer-in-pieces. We frame our questions of logic learning in terms of transfer to address the apparent paradox of how students can both reason about mathematics (semantic content) and reason about logical structure (ignorant of content). It should be pointed out that what we refer to as metaphors is often described in psychological literature as analogies. The term "metaphor" (treating one thing as if it were another) is used because it more closely matches our understanding of transfer in terms of assimilation. Our paradigm for metaphorical reasoning however closely parallels Holyoak's (2005) characterization of analogical reasoning.

The case study featured in this paper concerns the use of examples in proof-oriented mathematics, which has been extensively studied. Such examples can be used to refute a conjecture as a counterexample (e.g., Zazkis & Chernoff, 2008), to verify a conjecture or to understand a proof (Harel, 2001; Inglis, Mejia-Ramos, & Simpson, 2007; Weber, 2008; Weber, Porter, & Housman, 2008), to gain understanding of definitions (e.g., Alcock & Weber, 2008), and to elaborate concept images (e.g., Vinner, 1991). Whereas the example-related literature mentioned here often focused on examples students or mathematicians generate, our emphasis in this paper is on how professors can use metaphorical reasoning to guide their students' use of examples for formal mathematical activity.

Students' Reasoning about Logic versus the Logic of Students' Reasoning

Constructivism casts learning as organization of experience and action into schema that regulate human activity, rather than the acquisition of direct knowledge of external systems and phenomena. Research on students' logical activity indicates that their untrained reasoning (Evans, 2005) is not always consonant with formalized logic, but grows more sophisticated over time (Inhelder & Piaget, 1958). Logic then must be understood as an organization of human reasoning. Research upon logic must be careful not to deem reasoning as "illogical" when it differs from the formalized system of logic agreed upon by the mathematical or philosophical communities. Noting other researchers inherently sought to make "thought the mirror of logic," Piaget (1950) suggested, "simply to reverse the terms and make logic the mirror of thought, which would restore to the latter its constructive independence" (p. 30).

Students might be taught formalized logical systems to help them organize and systematize their naïve reasoning. In line with current research on "transfer" (Lave, 1988; Lobato & Siebert, 2002; Wagner, 2006) however, exposure to abstract systems does not guarantee formalized reasoning, especially because formal logic ignores semantic content. As Wagner (ibid.) said, "Abstraction [is] a *consequence* of transfer and the growth of understanding—not the cause of it" (p. 66). Thus in the paper, we investigate how students' learning about the logic of their reasoning "transfers" to their ongoing mathematical activity.

Description of the Study and Data Analysis

We embarked upon joint analysis of two real analysis classes in order to identify shared and effective tools or methods of instruction and learning that emerged in these classrooms. Both classes were inquiry-oriented in the sense that definitions, theorems, and proofs were treated as something to be constructed from intuitive meanings rather than pre-existing knowledge to be presented and internalized. Students were expected to play an active role in learning the materials by raising conjectures, justifying their own arguments, and debating contrasting claims within their discourse communities. Comparison of the two courses revealed that neither course dedicated time to "teaching logic" in isolation, but rather used a more localized or integrated approach to fostering logic learning. In line with the inquiry-oriented approach, the way the professors addressed logic guided students to explicitly examine logical structure such that they could reflect and guide their own logical activity.

Our data analysis followed a "grounded" approach (Strauss & Corbin, 1998) not in the fullest sense of that coding scheme, but in the sense of (1) developing a localized theoretical account that characterized the instruction and (2) testing such accounts by a constant comparative methods. We intend this model to appropriately characterize the instructional approach across different mathematical topics, logical structures, metaphor types (Dawkins, 2009), and the differing structures of the two inquiry-oriented classrooms (Dawkins & Roh, 2011). From our observations of teaching and learning episodes across the two classrooms, we abstracted a general model of teaching logical structure through metaphor or story. While two professors' practice does not constitute a large sample size, the model was vetted against a larger number of instructional episodes throughout the two classes (and multiple semesters). Due to space limitations, we present the general model in terms of a "thought experiment" in the sense of Freudenthal (1973, 1991) and a case study of one students' metaphorical reasoning about the logical (or meta-theoretical) structure of real analysis content.

The nature of the instructional practice we want to study poses a great methodological challenge because it is localized in the sense that logic was not taught as a topic, but rather as auxiliary to real analysis topics. Thus while the professors employed these tools throughout the semester, it is hard to separate the influence of that aspect of instruction from the range of other ongoing instructional activities related to the same mathematical topics. To trace the pedagogical influence of the metaphors, we must observe students' spontaneous and clear use of the metaphors for their mathematical activity. To avoid merely anecdotal evidence despite the methodological challenges, we sought "critical events" (Maher & Martino, 1996) from the two classes that satisfied the following criteria:

- C1. The students must spontaneously engage in metaphorical reasoning about logical structure or make a spontaneous mathematical (re-)discovery via metaphorical reasoning.
- C2. The students must elaborate the metaphor so as to influence their perception of the mathematical situation beyond simply using metaphorical "language."
- C3. The students must show evidence that they are reasoning about logical or meta-theoretical structure.

We shall justify how our featured case study satisfies each of these analytical criteria.

Results

Thought Experiment

To avoid what Wagner (2006) called "transfer by abstraction," students must somehow be led to explicitly examine the logic of their own mathematical activity. We claim both of the professors we studied successfully supported students' ability to reason about logic by the following instructional method. The instructor provided students with a metaphor that embeds the logical and/or quantitative structure within a quasi-real world context. The class explored the metaphor in conjunction with mathematical activity to induce a metaphorical mapping between the elements within the mathematical and metaphorical contexts. The context or story must be carefully created to display and motivate the logical structures within some frame of reference other than linguistic logic (such as deontic or rule-based reasoning does in the Wason, 1968, card task as discussed in Evans, 2005). That is to say the metaphor must have resonance, meaning it allows deep elaboration of the connections between the two domains (Black, 1962, 1977 as discussed in Oehrtman 2003, 2009). This distinguishes such instruction from that which Dubinksy & Yiparaki (2000) criticized where a mathematical statement's logic is simply compared to the logic of an everyday statement (which their research shows to be unreliable as a source of appropriate logical structure).

The metaphor then establishes a logical schema into which novel mathematical tasks may be assimilated. On such tasks, students assimilate the interrelationships and structure of a task into the schema of the metaphorical context, matching the known conditions that maximize transfer between superficially dissimilar tasks (Kimball & Holyoak, 2000). Figure 1 displays this form of instruction's intended avenue for students' logical transfer. In line with Wagner (2006), transfer occurs when a student has "constructed a framework of knowledge that was sufficiently complex [rather than abstract] to permit her to structure the two situations similarly" (p. 64). This differs from teaching formalized logics because it assumes students will see logical structure *within* the mathematical context rather than *abstracted from* it.

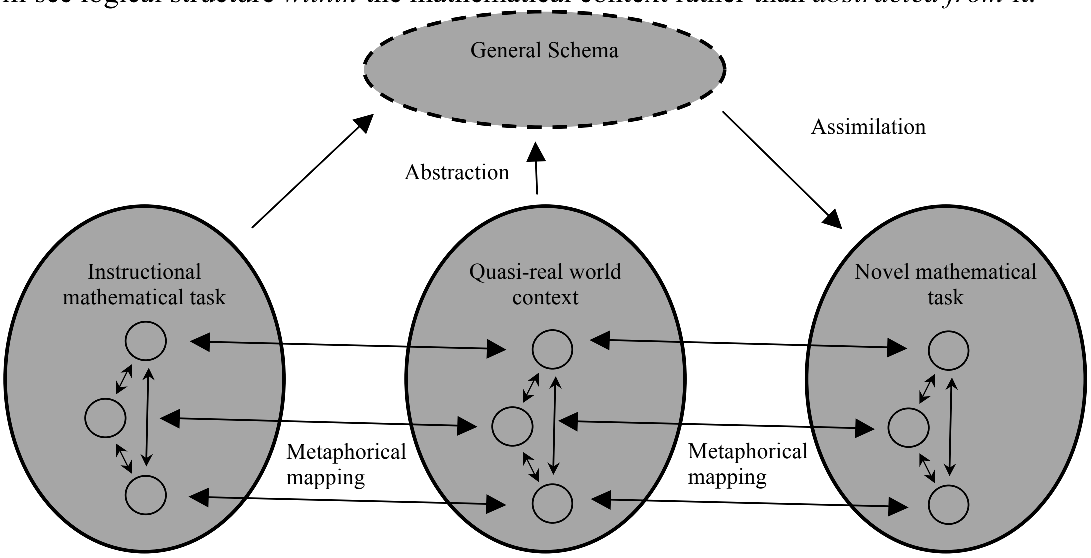

Fig 1. Pathway to transfer under metaphorical logical instruction.

Case study of Vincent's use of the Platypus metaphor

The real analysis class featured here was taught at a mid-sized, research university in the United States in the spring of 2008. A mathematician specializing in differential geometry taught the course. She had taught this real analysis at least 2 previous times and was awarded multiple teaching awards based upon her students' nominations. Most of the students were mathematics majors, a large portion of whom proceeded to pursue graduate degrees in mathematics or related sciences. The course met for 75 minutes twice per week over the course of a 15-week semester. The first author was present as an observer during all class meetings and conducted weekly task-based interviews with a small group of volunteers from each class. The researchers' interpretations of the professor's pedagogical intentions were vetted against her own articulations during bi-weekly interviews.

The professor used a large number of examples to guide students' reasoning about constructing definitions and theorems. She paid particular attention to the common student misconception that sequences only converge or tend to infinity if they are monotonic. To address this issue, she introduced the examples Penguin -{1,1,2,1,3,1,4,1,5,1…} and Platypus - {2,1,4,3,6,5,8,7…}. While she agreed that both appeared to tend to infinity, they both displayed strange properties. She specifically stated that biologists had to decide whether Platypus was a mammal because it did have hair, but it laid eggs instead of having live young. Ultimately, biologists agreed that platypus was a mammal just like the Platypus sequence does tend to infinity. Penguins look like they have fur, but they are birds and not mammals. Similarly the Penguin sequence is unbounded, but does not tend to infinity.

Once the class ratified a definition for a sequence tending to infinity, they verified that Platypus satisfied the definition and Penguin did not. The professor then provided the following true/false questions for the students to consider in groups and then discuss as a class (note that if a sequence tends to infinity or negative infinity, then it "properly diverges"): "If $\lim_{n\to\infty} x_n = +\infty$, then $\{x_n\}$ is unbounded and increasing." and "A sequence properly diverges if and only if it is unbounded." The students pointed out that the first statement is false because of platypus and the second statement "only works in one direction" because of penguin. Members of the class used the name platypus to refer both to the sequence itself and to its role as an atypical example of a sequence that tended to infinity. In contrast to *prototypes,* which are key examples that display the "standard" properties of a class of objects, we use the term *wedge* to denote examples that distinguish easily conflated properties. The wedge examples described in this paper are also different from the notion of boundary examples (Watson & Mason, 2001), which are examples used to make clear why a condition is required to define a concept by showing the definition would fail to describe the concept without the condition. While boundary examples relate a property and the category it describes, wedges are intended to relate multiple properties or definitions to each other.

For instance, the absolute value function is a standard wedge between the properties of continuity and differentiability. Platypus became the class' wedge between monotonicity and tending to infinity. Penguin acted as a wedge between unboundedness and tending to infinity. Students consistently referred to the sequences by their animal names without having to explain indicating the names became taken-as-shared.

An interview with Vincent two weeks later revealed he held a (nonstandard) personal concept definition (PCD) of proper divergence: "For every $K \in \mathbb{N}$… pick some term x_K right here, then for every term… $n > K$, $x_n > x_K$. So all the x_n's got to be in that interval [(x_K, ∞)] here." This definition would imply that sequences tending to infinity are monotone. However, when Vincent tried to describe Platypus, he recalled because of the name that it was a strange example of a sequence that tended to infinity. He thus modified his PCD to say you can only use even values of *K*. He also stated directly that he had previously thought sequences tending to infinity should be monotone, but that Platypus rendered that false.

When the interviewer asked Vincent one month later about various example sequences, Penguin was presented first. Vincent began listing properties of the sequence before citing its metaphorical name. When presented with Platypus, he immediately named the sequence explaining it was a "weird looking mammal." With some work, he elaborated the metaphor to say that mammals were sequences tending to infinity, but Platypus did so in a weird way. With some hesitance Vincent acknowledged that Platypus meant that not every sequence tending to infinity was monotonic. He restated his idiosyncratic PCD and noted again that Platypus limits the indices that "will work." It appeared that he "rediscovered" the distinction between the two properties via Platypus acting as a wedge.

Discussion

The Platypus and Penguin metaphors allowed the professor to simultaneously refer to the sequences themselves and their logical role as wedges between oft-conflated properties. The metaphors were localized tools for drawing attention to the logic of sequences categorization. The *logical metaphor* (Dawkins, 2009) induced a biological *structural metaphor* comparing the classification of sets to the classification of animals in biology. Vincent's repeated expression of the targeted misconception validated the professor's instructional intervention.

We argue that this episode satisfies all of our analytical criteria (denoted above by C1-C3) for evidence that the metaphor directly contributed to Vincent's mathematical learning. During the second interview, Vincent spontaneously recalled the sequence names and appeared to discover anew that sequences tending to infinity need not be monotone (C1). Without the metaphor, Vincent might likely have described Platypus as not tending to infinity according to his PCD. However, the name seemed to fix in his mind the fact that Platypus was a non-standard example of tending to infinity. Because he reevaluated his mathematical understanding, we claim that Vincent elaborated the metaphor beyond simple language use (C2). Third, though Vincent began the discussion in the latter interview *describing* sequences, he shifted to *categorizing* classes of sequences before actually *relating* sequence properties to one another. While previous research (Alcock & Simpson, 2002; Edwards & Ward, 2008) indicates that many students use definitions to *describe* examples, Vincent shifted to using examples to *relate* properties. This constitutes a sophisticated shift in the nature of his mathematical activity. Because he shifted from examining Platypus to the logical activity of relating properties, we claim that he was directly reasoning about the logic of sequence classification (C3).

This final claim also reveals how the instruction in this episode exemplifies the model described in our thought experiment. The metaphor guided Vincent to shift his attention to logical structure while still examining the particular properties of the mathematical objects at hand. His attention to the logical role of Platypus as a wedge allowed him to examine and assess his mathematical reasoning about particular sequence properties and his understanding of their definitions. Rather than abstracting his attention away from particular examples to talk about sequence properties, the metaphor elevated Platypus from a (non-)example of certain classes to a *wedge* with meta-theoretical significance. Figure 2 portrays Vincent's shift from seeing properties describing sequences to sequences relating various properties.

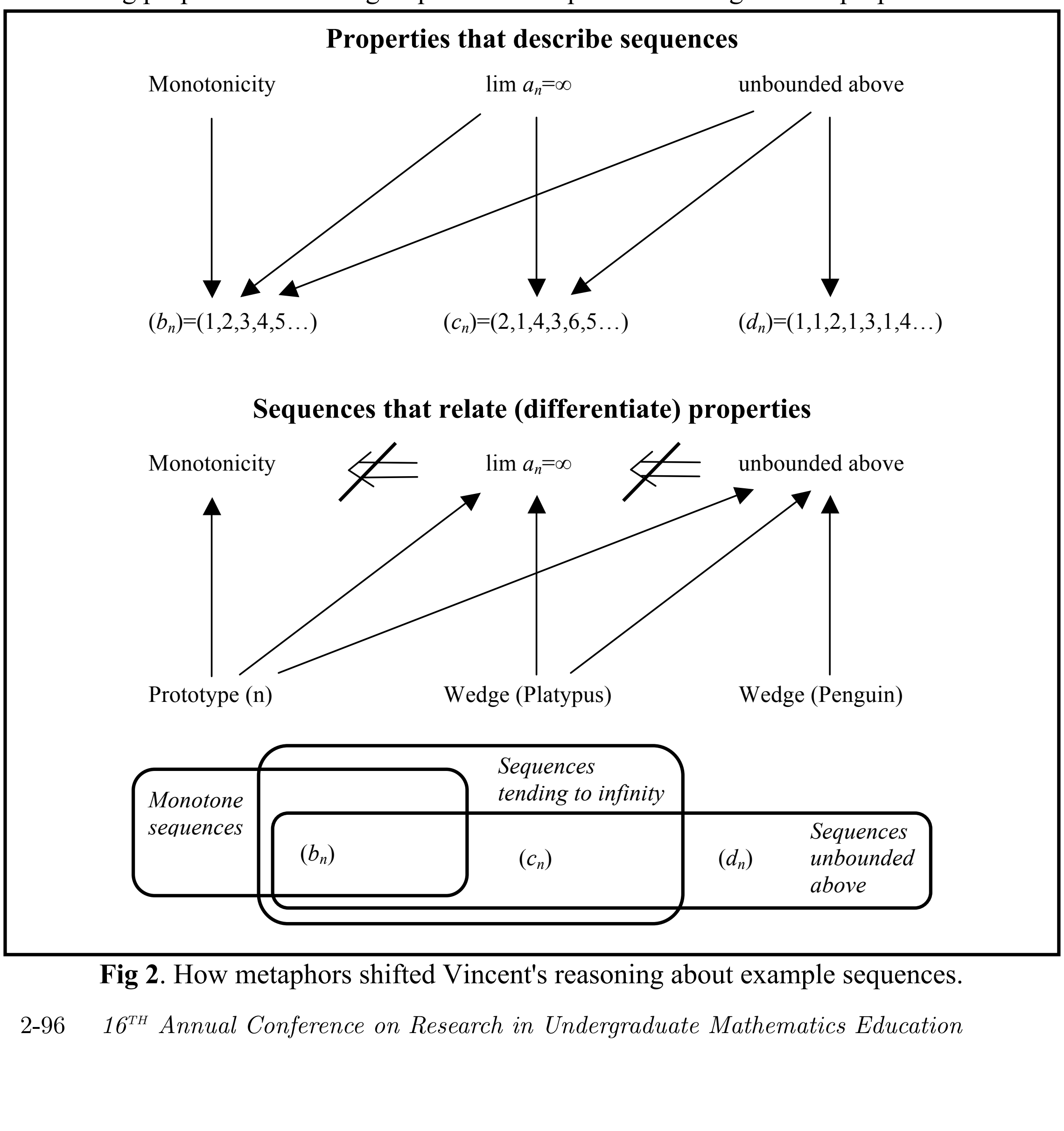

Fig 2. How metaphors shifted Vincent's reasoning about example sequences.

The form of instruction described here provided a valuable tool in the inquiry-oriented real analysis classrooms we observed for bringing logical structure into the consensual domain. The metaphors allowed students to examine and assess the logic of mathematical arguments. The method is somewhat limited because the metaphors model particular logical structures (such as wedges) rather than comprehensive logical systems such as propositional logic. They were generally used to support students' reasoning about problematic logical structures. We think one main analytical contribution is to draw attention to the issue of whether students are reasoning about logic at all. Future studies on students' logical reasoning should carefully delineate whether they are observing the logic of students' reasoning or students' reasoning about logic.

Acknowledgement

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SWITCHER AND PERSISTER EXPERIENCES IN CALCULUS I

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Previous reports show that not only are too few students pursuing Science, Technology, Engineering, or Mathematics (STEM) fields, but also many who originally intend to pursue these fields leave after their experiences in introductory STEM courses. Based on data gathered in a national survey, we will present an analysis of 5381 STEM intending students enrolled in introductory Calculus in Fall 2010, 12.5% of whom switched out of a STEM trajectory after their experience in Calculus I. When asked why these students no longer intended to continue taking Calculus (an indicator of continuing their pursuit of a STEM major), 31.4% cited their negative experience in Calculus I as a contributing factor. We analyze student and their instructor survey responses on various aspects of their classroom experience in Calculus I to better understand what aspects of this experience contributed to their persistence.

Key words: [Calculus, Retention, Instruction, Persistence.]

The number of students completing degrees in Science, Technology, Engineering, or Mathematics (STEM) continues to fall short of the demand for workers in these fields and hence is a national problem of great importance. Not only are too few students pursuing STEM fields, but also many who originally intend to pursue these fields leave after their experiences in introductory STEM courses (Seymour & Hewitt, 1997). Thus, one integral aspect to increasing the number of STEM graduates is to increase the *retention* of STEM students. The most recent report from the President's Council of Advisors on Science and Technology (PCAST, 2012) predicts that simply increasing the retention of STEM majors by ten percentage points would go a long way to meeting the need for the targeted 1 million more STEM graduates.

In the seminal book centered around student persistence in STEM fields, *Talking About Leaving*, Seymour and Hewitt (1997) noted that students leave STEM majors primarily because of poor instruction in their mathematics and science courses, with Calculus often cited as a primary reason. Using the same dataset that we use for this report (see details below), our previous analysis found that 12.5% of the STEM intending students were identified as *switchers,* those students who chose not to continue onto Calculus II after their experiences in Calculus I. When asked why these students no longer intended to continue taking Calculus (an indicator of continuing their pursuit of a STEM major), 31.4% cited their negative experience in Calculus I as a contributing factor (Rasmussen, Ellis, Duncan, Bressoud, and Carlson, in preparation). In this analysis, we investigate the Calculus I experience as reported by both students and instructors to better understand how Calculus I experiences relate to student persistence.

Methods

Data for this study come from a large-scale national survey of mainstream Calculus I instruction that was conducted across a stratified random sample of two- and four-year undergraduate colleges and universities during the Fall term of 2010. The survey was sent to a stratified random sample of mathematics departments following the selection criteria used by Conference Board of the Mathematical Sciences (CBMS) in their 2005 Study (Lutzer et al, 2007). In all, we selected 521 colleges and universities, 222 of which participated: 64 two-year colleges (31% of those asked to participate), 59 undergraduate colleges (44%), 26 regional universities (43%), and 73 national universities (61%). There were 660 instructors and over 14,000 students who responded to at least one of the surveys. For the purpose of this analysis, we focus only on STEM intending students who responded to both pre and post term surveys and whose instructors did as well, resulting in a data set of 5345 students from 421 instructors from 145 institutions.

We determined students to be STEM intending as the students that indicated intent to take Calculus II at the beginning of the Calculus I term. Students were again asked to report their intention to take Calculus II at the end of the term, and based off of their responses we classified them as either *Persisters* or *Switchers*. Persisters were those students who initially intended to take more Calculus and did not change from this intention at the end of the term (or one year later). Switchers, on the other hand, were those students that started Calculus I intending to take more Calculus, but then by the end of the term (or one year later) changed their plans and opted not to continue with more Calculus.

Before conducting this analysis, we wondered if switchers and persisters were in the same classes, or if they are in the same classes but experiencing them differently. In order to determine this, we looked at the subset of instructors for which we had at least ten students' end of term survey data[1]. Because of this clustering, our significance values are lower than they would be under complex sampling analysis. This left 181 instructors accounting for 4,280 of the students. The percentage of switchers per class ranged from 0% to 71.4%, suggesting that some of the classes have a mix of switchers and persisters, and some do not.

Student Behavior in Class

To understand how students reported their in class behavior, we examined four questions from the end of term survey. Students were asked to report how frequently they did each of the following activities during class, from never (1) to every class session (5): contributed to class discussions, were lost and unable to follow the lecture or discussion, asked questions, and simply copied whatever was written on the board. For each of these questions, we conducted an independent-samples t-test to compare responses for switchers and persisters. As can be seen in Table 1, there was a significant difference in the responses for the amount of time spent contributing to class discussions between switchers and persisters, time spent lost and unable to follow the lecture or discussion, and time spent simply copying whatever was written on the board, but there were not significant differences between switchers and persisters on time spent asking questions. These results indicate that switchers report spending less time in class contributing to class discussion, more time lost and copying down what is written on the board, and the same amount of time asking questions as reported by the persisters. Taking these together, switchers report being *less engaged* than persisters during class.

[1] Because overall 12.5% of STEM intending students were switchers, if an instructor was linked to 10 students this would provide on average 1 switcher per instructor.

Table 1. Student reports of in-class behavior.

During Class:	**Persister**	**Switcher**
I contributed to class discussions.**+	2.69 (1.25)	2.47 (1.17)
I was lost and unable to follow the lecture or discussion.**	1.89 (0.99)	2.18 (1.02)
I simply copied whatever was written on the board.**	2.86 (1.36)	3.26 (1.32)
I asked questions.+	2.38 (1.12)	2.34 (1.07)

Note. * = p ≤ .05, ** = p ≤ .001, + = Persister mean greater. Standard Deviations appear in parentheses below means.

Instructor Behavior

In the above section we investigated whether differences existed in students' behavior based on persistence. In this section we focus our attention on instructor behavior as reported by their students, and by instructors (when available).

The first set of 15 questions asked students to report their level of agreement, on a six-point scale from strongly disagree to strongly agree, to various statements of instructor actions, such as "made class interesting" and "discussed applications of Calculus." Table 2 shows that persisters agreed that their instructors did all but four of these actions significantly more than switchers agreed. The actions that persisters reported their instructors doing in class more include "asking questions to determine if I understood what was being discussed", "allowed time for me to understand difficult ideas", and "made class interesting." The actions that switchers reported their instructors doing in class more included "made students feel nervous during class" and "discouraged me from wanting to continue taking Calculus."

Table 2. Student reports of instructor actions.

My Calculus instructor:	**Persister**	**Switcher**
asked questions to determine if I understood what was being discussed.**+	4.44 (1.21)	4.20 (1.32)
listened carefully to my questions and comments.**+	4.79 (1.14)	4.53 (1.31)
allowed time for me to understand difficult ideas.**+	4.39 (1.27)	4.03 (1.44)
helped me become a better problem solver.**+	4.43 (1.25)	4.02 (1.37)
provided explanations that were understandable.**+	4.63 (1.25)	4.27 (1.41)
encouraged students to enroll in Calculus II.**+	4.30 (1.25)	3.78 (1.35)
acted as if I was capable of understanding the key ideas of calculus.**+	4.83 (1.02)	4.57 (1.19)
made me feel comfortable in asking questions during class.**+	4.70 (1.21)	4.39 (1.36)
encouraged students to seek help during office hours.**+	4.98	4.82

	(1.08)	(1.21)
presented more than one method for solving problems.**+	4.65 (1.19)	4.34 (1.36)
made class interesting.**+	4.33 (1.42)	3.96 (1.54)
was available to make appointments outside of office hours, if needed.*	5.10 (0.99)	4.96 (1.00)
discouraged me from wanting to continue taking Calculus.*	2.24 (1.44)	2.66 (1.51)
made students feel nervous during class.**	2.07 (1.24)	2.22 (1.37)
discussed applications of calculus.+	4.73 (1.12)	4.63 (1.21)

Note. * = $p \leq .05$, ** = $p \leq .001$, + = Persister mean greater. Standard Deviations appear in parentheses below means.

The second set of questions asked both students and instructors to report how frequently their instructor (or they) did various activities, using a six-point scale from never to very often. These activities included frequency of lecture, having students give presentations, students working together in class, and having whole class discussions. Two analyses of this data are presented below. Table 3 provides both student and instructor reports of these frequencies, and the significance of the difference between the switcher and persistence reports from the students only. Table 4 takes a closer look at the difference between what the instructors report and what the students report.

Table 3 shows that persisters reported these all but two of these activities occurring significantly more frequently than switchers; lecturing and being assigned to read sections of the text before class were the two actions excluded. Taken together, the description of the class provided by the persisters appears to be more student-centered and engaging than the description of class presented by the switchers. The last two columns show instructor responses for these activities, which for the most part tell a similar story to what the students report. However, there are some discrepancies. Table 4 shows these in more detail.

Table 3. Student and Instructor reports of in-class activities.

How frequently did your instructor (you) do the following:	**Student Report**		**Instructor Report**	
	Persister	**Switcher**	**Persister**	**Switcher**
hold a whole-class discussion? **+	3.48 (1.81)	3.13 (1.80)	3.36 (1.61)	3.36 (1.77)
ask students to explain their thinking?**+	3.83 (1.63)	3.53 (1.70)	3.96 (1.39)	4.03 (1.56)
prepare extra material to help students understand calculus concepts or procedures?**+	3.94 (1.53)	3.67 (1.60)	4.08 (1.66)	3.83 (1.674)
require you to explain your thinking on exams? **+	4.11 (1.67)	3.82 (1.77)	4.32 (1.64)	4.05 (1.64)
show how to work specific problems? *+	5.01 (1.09)	4.80 (1.20)	5.01 (1.09)	5.08 (1.09)

have students work with one another? *+	3.28 (1.91)	3.09 (1.83)	3.67 (1.91)	3.50 (1.84)
have students give presentations? *+	1.83 (1.40)	1.67 (1.26)	1.74 (1.10)	1.76 (1.20)
have students work individually on problems or tasks? *+	3.78 (1.64)	3.60 (1.73)	2.91 (1.54)	3.12 (1.70)
ask questions? *+	4.63 (1.20)	4.50 (1.25)	5.23 (.949)	5.12 (1.20)
require you to explain your thinking on your homework? *+	3.38 (1.76)	3.15 (1.77)	3.87 (1.82)	3.79 (1.75)
assign sections in your textbook for you to read before coming to class? +	3.67 (1.96)	3.59 (1.99)	3.74 (2.11)	3.65 (2.02)
lecture?	5.02 (1.26)	5.09 (1.25)	5.12 (1.13)	5.18 (1.08)

Note. Student Report: * = $p \leq .05$, ** = $p \leq .001$, + = Persister mean greater. Standard Deviations appear in parentheses below means.

Table 4 shows that instructors and students report some activities occurring with different frequencies, and that this difference is not always consistent between switchers and persisters. For clarification, the numbers close to zero (either positive or negative) indicate agreement between the student and the instructor. Thus larger numbers in absolute value represent more disagreement between student and instructor. Positive numbers indicate that the instructor reported higher frequencies and negative numbers indicate that the student reported high frequencies.

Table 4. Difference between student report and instructor report on in class activities.

How frequently did your instructor (you) do the following:	**Persister**	**Switcher**
show how to work specific problems? **	-.019 (1.45)	.233 (1.43)
hold a whole-class discussion? **	-.163 (2.14)	.243 (2.28)
ask students to explain their thinking?**	.051 (1.85)	.457 (1.94)
have students work individually on problems or tasks? **+	-.893 (1.99)	-.528 (1.99)
have students work with one another? +	.326 (1.68)	.292 (1.70)
have students give presentations? *+	-.114 (1.54)	.058 (1.29)
lecture? +	.103 (1.38)	.086 (1.44)
prepare extra material to help students understand calculus concepts or procedures?+	.170 (2.15)	.106 (2.11)
require you to explain your thinking on exams? +	.192 (2.19)	.160 (2.33)
require you to explain your thinking on your homework?	.487	.640

	(2.21)	(2.18)
ask questions?	.568	.588
	(1.47)	(1.61)
assign sections in your textbook for you to read before coming to class?	.045	.073
	(2.11)	(2.11)

Note. * = p ≤ .05, ** = p ≤ .001, + = Persister mean difference greater (in absolute value). Standard Deviations appear in parentheses below means.

Activities for which there was consistent agreement between students and instructors, for both switchers and persisters, include lecture, requiring students to explain thinking on exams, and assigning reading form the textbook before class. The activities that the instructor reported occurring much more frequently than students, for both switchers and persisters, include asking questions, having students work with one another, and requiring students to explain thinking on homework. Thus instructors appear to uniformly overestimate the amount of time they spent on these activities, as compared to their students.

Activities for which switchers disagreed with their instructor more than persisters did include the instructor showing how to work specific problems, holding an in-class discussion, and having students explain their thinking. Taken together, these three activities reflect student engagement in class. Thus, though their instructors believed they were engaging students, and persisters were engaged, switchers did not report being similarly engaged. The only activities for which persisters disagreed more with their instructor than switchers was having students work individually on problems or tasks, although both switchers and persisters reported this happening more frequently than their instructors.

Conclusion

Taken together, the analyses reveal a more complete understanding of the Calculus I experience as told by switchers, persisters, and their instructors. Across the board, switchers report being less engaged during class than persisters; the switchers report contributing to class discussions less, felt less comfortable asking questions during class, found class less interesting, were asked to explain their thinking less, and reported working with other students in class less than persisters. Based on the percentage of switchers per instructors, in some cases this may be because they were simply in classes with instructor who taught differently. However, in some cases they were in the same class as persisters and experienced class differently. Table 4 highlights these students; although persisters agreed with their instructor on the frequency of being engaged, switchers report various engaging activities occurring less. In either class situation, the level of engagement of the students is a large component of their classroom experience, and is related to their eventual persistence.

These findings have implications both for research and for teaching. First, because there are differences between what the instructor reports occurring in class and what their students report occurring, when discussing classroom activities in research the researcher must be aware that differences exist between what the students report and what the instructors report, and that these differences vary among students. Second, instructors should be aware that students in their classes are experiencing their class differently from one another. By being aware of this, instructors can be more intentionally treat students similarly and actively engage *all* students. With the goal of increasing retention in the STEM fields by increasing retention in introductory classes such as Calculus I, it is necessary to improve students' experiences in these courses. By understanding that these experiences vary across students, we can begin to understand what aspects of their Calculus I experience negatively affect retention in Calculus.

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SECONDARY TEACHERS' DEVELOPMENT OF QUANTITATIVE REASONING

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This study was designed to document the development of teachers' ways of thinking about quantitative reasoning, one of the standards for mathematical practice in the Common Core State Standards. Using a Models and Modeling Perspective, the authors designed a model-eliciting activity (MEA) that was implemented in a graduate mathematics education course focusing on quantitative reasoning. Teachers were asked to create a quantitative reasoning task for their students, which they subsequently revised three times in the course after receiving instructor, peer, and student feedback. The MEA documented the development of the teachers' models of quantitative reasoning, and this report details one group of three teachers' development over the course. Findings include an overall model of teachers' development that is both generalizable and sharable for other researchers and teacher educators.

Key words: In-Service Mathematics Teachers, Model Eliciting Activity, Quantitative Reasoning

Introduction and Literature Review

The Common Core State Standards (CCSS) are set to bring a new wave of reform measures for classrooms across the United States, and with this comes new goals for teachers and teacher education programs. Research about how teacher education programs can support these goals are lacking, especially concerning how programs align with the CCSS in ways that productively impact teacher practice (Confrey & Krupa, 2010; Sztajn, Marrongelle, & Smith, 2011). Literature addressing the gap between teacher practice and teacher education efforts has described how a models and modeling approach to in-service teacher education can challenge teachers to develop ways of thinking to help their students while simultaneously documenting the development for research purposes. This approach uses Model Eliciting Activities (MEAs), which are tasks that engage teachers in thinking about realistic and complex problems embedded in their practice in order to foster ways of thinking that can be used to communicate and make sense of these situations (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). MEAs have been shown to contribute to teacher development because these activities make teachers engage in applicable mathematics, consider student reasoning more deeply, and reflect on beliefs about problem solving (Chamberlin, Farmer, & Novak, 2008; Lesh, 2006; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003).

While these studies have implemented successful MEAs for teachers, there is a need for additional activities given the recent demands the CCSS place on teacher education programs (Confrey & Krupa, 2010; Garfunkel, Reys, Fey, Robinson, & Mark, 2011). For instance, no MEAs currently exist that aim to identify and document the development of teachers' thinking about the CCSS standards for mathematical practice or the related area of quantitative reasoning. The purpose of this study is to investigate teachers' ways of thinking about quantitative reasoning by implementing a MEA in a graduate setting for secondary mathematics teachers. The specific research questions were (a) how do teachers' models of quantitative reasoning develop through a MEA grounded in their classroom practice? and (b) What researcher model can be developed from this process in order to produce generalizable and sharable findings for others?

Methods

The theoretical perspective we used for the study is a Models and Modeling Perspective, as described by Lesh and colleagues. In addition to having a powerful lens for examining teacher education, a Models and Modeling Perspective also provides guidelines for the methods that support significant findings given the current research questions. Given these methods, a Models and Modeling Perspective offers a framework for understanding teachers' ways of thinking, their development, and provides a mechanism for analyzing and piecing together findings (Koellner-Clark & Lesh, 2003; Hiebert & Grouws, 2007; Silver & Herbst, 2007; Sriraman & English, 2010).

The setting for this study was within a master's program in mathematics, where teachers took a combination of mathematics and mathematics education courses over two years; however, part of the study involved piloting the teachers' MEA in summer undergraduate mathematics courses. We focused the study on a newly developed mathematics education course in the program, called Quantitative Reasoning in Secondary Mathematics, which was offered Summer 2012. The authors designed the MEA, worth 50% of the course grade, to have the 21 teachers enrolled create and refine a quantitative reasoning task for their students with the intention of implementing the task in the fall. Teachers worked in groups of three or four and received feedback about their task during the summer from several sources, including the instructor, from each other, and from undergraduate students who completed the task. Each type of feedback prompted an updated iteration of the task and supporting documents that captured how the teachers' ways of thinking develop. Data collection consisted mostly of the iterations of documents generated by the MEA (see Table 1), with observations of Group 1 during in-class time devoted to the MEA. Using content analysis on the documents, the researchers identified patterns in the ways teachers' thinking about quantitative reasoning tasks developed due to this process. While the full study will analyze all 6 groups, here we present an analysis of one of the groups, which we call Group 1.

Table 1. Summary of Quantitative Reasoning (QR) documents analyzed

Assignment Name	**Short Description of Components**
Pre-Assignment	Document including initial models of QR, QR tasks, QR course
Version 1	Four documents including (a) Quantitative Reasoning Task; (b) Facilitator Instructions; (c) Assessment Guidelines; (d) Decision Log
Instructor's Feedback	Instructor's comments and suggestions to Version 1.
Version 2	Updated Version 1 in response to the instructor's feedback.
Teachers' Feedback	Groups swap Version 2 and offer comments and suggestions
Version 3	Updated Version 2 in response to the teachers' feedback.
Undergraduate Work	Student work after completing QR task (part (a) of Version 3)
Version 4	Updated Version 3 in response to student work, plus evaluation of student work.

Findings

Originally in their pre-assignment, the three teachers in Group 1 had wildly significant different definitions of quantitative reasoning, from drawing justified conclusions in real-world problems (Nicholas), visualizing amounts and number sense (Joyce), to using logical mathematical statements and deductive reasoning to reach conclusions using algebra (Percy). After reading the CCSS definition, all three teachers were asked to interpret what they thought this definition meant, for example by explaining what contextualization and decontextualization could look like in a secondary mathematics classroom. Nicholas chose to adopt this kind of language when asked to describe what quantitative reasoning looked like in his own classroom by saying “my students analyzed real-world problems and/or visual representations using contextualized mathematical representations.” This statement reflects his earlier model of quantitative reasoning which he believed was evidenced by students’ ability to determine the reasonableness of answers when solving real-world problems. The other two teachers expanded their original models without this type of language, identifying quantitative reasoning as focusing on relationships between quantities and making sense of what is being done when solving problems.

The group statement about quantitative reasoning given in Version 1 reflected the commonalities among the individual interpretations of the CCSS definition. Quantitative reasoning was defined as focusing on quantities when solving a problem, creating relationships between quantities, understanding why something works, and justifying the solution with units. These descriptions were all framed in terms of the logarithmic context they chose to develop for their MEA. All of these components were part of at least one teacher’s definition of quantitative reasoning in the pre-assignment, with the exception of the incorporation of units, which Joyce was observed to adopt during the first week of class. The ideas about quantitative reasoning that did not persist into Version 1 were ones that only one of the three mentioned in the pre-assignment, such as Joyce’s notion of number sense and Percy’s incorporation of argument and deductive logic. The group connected their task to quantitative reasoning by having students demonstrate “what the quantities associated to a logarithmic function represent” and compare how these quantities compare in terms “big changes vs. small”. The instructor feedback to these teachers largely encouraged the task’s alignment to MEA principles, and the group was encouraged to continue documenting how they thought quantitative reasoning related to the task.

The teachers made a number of changes to the task in response to the instructor feedback, documenting the details and rational for the alterations in Version 2. The teachers claimed their Version 2 task “will encourage students to think quantitatively (about how quantities relate to each other in different scenarios – exponential vs. logarithmic) and also develop an understanding of logarithms” by having students model about interest rates by modeling the situation using both exponential and logarithmic functions. This model of quantitative reasoning was observed to develop during group discussions and may perhaps be a result of the instructor’s in-class encouragement for the group to connect inverse functions and compare the size of various quantities.

The peer feedback provided information on how the teachers thought about quantitative reasoning through their evaluation of another group’s task. Comments from the small group discussion that occurred in class were echoed in the peer evaluation document; overall, Group 1 thought Group 6’s task was “a great opportunity to provide evidence of quantitative reasoning”, particularly through the prompting questions included in the task. These questions included “What quantities would be represented in your explanation?” and “How are the quantities related to each other?” Group 1 concluded that after the evaluation process, they began “thinking differently about how we would like our students to work.” Group 2 gave Group 1 feedback focusing on making the role of quantities more explicit in the task, and to make the students “look at all of the quantities more in depth…have them talk about each quantity and how it will relate to the situation and formula they’re supposed to come up with.” Aside from advice on how to improve the facilitator guidelines, Group 2 requested “a little bit more about your thought process regarding how you decided that this task could show quantitative reasoning in each of the students…maybe you could be more specific about this."

In response to the peer feedback, Group 1's Version 3 included changes addressing the comments, such as editing a question to ask students to "describe the co-varying relationship between the quantities that your identified variables represent". These and other changes indicate Group 1 incorporated suggestions from their peers while maintaining the ideas they previously had about quantitative reasoning. When addressing the comment asking how the task incorporated quantitative reasoning, Group 1 indicated a quantitative understanding of logarithms looked "like 'the answer to the logarithm is what exponent I would need to use on this base to make it into this number (the argument); or our visualization of the behavior and characteristics of logarithmic graphs; etc." This indicated the group model of quantitative reasoning now included a conceptual understanding of a mathematical topic, incorporating multiple representations of a contextualized nature. This change reflected the real-world context that had been part of the group's definition since Version 1, and the multiple representations that had always been a part of the task.

After piloting Group 1's MEA with three undergraduate students in a Concepts of Calculus course, the group's reaction to the work provided some of the richest data about how their individual models had developed. The in-class discussions typically began with a teacher bringing up a comment about an area where students had difficulties, errors, or gave unexpected responses. The group's conversation then tended to move towards identifying changes they would want to make in response to student difficulties. This evaluation process encouraged teachers to reveal the intent of each question and whether the teachers thought the student response addressed the goal of the question and the overall MEA to show student thinking about quantitative reasoning. The teachers made comments such as "[I'm] trying to put ourselves in the mind of the learners", indicating their transition to seeing the activity from a student perspective. These observational findings were triangulated through Version 4, particularly in their comments that "as we looked over our student feedback, it became apparent that our main goal was to improve questions that did not [elicit] the desired response from the students." These revelations from the teachers often resulted in changes being made to the questions to better meet the MEA and question goals.

Group 1 stated the changes in Version 4 were

…making the table go up by more consistent increments, streamlining language somewhat (rule vs. model vs. function vs. relationship vs. co-varying), and providing more guidance for the process of estimating an exponent solution. We also had discussion over whether this activity actually promotes and assesses quantitative reasoning, and we believe that ultimately it does. Students model their understanding of exponential functions with a table, graph, and function rule. They also do the same with the inverse, the logarithmic function. A significant motivator for creating this activity is students' anemic procedural understanding of logarithms.

The definition of quantitative reasoning and its connection to the task were similar to that stated in Version 3; however, their interpretation of student work included newer components. By evaluating students' quantitative reasoning, the teachers revealed the following attributes constituted evidence of the term: writing the relationships of quantities in words, explaining relationships between functions, algebraically working to contextualize and articulate quantities, and explaining mathematical observations through quantities. The teachers were also able to articulate what quantitative reasoning would look like in their own classrooms:

"Overall, the revision process was very valuable in creating the lesson. In particular, working with peers and discussing the student feedback was very worthwhile. The realistic and honest answers demonstrated how the students interpreted the investigation, and therefore created an opportunity to create a better lesson. We believe that the quantitative reasoning skills we are trying to develop are not only dependent on well conceived lessons, but almost more importantly, well conceived classroom attitudes and expectations."

Discussion

The development of this group's model of quantitative reasoning began with the consolidation of different definitions from each individual, where common characteristics between ideas were preserved and non-shared ideas were largely abandoned. These changes likely resulted from the classroom readings and activities describing frameworks for thinking about quantities and quantitative reasoning. For example, the instructor presented Moore, Carlson, and Oehrtman's (2009) definition for quantitative reasoning as attending to and identifying quantities, representing the relationships between quantities, and constructing new quantities during the first week of class; this definition is similar to the one submitted in Version 1. Similarly, the inclusion of Thompson's (1989; 1994; Smith III & Thompson, 2008) framework for quantities can be attributed to Group 1's inclusion of units in this definition. While characteristics of identifying and comparing quantities became clear in their definition, how these components were operationalized in the task were unclear.

The peer feedback process allowed the group to see other teachers' ideas of quantitative reasoning, and allowing the introduction and adoption of new ways of thinking. At the same time, the group was challenged to improve their task with comments from the peer feedback asking for further articulation of the task's connection to quantitative reasoning. Some of these changes were observed to be the result of instructor suggestions in class about how to operationalize these characteristics in terms of the group's MEA. By the third model, group expanded their definition to include core characteristics of the task they had included, such as conceptual understanding and multiple representations. The group more clearly stated the relation of their task to quantitative reasoning through these characteristics. This addition may be the result of the readings and assignments Group 1 completed from the Carlson and Oehrtman (2011) precalculus textbook, as prior to submitting this version Group 1 was exposed to ideas of multiple representations being used within a single problem about proportional reasoning and average speed, similar to the problems in Thompson (1994).

The student feedback was invaluable in promoting teacher development of models about quantitative reasoning, and about the task itself. These responses indicated the teachers' new ability to apply their definition of quantitative reasoning to a specific task for their students, in this case logarithms. The increased emphasis on connecting logarithmic and exponential functions may have resulted from the in-class activities that had teachers applying proportional reasoning to exponential functions, and contrasting these with linear functions.

These results have implications for both researchers and teacher educators. The use of this MEA pushed teachers to develop their model of quantitative reasoning. By the end of the course, teachers had moved from a range of definitions of quantitative reasoning to a more clearly defined model that connected this term to quantities, relationships of quantities, how these ideas were important to a context (financial planning using logarithms), and how these ideas could be developed in their students. Also, the role the instructor played by selecting reading and activities that introduced teachers to different frameworks about quantities, proportional reasoning, and exponential functions seemed to influence teachers' model development.

Given the influence of the CCSS on assessments taking place in the 2014 school year, teachers will be expected to include the standards for mathematical practice as a daily part of the mathematics classroom and connect these practices to content (CBMS, 2012; CCSS, 2010). The connection between quantitative reasoning and these practices mean teachers need to be able to interpret and instruct these ideas in meaningful ways. However, the findings indicate teachers have little experience with these terms, and interpret them in ways that are different and disconnected to classroom activities. From a practical stance, this MEA structure was successful for the goals of the course in that teachers were able to analyze the mathematical and conceptual structure of quantities and the relationships between quantities in secondary mathematics courses. For example, including seminal readings about this topic can contribute to encourage new ways of thinking about quantitative reasoning in teachers. Structuring teacher education and professional development to help teachers overcome these gaps in productive ways continues to be a major focus in mathematics education. This study may offer teacher educators and researchers a potential starting point for shaping teacher education in ways that support development of teachers' way of thinking about quantitative reasoning and other standards for mathematical practice.

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UNDERGRADUATE STUDENTS' MODELS OF CURVE FITTING

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The Models and Modeling Perspectives (MMP) has evolved out of research that began 26 years ago. MMP research uses Model Eliciting Activities (MEAs) to elicit students' models of mathematical concepts. In this study MMP were used as conceptual framework to investigate the nature of undergraduate students' models of curve fitting. Participants of this study were prospective mathematics teachers enrolled in an undergraduate mathematics problem solving course. Videotapes of the MEA session, class observation notes, and anecdotes from class discussions served as the sources of data for this study. Iterative videotape analyses as described in Lesh and Lehrer (2003) were used to analyze the videotapes of the participants working on the MEA. Results of this study discuss the nature of students' models of the concept of curve fitting and add to the introductory undergraduate statistics education research by investigating the learning of the topic curve fitting.

Keywords: [Problem solving, Curve fitting, Modeling, Models, Statistical reasoning]

Introduction

Research on the teaching and learning of curve fitting is meager when compared to other statistical topics, such as the measures of central tendency or variability. Curve fitting, as a topic, is important, not only in general mathematics, but also in specific areas that correspond with the subject, including engineering and science. Situations in which curve fitting is required frequently arise in every day life in which we are given a set of data about which we would like to make a prediction. Only few studies discussed *how to learn* and *how to teach* this topic. But even those studies simply discussed what the students understand and do not understand about curve fitting. There was no literature that discussed, in depth, how students learn this topic. Therefore, this study helps to fill this void in the curve fitting literature, and consequently the undergraduate statistics education research literature.

In order to conduct research on the way in which someone learns mathematics, the learning must actually occur on the part of the participants. Extensive research exists on students' mathematics learning through clinical interviews (Hunting, Davis, & Pearn, 1996; Hunting & Doig, 1992) and teaching experiments (Cobb, 2000; Cobb & Steffe, 1983; Simon, 1995; Steffe & Thompson, 2000). In addition, literature exists in regard to students' learning through problem solving (Clement, 2000). Learning via problem solving was used as a method for this study as approaching mathematics through problem solving can create a context which simulates real world and, therefore, justifies the mathematics rather than treating it as a means to an end.

The research that has been successful in investigating students' learning via problem solving is the models and modeling perspectives (MMP) developed by Lesh and his colleagues over the past 26 years (Lesh & Doerr, 2003; Lesh, Hamilton, & Kaput, 2007; Lesh, Hoover, & Kelly; 1993; Lesh & Lamon, 1992; Lesh, Landau, & Hamilton, 1983). MMP uses the notion of students' models to study learning of a particular mathematics topic. MMP researchers have changed the notion of problem solving as *solving* difficult mathematical problems to *modeling* complex mathematical activities. Detailed discussion on MMP is provided below.

Models and Modeling Perspectives

Models are defined as "conceptual systems that are expressed using external systems and that are used to construct, describe, or explain behaviors of other systems" (Lesh & Doerr, 2003, p. 10). MMP is the name given to the theoretical perspectives that have evolved from the research first utilized more than 26 years ago by Lesh and his colleagues (Lesh & Doerr, 2003; Lesh, Hamilton, & Kaput, 2007; Lesh, Hoover, & Kelly; 1993; Lesh & Lamon, 1992; Lesh, Landau, & Hamilton, 1983). As this study will focus on studying the nature of students' models of curve fitting through a methodology that uses MEAs (Model Eliciting Activities), MMP will serve as a useful conceptual framework as it brings together two important, but separate research traditions: problem solving and conceptual development, in mathematics education research.

From MMP, problem solving and conceptual development in mathematics can be seen as co-developing as modeling can be seen as local conceptual development. Local conceptual development refers to the, "development of powerful constructs in artificially rich mathematical learning [problem solving] environments" (Harel & Lesh, 2003, p. 360). Lesh and Harel (2003) state that:

> [W]hen problem solvers go through an iterative sequence of testing and revising cycles to develop productive models (or ways of thinking) about a given problem solving situation and when the conceptual systems that are needed are similar to those that underlie important constructs in the school mathematics curriculum, then these modeling cycles often appear to be local or situated versions of the general stages of development that developmental psychologists and mathematics educators have observed over time periods of several years for the relevant mathematics constructs. (p. 157)

In other words, during an MEA session, students go through several modeling cycles that lead to students' conceptual development.

Objectives of the study

The research goal of this study was to investigate the nature of undergraduate students' models of curve fitting.

Methods

MMP research uses MEAs, which are designed specifically for research purposes (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly, & Post, 2000). MEAs are simulations of real world situations which are often used in research for their *model-eliciting* properties. The MEAs are designed using six principles (Lesh et al., 2000). Model-eliciting property refers to the way in which MEAs are designed to encourage students to clearly express, not only their final models, but, also, the numerous models that they create, revise and reject along the way. MEAs are different from typical mathematical modeling activities in that they not only require students to clearly express their final models, but, also, elicit the students' intermediary models.

A typical MEA session involves three distinct phases, summarized in Figure *1* below.

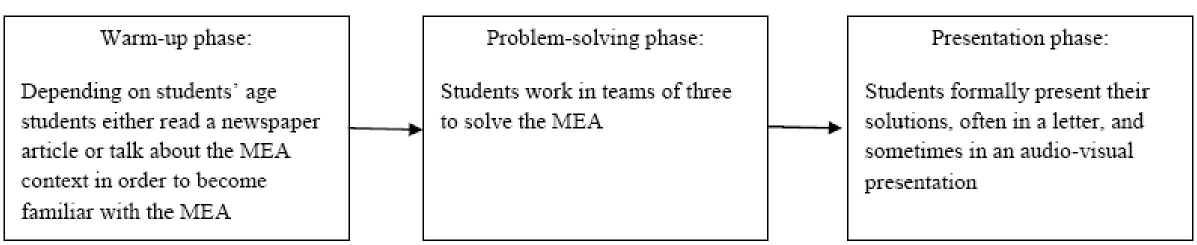

Figure 1. A typical MEA session

This study used the Time on Drill and Test Scores problem (TDTS) which is an MEA and has been used, tested and revised with different populations of students, both undergraduate (at all different levels of their program) and graduate. The core statistical ideas of this MEA are centered on the notion of fitting a line or curve to make a prediction about the situation in the MEA. The students in this study had no specific formal exposure to or instruction in these ideas prior to this MEA. Rather, this MEA was designed so that the students could readily engage in meaningful ways with the problem situation and create, use and modify the quantities in ways that would be meaningful to them and could be shared, generalized and reused in new situations. An excerpt from the TDTS problem appears in Table 1. In the TDTS problem, the problem solvers are supposed to provide the school administrators with a prediction of the test scores of students in several schools based on the time that the schools spend on the drill that teaches the information on the test. Problem solvers are given data on the time spent on the drill by 26 different schools and their respective average student test-scores. The TDTS problem was designed to elicit the notion of curve fitting for the purpose of making a prediction about the test scores.

Table 1. *Excerpts from the TDTS Problem*

Time Spent on the Drill as a Predictor of Performance on a Standardized Test		
School	Time Spent on the Drill (in minutes)	Test Scores
A	17.49	49.55
B	28.02	48.76
C	23.62	64.69
D	36.6	57.74
E	39.92	55.55

Table reports the amount of time spent on the drill each week. Then, it shows each school's average score on the end of the year test. The school administrators want to know whether the time spend on the drill is helpful in improving the test scores. For example, the administrators want to predict the end of the year test score for a school that spends 120 minutes per week on the drills. The school administrators need your help in making this prediction. In addition to making a prediction, they need you to write a letter describing how you made this prediction. They will use your process to make predictions about the test scores given any time on drill. Thus, you need to make sure that your process for making predictions will work for any time-on-drill.

Settings

The study took place in March of 2007 in an undergraduate mathematics teacher education classroom at a large mid-western university. The student demographics for a typical undergraduate mathematics teacher education classroom at this University are female and 90% white.

Procedural Details and Data Sources

Table 2. *Procedural details*

Class session #	**Task**	**Data collected**	**Number of students**
1(~30 minutes)	Explaining the study and its purpose to the students	N/A	14
2(~75 minutes)	TDTS problem	Artifacts produced by the students, observation notes, anecdotes shared by AIs, audiotapes and videotapes of the session.	9
3(~75 minutes)	Class Presentation	Artifacts produced by the students, observation notes, audiotapes and videotapes of the session.	12

The data sources from the class included

1. audiotapes and videotapes of all of the classroom sessions,
2. the students' worksheets and final reports detailing the development of their models, detailed field notes, and anecdotes.

Data Analysis

Iterative videotape analysis (Lesh & Lehrer, 2000) was used to analyze the videotapes. Lesh and Lehrer described multiple windows through which to view a given video. For example, each video of students' work has theoretical and physical aspects. These aspects can be analyzed in a variety of ways including analysis of isolated sessions, analysis of one group across several sessions or analysis of similar sessions across several groups, where a session implies a MEA session. This study focused only on students' models, therefore, only the mathematical perspective of the theoretical aspect was focused and field notes and the transcripts of the audiotapes and videotapes were analyzed. As only one MEA was used and focused on only one particular group in this study, only an isolated session was analyzed, see Figure 2.

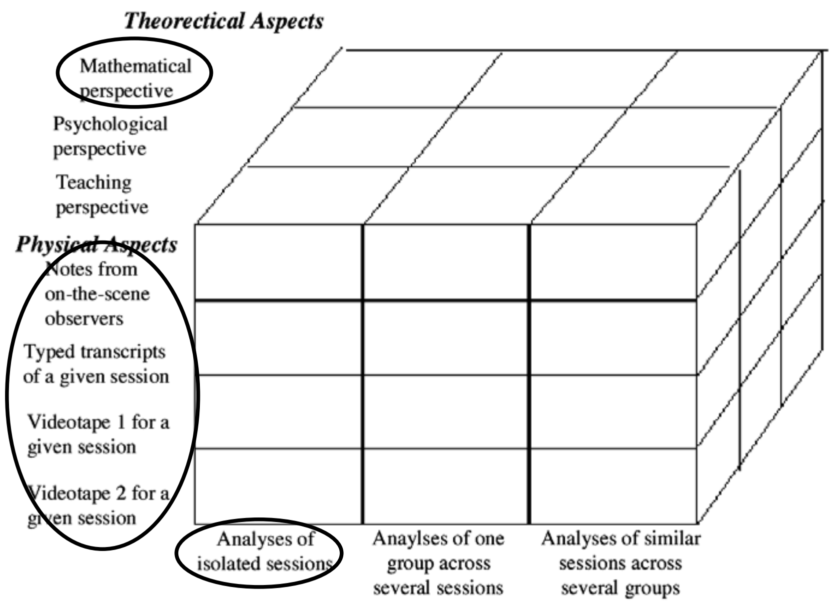

Figure 2. Multiple windows for viewing given sessions (the ovals represent the aspects that I focused on). (Lesh & Lehrer, 2000, p. 679)

Results

The results of the fine grained analysis of the cycles of conceptual development of curve fitting as displayed by the participants are given. In tracing the nature of the models of one group of 3 students, namely, Adam, Beth and Cathy, multiple cycles of increasing coordination and stability of conceptual systems that were observed in the students' responses are reported. Each cycle represents a shift in the students' thinking, providing powerful forms of information about the nature of students' models. In the present case, the cycles ranged from applying standard procedures without detailed analyses, to thinking about chunks of data, to sophisticated models of curve fitting that predict the situation. The analyses below shows that as students' work progressed on the TDTS problem, their conceptual systems evolved from being uncoordinated and unstable to becoming increasingly coordinated and stable. As the students' conceptual systems evolved along this line of coordination and stability, they were developing the concept of curve fitting.

First Cycle: Finding Data Summary.

When the students looked at the TD and TS scatterplot, they came to the conclusion that "no relationship" existed between the TD and TS and that "no pattern" was apparent in the scatterplot. As Adam's group had a limited, but useful background, in statistics, it was natural for them to resort to the usual methods of finding mean and standard deviation, even though these numbers were not helpful in the context of the problem.

The group spent a short amount of time finding the standard deviation using the summary table (students can do a summary table in *Fathom™* to find standard deviation for a data set) provided by *Fathom™*. Then, they discussed their results and concluded that finding the standard deviation did not help them solve the problem. That is, standard deviation did not help them making the desired prediction about TS.

The students then decided to consider both sets of data, TD and TS, in an attempt to find a relationship between them because the students were unsure whether finding individual values such as mean and standard deviation for each TD and TS would help.

In this modeling cycle Adam's group *described* the scatterplot as having almost no correlation and found certain statistical values like standard deviation and mean. When they *translated* back to the original problem of making a prediction and tried to *verify* the usefulness of their results, they realized that finding such values did not make any sense in the context of the problem. Eventually, they started a new modeling cycle as described below.

Second Cycle: Applying Standard Procedures.

After making a scatterplot, the group spent time looking at the graph. As Caty had previously taken a traditional statistics course, in which she was taught about lines and curves of best fit, she suggested a line as a best fit for the data. It is interesting to note that while Caty thought that the shape of the scatterplot was not linear, she still suggested line as a best fit, as seen in their exchange below. One of the possible explanations could be that in her introductory statistics course she almost always used lines as a best fit for any given scatterplot without thinking about the underlying assumptions that a line of best fit makes (e.g., a strong linear relationship between the variables).

The group plotted a movable line, least squares line and median-median line, which are all built-in functions in *Fathom™*, but were not convinced that the lines represented the best fit in regard to making the desired prediction. Therefore, they rejected the notion of linearity and moved on to discussing which curve would make the most sense. During the second cycle Adam's group *described* the relationship between TD and TS as linear and hence plotted the best-fit lines available in *Fathom™*. They even plotted a movable line and *manipulated* it to make it fit to the scatterplot. When they *translated* back to the original problem and tried to *verify* their results they concluded that a linear relationship between TD and TS did not make any sense for the TDTS problem. So they shifted to another interpretation of the problem, that is, the third modeling cycle.

Third Cycle: Thinking About Correlation.

This group now started thinking about several functions (linear, polynomial etc…) in this cycle. They began discussing the different graphs in the third cycle in order to figure out which shape would best describe the scatterplot after rejecting the notion of linearity. After introducing the sliders[1], which are built in to *Fathom*^{TM}, Cathy and Beth attempted to move the sliders to fit the curve to the data, when Adam *translated* his focus back to the scatterplot he was not convinced that there was enough correlation between the TD and TS in order to make an appropriate prediction. Therefore, upon his suggestion, they returned to the original problem and started a new modeling cycle.

Fourth Cycle: Focusing on Small Chunks of Data.

The group again looked at the scatterplot and began focusing on individual data points and small groups of data points after Adam pointed out that "not enough correlation [existed] to make a prediction." They then realized that some points on the scatterplot were "throwing off" any pattern in the plot and began concentrating on the individual points and whether each point was throwing off a potential pattern in the scatterplot.

When no obvious pattern was apparent in the scatterplot, the group decided to concentrate on the small groups of data points in order to see if a pattern existed. Then, they decided to make another scatterplot with the TDs between 20 and 30 minutes. This scatterplot did not help them because no pattern was obvious in the new scatterplot. In fact, this plot was less organized than the original scatterplot. They made the table and plotted the graph, however, the new scatterplot did not help them finding a pattern. After trying out small groups of data points, Adam's group was convinced that they needed to look at all of the data points in order to find any patterns.

Fifth Cycle: Making a Prediction Using a Model.

The group then began arguing about patterns in the scatterplot of the whole data. Adam suggested that there was "no correlation" in the data and no obvious pattern, while Caty suggested that unless they found a pattern, they could not make the desired prediction. This argument started a discussion about the correlation between the TD and TS. Each of the students had a different idea about the correlation. Adam stated that there was "no correlation" at all, Beth stated that there may be some "coincidental correlation" and Caty stated that there was "some correlation" that may help in finding a pattern and making a prediction. As the problem asked them to make a prediction, they had to agree with Caty in order to proceed with the problem.

After the group decided that they *have to* look for a pattern in order to make a prediction, they started investigating the scatterplot more closely for a pattern. Caty suggested that they should look for "curvy lines" because the TSs increased "gradually" with the TDs. They came up with two curves on the scatterplot because, according to them, no single curve described the data the best. They discussed "somehow" combining the two curves in order to come up with a single curve to make a more accurate prediction, but did not have enough mathematical tools or skills to do that.

In their final solution, they used these curves to make their prediction. They also used the sliders to shrink and stretch the curves to fit the scatterplot. This modeling cycle culminated in their final model. During this cycle all the members of Adam's group had different *descriptions* of the scatterplot. While *translating* back and forth from the original problem of making a prediction to the scatterplot they came up with a single idea of plotting two curves. Finally, when they *verified* their result with the original problem it made sense to them that the final prediction would lie between the two curves.

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[1] *Fathom*^{TM} sliders are used to vary the values of coefficients in the functions. You can interact with the sliders and note the changes in the functions.

From the analysis of the data, it is clear that the idea of curve fitting not only evolved, but changed significantly in the students' thinking throughout the TDTS problem session. The students encountered ideas about correlation; how to shrink, move and stretch curves; how to decide the "fit" and how to combine formulas. This simultaneous awareness of different concepts caused their idea of curve fitting to evolve.

Discussion

Results of this study provide insights into the nature of students' developing models of curve fitting. The data supports the claim that models evolve from being uncoordinated, unstable and undifferentiated to being increasingly coordinated, stable and differentiated as their work on the TDTS problem progressed.

Significance of the Study

This study contributes to several areas of the mathematics education research, including statistics education research and mathematical modeling, and problem solving in undergraduate mathematics. The significance of the products of this study can be assessed in the following ways:

1. This study adds to the existing statistics education research by investigating students' learning of a statistical topic, curve fitting, which has not been the subject of much research until now.
2. This study also introduces the use of a conceptual framework, MMP, which can be used to investigate the learning of other statistical topic areas. It offers an in-depth analysis of how students learn a particular topic via solving a real world problem.

Finding the nature of students' models also lays the groundwork for activities that could enhance students' understanding of the topic under investigation and, ultimately, improve instruction at college- and school-level mathematics and statistics.

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CRITIQUING THE REASONING OF OTHERS: DEVIL'S ADVOCATE AND PEER INTERPRETATIONS AS INSTRUCTIONAL INTERVENTIONS

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This study investigated the ways in which college mathematics teachers might encourage the development of student reasoning through critiquing activities. In particular, we focused on identifying situations in which the instructional interventions were implemented to encourage the critiquing of arguments and in which students explained another's reasoning. Data for the study come from two teaching experiments – one from the domain of combinatorics and the other from real analysis. Through open coding of the data, Devil's Advocate and Peer Interpretations emerged as effective interventions for the creation of sources of perturbation for the students and for assisting in the resolution of a state of disequilibrium. These two interventions differ in design and in the type of reasoning students evaluate, but they both provoke students to further develop their reasoning, and therefore their understanding. We discuss the implications of these interventions for both research and teaching practice.

Key words: Combinatorics, Perturbation, Real Analysis, Student Reasoning, Teaching Practice

Introduction and Research Questions

The purpose of this study is to examine how mathematics teachers may leverage college students' reasoning and understanding of advanced mathematics by engaging them in critiquing the reasoning of others. Critiquing activities may perturb students who are not familiar with the ideas employed by the argument given or those whose reasoning might have a different base. It is hoped that through such activities students can grasp the essence of the argument, develop their ability to distinguish correct logic from flawed, and explain the flaws if they exist. Hence activities of critiquing others' arguments would provide students a way to resolve their perturbation and to develop their mathematical reasoning. In fact, it might be difficult for a student to find flaws in his or her own reasoning. However, critiquing arguments written by others or fictional characters might enable a student to reflect on his or her own reasoning and understanding. In line with the standpoint, the recently released Common Core State Standards (CCSS, 2011) state that mathematically proficient students of all levels develop the skills to construct viable arguments and critique the reasoning of others. The mathematics education community also has demonstrated increasing interests in using critiquing activities as a research method (Lockwood, 2011; Selden & Selden, 2003) or an instructional intervention (Halani, 2012; Kasman, 2006; Roh & Lee, 2011).

This paper focuses on teachers' creation of the source for student perturbation and their ways to facilitate students' understanding of new ideas through critiquing activities in the domains of combinatorics and real analysis. We consider that such understanding is a result of what Steffe et al. (1983) called "second-order models." This refers to an observer's model of the subject's knowledge, or the models that "observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (p. xvi). While a body of research explored teachers' models of their students' mathematics (e.g. Courtney, 2010; Silverman & Thompson, 2008), this study emphasizes students' construction of the second-order models of their peers or fictional characters and teachers' role in the development of such a construction. This study addresses the following research question: How might an instructor use critiquing activities to create sources of potential student perturbations along with the ways to resolve such perturbations?

Theoretical Framework

Under the perspective of constructivism (Von Glasersfeld, 1995) adopted in this study, the role of an instructor is to orient students' cognitive processes and aid students with their construction of mathematics. One way that a teacher might do exactly this is to create sources of potential perturbation along with ways to resolve it in order to encourage students to develop their reasoning, which is what we call *instructional provocations* in this paper.

Devil's Advocate (DA) and Peer Interpretations (PI) are two instructional provocations which extend Rasmussen and Marrongelle's (2006) "generative alternatives" and which are designed to encourage student explanation and justification. The first, *Devil's Advocate* (DA), refers to an incorrect or atypical argument provided to students by the instructor for evaluation. The purpose of this provocation is to highlight cognitive conflicts or to raise awareness of certain aspects of a topic. The students would refute the argument if they disagree or provide justification for parts of the argument otherwise. The second, *Peer Interpretations* (PI), refers to a student's interpretation of a peer's argument, at the prompting of the instructor. The purpose of this provocation is to highlight similarities and differences in thinking and to allow students to learn from each other. The student interpreting the peer's argument would often include his or her own reasoning and thinking into the interpretation.

Both DA and PI have the potential to create sources of perturbation or help resolve such perturbation. However, DA and PI differ from two aspects: First, when these provocations are designed is different – while there is often a predefined argument with using DA, PI comes about in the classroom as the interactions between students dynamically develop. Second, the types of reasoning presented are usually different. Since DA is typically prepared by the instructor prior to the lesson, DA is well-formed so that students analyze the essence of an argument. On the other hand, through PI, students analyze their peer's reasoning which is often in a formative stage so that students pull out the essence of the argument.

Methods and Analysis

Data for this study come from two teaching experiments conducted at a large southwestern university - one from the domain of combinatorics, the other from real analysis. Each teaching experiment constituted of 9 or 10 teaching sessions, and involved a teaching agent and two high-performing undergraduate students, with no prior experience in the subject. The rationale for including two teaching experiments is straight-forward: The content of mathematics courses typically involves the study of either discrete or continuous structures, yet in both cases, it is imperative to push students to further develop their reasoning and understanding by critiquing the reasoning of others.

The analysis of the data was conducted in several phases. Content logs and full transcripts of all videos were first created. We, as the research team, then used an open coding system (Strauss & Corbin, 1998) to identify situations where a student explained someone else's reasoning. Following this, we used open coding to classify the instructional interventions (e.g., DA and PI) which had the intention of encouraging students to critique the reasoning of others. The coded data were then reviewed for consistency.

Results

We found that in both teaching experiments, each instructor pushed a student to further develop his or her reasoning and understanding by critiquing the arguments of others. In particular, two provocations, DA and PI, emerged from the data analysis. Both DA and PI were implemented throughout the study, but only a few illustrative examples are provided here.

Case 1: PI in the Combinatorics Study - The ARIZONA Activity

In the combinatorics teaching experiment, the instructor, the first author of this paper, asked Kate and Boris to determine the number of ways to rearrange the letters in ARIZONA. First, Kate explained her idea:

"*I disregarded the facts that there's a repeated letter and I just said 'how many ways can* […] *you arrange these seven letters?' and that's going to be 7!. But, um, you're going to have to take some of those out.* […] *I think for every* […] *one possible order of the letters, you're going to have another* […] *that's the same because there's only one letter that is repeated. So like, if we had like just a random RZIANOA there's going to be two ways. By this, there's 7!, which count that* [RZIANOA] *twice. So I think you just divide 7! by 2 to take those out.*"

Kate determined her solution of 7!/2 by first imagining that she was permuting 7 distinct letters, though she did not use the term "distinct." She recognized that the repeated A's would actually mean that she had counted twice as many permutations as she wanted. Boris also tried to permute distinct objects first, but he tried to take away one of the A's before permuting the other 6. He then tried to insert the remaining A into the permutations he had just created, determining a solution of $6!\times 6$. The instructor asked Boris to explain Kate's argument. He responded, "*Well she went and found the total number of ways that you could arrange seven unique letters, which would be seven factorial, and she said that for each of those* […] *you're counting twice as many possibilities as you should, because of the two different A's you're assuming that those are unique letters. Like A_1 or A_2 when they're really just both A's. So you have to take out half of those*."

Notice that Boris did not repeat Kate's reasoning verbatim and instead reinterpreted it while adding further justification, thus indicating that he had built a second-order model of his peer's argument in order to extract its essence. Boris experienced disequilibrium when he realized that the two solutions the students had created could not both be correct and indicated that he believed Kate's argument to be correct by stating that he was not sure what he was counting twice. Boris eventually resolved his perturbation by recognizing mistakes in his own argument through comparing it with Kate's idea for dealing with duplicates. Thus it seems as if the instructor's request that Boris explain Kate's argument was an effective implementation of PI – it not only created a source of perturbation, but helped Boris resolve it as well. In the next session, Boris assimilated this same way of thinking in order to determine that there were 4!/2 ways to permute two blue, one red, and one black counter.

Case 2: PI in the Real Analysis Study – Proofs involving Inequalities

The students in the real analysis study, Sam and Jon, attempted to prove "For any $a,b\in\mathbb{R}, |a-b|\geq||a|-|b||$." In order to do so, they were directed towards first proving two lemmas: "Let $a,b\in\mathbb{R}$, then (1) $|a|-|b|\leq|a-b|$ and (2) $-(|a|-|b|)\leq|a-b|$." The students were already familiar with the triangular inequality and what they called Theorem 1 (*iii*): "Let $a,b\in\mathbb{R}$, then $|ab|=|b|\cdot|a|$." After the students had written a proof for Lemma 1 together, they spent some time thinking about Lemma 2 separately. Sam's written work can be seen in Figure 1. He wrote down Lemmas 1 and 2 on top of his paper, along with a theorem he thought he might want to use T2 (*ii*) in Figure 1. This written scratch work hinges on the idea that $|b|=|b-a+a|\leq|b-a|+|a|$ by the triangular inequality ("TE" in Figure 1). Sam stated that $|b-a|=|a-b|$ by Theorem 1(*iii*) and by the fact that $b-a=(-1)(a-b)$. After Sam explained his thought process, the instructor of the session, the first author of this paper, asked Jon to explain what Sam had just said. Jon indicated that he thought Sam had used Lemma 1, but Sam interrupted and pointed out that he had applied the triangular inequality. When the instructor asked Jon to state his understanding of Sam's argument, Jon realized that he did not fully understand Sam's argument. In fact, in his reflection that evening, Jon wrote, "[*the instructor*] *did ask us to explain our thinking several times, to articulate our logic. This helped me see some holes in my understanding. For example, before she asked me to explain what Sam did for Lemma* [2], *I thought he was manipulating* *the inequality in order to use Lemma 1. Instead, he was using the "adding zero" technique and applying the triangular inequality in order to set up it up for Theorem* [1(*iii*)].” This exemplifies a case when the instructor requested Jon his interpretation of Sam’s argument, Jon realized that his model of Sam’s argument was not what Sam intended. The instructor’s implementation of the PI was therefore effective in creating Jon’s perturbation which was later resolved as the two students collaboratively worked to complete their proof of the lemmas.

$$|b| - |a| \geq -|a-b| \quad \text{by Lem 1}$$

$$\text{To show / WTS} \quad |b| - |a| \leq |a-b| \quad \text{Lem 2}$$

$$|a-b| \leq |a-c| + |c-b| \quad \text{T2 (ii)}$$

$$|b| = |b-a+a| \leq \underline{|b-a|} + |a| \quad \text{by T.E.}$$

Figure 1: Sam's scratch work to prove Lemma 2: $-(|a|-|b|) \leq |a-b|$

Case 3: DA in the Real Analysis Study - The Vice of Inequality

At the third session of the real analysis study, the instructor, the third author of this paper, presented an alternative argument to the students in an attempt to highlight the importance of the order of quantifiers. First, she asked Sam and Jon the following question: “Would there be $x \in \mathbb{R}$ satisfying $\forall \varepsilon > 0, |x| < \varepsilon$?” After the students were given a few moments to think, the instructor asked Sam to share his thoughts. He responded “*Okay, so I was thinking that the only x that will work for this* […] *would be 0, because* […] *you could get x really small* (pinches fingers together)*, give it a really small value, but it's still not gonna work for any ε greater than 0 because the limit of that is 0. So, it's* [*x* is] *always going to be infinitessimally larger than 0, which means it* [ε] *can always be smaller than any positive x* [which is a contradiction]. *So, it would only work for 0.*” The instructor then presented an alternative solution to the given statement, asserting that there are infinitely many possible values of *x* based on the following theorem: If *x* is between $-\varepsilon$ and $+\varepsilon$, then $|x| < \varepsilon$. The alternative argument included an error in the order of quantifiers by assuming that *x* can be chosen based on the value of ε, which is not the case in the original statement. The instructor asked Sam to discuss his reasoning about the alternative argument and he responded that the alternative argument cannot be a valid argument for the given statement. When pressed to discuss his reasoning for how he could tell, Sam had difficulty in doing so. This difficulty caused the perturbation necessary for Sam to create models of both the given statement and the instructor’s alternative argument. After prompting from the instructor Sam presented his model of the given statement as there is “*one value of x, for which the value of |x| is always less than ε.*” His representation of the instructor’s alternative argument was that “*basically, you pick some value of ε and then it tells you for what values of x,* (that)$|x| < \varepsilon$.” Once he built these models, Sam resolved his perturbation. When prompted by the instructor, Sam was able to describe the difference between the two statements, in which the order of the quantifiers had an impact on the meaning of the statements. The instructor’s introduction of the alternate argument raised a cognitive conflict that was resolved by the student Sam, which indicates the instructor’s use of the alternative argument was an effective use of DA.

Case 4: DA in the Combinatorics Experiment - Tree Diagrams

In the combinatorics teaching experiment, the instructor, the first author of this paper, also often provided alternative arguments to the students for evaluation. One example where she did so was when Kate and Boris were asked to solve the following task, which is adapted from Batanero et al.’s (1997) questionnaire:

Situation: Four children: Alice, Bert, Carol, and Diana go to spend the night at their grandmother’s home. She has two different rooms available (one on the ground floor and another upstairs) in which she could place all or some of the children to sleep.

Question: In how many different ways can the grandmother place the children in the two different rooms?”

Boris determined the answer to be 2^4, explaining that there were two rooms that the first person could go to, for each of those possibilities, there were two possibilities for where the second person could go, and so forth. Then the instructor provided the tree diagram shown in Figure 2 as a solution provided by a supposed former student, Annette. At first Kate was confused by the representation and stated, “*I don’t even know what that means*.” After examining the tree diagram for a while, Boris stated, “*So I guess it's like doing it per person.* […] *She [is] pulling it apart like one person at a time. For the first person, they can either go to the ground floor or the upper floor. So like, you hold one constant. Say the first goes to the ground floor*. […] *And then the next person could go to the ground floor or the upper floor. So then, they both go to the ground floor for those* […] *four possibilities* (points to the top four leaves of the tree). *After that point* (points to the vertex G G _ _) *they* [*the third person*] *can go to the ground floor or the upper floor. So if they go to the ground floor* […] *and again there are two more possibilities for each of those. So there's two more there*.”

Boris had made a connection between Annette’s solution and the idea of holding something constant. He was able to pull out the essence of Annette’s argument and explain it in his own words. Following Boris’ interpretation of Annette’s solution, Kate immediately responded, “*so this is just a graphic representation of what you* [Boris] *were saying*.” This indicates that Kate, as well, was able to grasp the essence of Annette’s solution and connect it to Boris’ original solution even though she originally experienced some perturbation and did not immediately understand the tree diagram. The instructor’s intention in providing Annette’s solution was to raise awareness of the existence of visual representations for their current ways of thinking. Since the students were successful in building connections between Annette’s solution and Boris’ original solution, therefore further developing their reasoning and understanding, we consider the instructor’s introduction of Annette’s solution to be an effective implementation of DA.

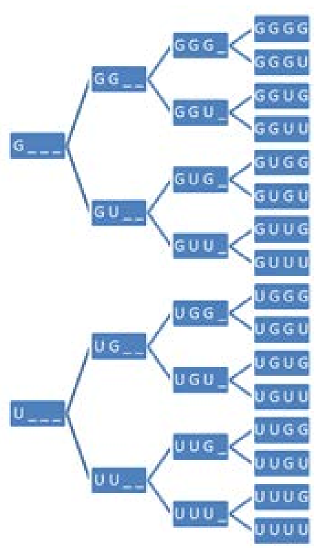

Figure 2: “Annette’s Solution” provided through Devil's Advocate

Later in that session, Kate and Boris were attempting to determine the number of 3-letter "words" that could be formed from the letters *a*, *b*, *c*, *d*, *e*, and *f* if repetition of letters were allowed and the letter "*d*" must be used. They first over counted and found the answer to be $3\times6\times6$. The instructor provided a DA that determined the solution to be $6^3-5^3=91$. The students realized that both solutions could not be correct but they both had trouble identifying which solution was correct and which involved a flaw in reasoning. Boris and Kate used the tree-diagram in Figure 3 to solve the problem using a third method and confirm that the alternative solution provided was correct. Their tree-diagram differs vastly from the one supposedly written by Annette in the earlier task – the leaves in Figure 2 each represent an element of the solution set, but in Figure 3 all of the leaves are missing, many of the trees have only a root, and the use of slots to indicate where other items would be placed is inconsistent. However, Annette's idea of using a tree diagram to visually represent the elements being counted was adopted by the students. This seems to be an example of actor-oriented transfer (Lobato & Siebert, 2002).

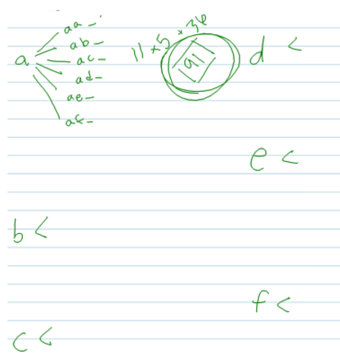

Figure 3: Kate and Boris's Transfer of Tree Diagrams

Discussion

We found that both Peer Interpretations (PI) and Devil's Advocate (DA) were effective instructional interventions designed to encourage students to critique the reasoning of others. Such activities challenged the students to understand the mathematics of a peer, fictional or otherwise, and provided opportunities to deepen conceptual understanding or reasoning. Indeed, these provocations could create sources of perturbation or assist in the resolution of such perturbation. In some cases, like in PI example from the real analysis study, the provocation may simply accomplish one of these tasks. In other cases, like both combinatorics examples and the DA example from real analysis, the provocation could both create the perturbation and help with its resolution.

This study has implications for both research methods and teaching practice. As shown, both DA and PI were effectively implemented in teaching experiments (Steffe & Thompson, 2000) and provided opportunities for further discourse, thus allowing the researchers to better understand the students' reasoning. We contend that DA could be implemented in clinical interviews (Clement, 2000) in a similar manner to help the researcher confirm his or her model of the student's mathematics. Because it is possible that a student's mathematics may change as a result of DA, we recommend the use of such provocation at the end of an interview. In a classroom, a teacher could implement either DA or PI to highlight differences in reasoning and raise or resolve cognitive conflict. In both cases, the reinterpretation by a student can include the student's own thinking and reasoning, while also including his or her own interpretation of the original argument. The interpreting student may adopt the meaningful aspects of the other argument into their own model of the situation. Indeed, we found evidence of this adoption in the student's assimilation of the idea to new situations in both combinatorics episodes discussed in this paper. Both DA and PI were effective in pushing students to further develop their reasoning by critiquing the reasoning of others.

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VENN DIAGRAMS AS VISUAL REPRESENTATIONS OF ADDITIVE AND MULTIPLICATIVE REASONING IN COUNTING PROBLEMS

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This case study explored how a student could use Venn diagrams to explain his reasoning while solving counting problems. An undergraduate with no formal experience with combinatorics participated in nine teaching sessions during which he was encouraged to explain his reasoning using visual representations. Open coding was used to identify the representations he used and the ways of thinking in which he engaged. Venn diagrams were introduced as part of an alternate solution written by a prior student. Following this introduction, the student in this study often chose to use Venn diagrams to explain his reasoning and stated that he was envisioning them. They were a powerful model for him as they helped him visualize the sets of elements he was counting and to recognize over counting. Though they were originally introduced to express additive reasoning, he also used them to represent his multiplicative reasoning.

Key words: Additive reasoning, Combinatorics, Multiplicative reasoning, Representations, Set Theory

Introduction and Research Questions

Piaget and Inhelder (1975) contend that children's combinatorial reasoning is a fundamental mathematical idea based in additive and multiplicative reasoning. However, the research indicates that students of all ages often struggle with counting problems (Batanero, Godino, & Navarro-Pelayo, 1997; Eizenberg & Zaslavsky, 2004; English, 1993; Hadar & Hadass, 1981). Though some studies have adopted counting problems as the backdrop for research in other aspects of student learning (Eizenberg & Zaslavsky, 2004; Fischbein & Grossman, 1997), very little research has been conducted on combinatorial reasoning. Shin and Steffe (2009) began to investigate how middle school students dealt counting problems based on their additive and multiplicative reasoning and determined levels of enumeration that appeared in the students' behavior: additive, multiplicative, and recursive multiplicative enumeration. Though they provide examples of students' visual representations of the elements being counting, Shin and Steffe (2009) do not focus on how students' reasoning can be expressed in their representations. Further, the problems their students encountered did not address operations more complex than permutations of distinct elements.

Research on Venn diagrams and their use in discrete math or probability courses seems to be of two minds. On one hand, Fischbein (1977) states that Venn diagrams are powerful models that can be used to solve a wide range of problems. Indeed, some combinatorics texts (e.g. Bogart, 2000) present Venn diagrams as a visual representation for basic counting problems. On the other hand, it has been reported that students have trouble using Venn diagrams and visualizing set expressions (Bagni, 2006; Hodgson, 1996) to the extent that some authors have recommended the removal of these representations from basic probability courses (Pfannkuch, Seber, & Wild, 2002) and some combinatorics texts (e.g. Tucker, 2002) introduce them only while solving complex counting problems. This study extends the current research by investigating combinatorial reasoning in relation to students' visual representations, particularly Venn diagrams. Thus, this study attempts to answer the following question: How could a student use Venn diagrams to express the additive and multiplicative reasoning he employs while solving counting problems?

Theoretical Framework

The primary tenant underlying this study is that mathematical knowledge is not received through the senses or communication but must be actively built up by the cognizing subject (Von Glasersfeld, 1995). According to Harel (2008), there are two categories of mathematical knowledge: ways of understanding and ways of thinking. The reasoning applied in a particular mathematical situation – the cognitive product of mental acts – is known as a *way of understanding*. On the other hand, *ways of thinking* refer to what governs one's ways of understanding and are the cognitive characteristics of mental acts. Ways of thinking are always inferred from ways of understanding.

The author developed a preliminary framework of students' ways of thinking about the set of elements being counted, known as the *solution set*, in basic counting problems (Halani, 2012a, 2012b). Three of these ways of thinking are relevant to this study and are summarized in Table 1. While engaging in the first, *Union* thinking, a student will first think globally and envision the solution set as the union of smaller subsets which he or she may believe to be distinct before taking the sum of the sizes of these subsets. The next two ways of thinking both involve answering a question that has not been asked. In *Deletion* thinking, a student will consider a related problem whose solution set contains a subset which is in one-to-one correspondence with the original solution set and then find an additive relationship between the sizes of the solution sets. In *Equivalence Classes* thinking, a student will consider a related problem whose solution set can be partitioned into equivalence classes, each of which correspond to an element of the original solution set, before finding a multiplicative relationship between the sizes of the solution sets. By their very definitions, Union and Deletion have their roots in additive reasoning and are therefore shown in orange in Table 1, whereas Equivalence Classes is multiplicative and shown in purple.

Way of Thinking	**Description**	**Example of a task whose solution could be driven by this way of thinking:**
Union	Envision the solution set as the union of smaller subsets. Add the sizes of the subsets.	How many 3-letter "words" can be formed from the letters *a*, *b*,*c*,*d*,*e*,*f* if repetition of letters is allowed and *d* must be used?
Deletion	Consider a related problem with solution set *C* which contains a subset, *B*, which has a bijective correspondence with the solution set of the original problem, *A*. Find an additive relationship between the the original and the new solution set.	How many 3-letter "words" can be formed from the letters *a*, *b*,*c*,*d*,*e*,*f* if repetition of letters is allowed and *d* must be used?
Equivalence Classes	Consider a given task with solution set *A*. Consider a related problem with a solution set *S* which can be partitioned into equivalence classes of the same size – each one of which corresponds to an element of *A*. Find a multiplicative relationship between the solution sets.	How many permutations of *MISSISSIPPI* are there?

Table 1: Some Ways of Thinking about Combinatorics Solution Sets

Methodology

Data for this study come from a teaching experiment (Steffe & Thompson, 2000) conducted at a large southwestern university in the United States. Al, a freshman enrolled in a second-semester calculus course, participated in nine teaching sessions with the researcher over a four-week period. Tasks for this study involved the operations of arrangements with and without repetition, permutations with and without repetition, and combinations. In addition, it is known that students do not always interpret combinatorial tasks in the same manner that the mathematical community does (Godino, Batanero, & Roa, 2005). As a result, tasks for this study were separated into two parts: a situation and a question (or questions). See Table 2 for the tasks discussed in this paper. Following the completion of many of the tasks, the researcher implemented the Devil's Advocate instructional provocation (Halani, Davis, & Roh, this issue) by presenting alternate solutions written by supposed previous students to Al for evaluation. He reinterpreted and justified the solution if he believed it to be correct, and refuted it if he disagreed. Through these alternate solutions, many visual representations such as tables, tree diagrams, Venn diagrams, slots, and mapping diagrams were introduced. Venn diagrams were first presented during the fourth session; however, set theoretic language was not employed. For example, the term "overlap" was used instead of a formal term such "intersection." The concept of a universal set was not introduced until the eighth session.

Session	Task	Statement
5	14(vi)	*Situation*: Suppose we have the letters a,b,c,d,e,f and we are forming three-letter strings of letters ("words") from these letters. *Question*: How many 3-letters "words" can be formed from these letters if repetition of letters is allowed and the letter "*d*" must be used?
6	16(iii)	*Situation*: A university decides that sorority names can be three-letters chosen from the following Greek letters: $\Gamma, \Delta, \Theta, \Lambda, \Pi, \Phi, \Psi, \Omega$. *Question*: How many sorority names can be formed from these letters if repetition of letters is allowed and the letter "Θ" must be used?
8	30(v)	*Situation*: Consider the word *WELLESLEY*. We will be forming "words" from these letters. *Question*: How many "words" can be formed from the letters in "WELLESLEY" if we need 4-letter words, each letter may be used multiple times, and we must use the letter "E"?
9	31(i)	*Situation*: Consider the word *MISSISSIPPI*. We will be forming "words" from these letters. *Question*: How many "words" can be formed from the letters in "MISSISSIPPI" if we need 11-letter words created by rearranging the letters provided?

Table 2: Relevant Tasks Implemented in the Study

There were a few phases of data analysis. Following each session with the student, content logs were created including summaries of the session along with observational, methodological, and theoretical notes (Strauss & Corbin, 1998). After the data were collected, each session was transcribed. Open coding (Strauss & Corbin, 1998) was used to identify the visual representations employed by the student. The student's ways of thinking and the type of reasoning in which the student engaged were also analyzed.

Results

After Venn diagrams were introduced during the fourth and fifth sessions of the study, Al demonstrated that he could visually represent his reasoning using Venn diagrams while engaging in the following ways of thinking: Union, Deletion and Equivalence Classes. As indicated in Table 1, the first two ways of thinking draw primarily on additive reasoning and the last draws upon multiplicative reasoning. He often employed Venn diagrams when asked to explain his reasoning and stated that he was envisioning them as he solved the tasks.

Venn Diagrams to Represent Additive Reasoning

Al recruited Venn diagrams as a tool to explain his additive reasoning employed while engaging in Union and Deletion thinking to solve counting problems.

Union. During the fifth session, Al was asked to complete task 14(vi) from Table 2. He first over counted and found that there were $3\times6\times6$ "words." Three alternative solutions were presented via Devil's Advocate (Halani et al, this issue),one driven by Deletion thinking and two which relied on Venn diagrams. In the next session of the study during a mid-study test, Al was asked to complete task 16 (iii) from Table 2. He first drew three sets of three slots and wrote 1 in the first slot of the first set, 1 in the second slot for the second set and 1 in the third slot for the third set. Each set of slots corresponded to a different subset of the solution set, based on the location of Θ. This indicates that he was engaging in Union thinking. His solution is shown in Figure 1. While determining the number of possible options in each case, Al was careful to avoid over counting by partitioning this union of sets. He multiplied the numbers in the slots for each set and then took the sum of these products to get 64+56+49.

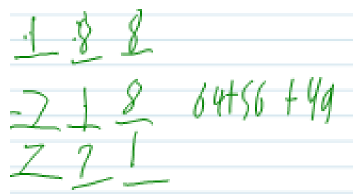

Figure 1: Al's solution to task 16(iii)

Al was asked about his confidence in his solution. In order to explain, Al referenced doing a similar problem during the fifth session and immediately drew a Venn diagram (not shown) to illustrate his additive reasoning. He explained his thinking:

"*I was trying to think, ok, we have each of these different I guess groups of where it can be. Like with this one I could tell that you have a group where it's* [Θ *is*] *the first letter, a group where it's the second letter, a group where it's the third letter* (draws three overlapping circles). […] *And I knew that for all of this* (indicates all of the first set)*, I can only count this much of this* (indicates the elements in the second set excluding the first set)*, and I can only count this much of this* (indicates the elements in the third set which have not yet been counted)."

Even though Al did not draw a Venn diagram during his counting, it seems as if he may have been envisioning one from his explanation. It is clear that while he was counting, he was attending to the intersection of the subsets based on the location of Θ. The first Venn diagram Al drew was hard to read so Al drew a second one (see Figure 2). When Al was asked to compare his current thinking about this type of problem to the similar problem he said they had encountered in the previous session, his response follows:

"*Well, I think before, I would list them all, or I guess I didn't have as clear of a way of understanding that repetitions occur in this type of problem.* […] [*Now*] *I'm using some way to define what these three sets are. And I'm defining* […] *the first set as places where the first variable is theta. Defining that group* (points to second circle in Figure 2) *as where the second variable is theta, and that group* (points to third circle) *where the third variable is theta. And by defining them, I guess I was kind of realizing that they overlap when both the first and the second requirements are met. Or when the first and the third. Or when all three are met. So by kind of knowing that the only place I'm going to have* *repetitions is where that’s true and that’s true* (points to intersections of two sets), *or when all three are true, then I could kind of look for it better*.”

Here, it is clear that he was envisioning this Venn diagram even though he did not originally visually represent his reasoning while solving the task. From his comparison of his current thinking to his previous thinking, it appears as if Venn diagrams helped him clearly picture what he was counting so that he was better able to avoid over counting.

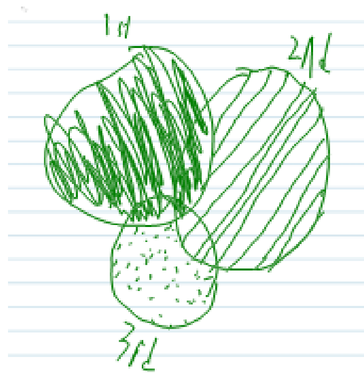

Figure 2: Venn diagram for Union thinking for task 16(iii)

Deletion. Al employed two different variations of Venn diagrams to represent his Deletion thinking – one in the fifth session and a second during the eighth.

Session Five. Devil’s Advocate (Halani et al., this issue) was used to provide the following argument for task 14 (vi) – it is driven by Deletion thinking and written by a supposed former student, Carrie: “We first determine the number of 3-letter ‘words’ possible regardless of whether ‘*d*’ is used: $6\times6\times6$. Then, we determine the number of ‘words’ which do not include ‘*d*’: $5\times5\times5$. Thus, there are $6^3-5^3=91$ ‘words’ which include the letter ‘*d*.’”

Al was asked if he had seen an argument like Carrie’s before. He had naturally engaged in Deletion thinking for previous problems; however, his response refers to Venn diagrams:

“*It’s kind of like the Venn diagram but it’s kind of not.* […] *It’s kind of like the Venn diagram, cause in the Venn diagram you have kind of these two circles* (draws the two circles in Figure 3), *but she was saying that is with* ‘d’ (writes “*d*” in the portion in the right circle that is not in the left circle) *and then this is with all the possibilities without* ‘d’ (writes “~~*d*~~” in the portion in the intersection of the circles). *So she just kind of ignored this* (scribbles in the portion of the left circle that is not in the right circle)*...this is all the possibilities with* ‘d,’ (indicates the entirety of the right circle) *then she subtract[ed] the* [*possibilities*] *without a* ‘d’ *to figure out how many just have* ‘d’”

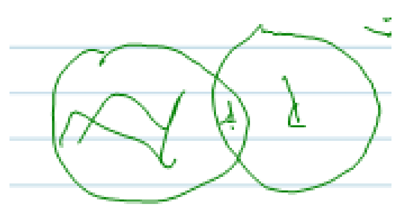

Figure 3: Venn diagram for Deletion thinking from session 5

At this point in the study, Venn diagrams had been introduced to solve other questions involved in task 14, but they all involved two sets with a non-empty intersection. As mentioned in the previous section, the concept of a universal set had not been introduced. Thus, Al’s Venn diagram for Carrie’s reasoning was based off the Venn diagrams he had seen before and therefore involved two sets with a non-empty intersection. His representation for Carrie's Deletion thinking involved counting everything in the right circle of Figure 3 and then subtracting the number of elements in the intersection. Thus, it seems as if Al understood that Carrie constructed a new problem (that of determining the total number of 3-letter words) and then found an additive relationship between the new solution set and the original solution set.

Session Eight. In the eighth session, Al tried to solve task 30 (v) from Table 2. At first, he over counted and found the answer to be $\binom{4}{1} \cdot 5^3$ because he considered places the E could go and then determined that there 5 choices for each of the remaining spots. The researcher reminded Al that he should ensure that he had counted everything he wanted to count and that he had not counted the same thing more than once. He quickly realized his mistake and determined the solution to be $5^3 + 5^2 \times 4 + 5 \times 16 + 4^3$ by engaging in Union thinking with subsets determined by the location of E and then carefully ensuring he does not over count the intersections of these subsets. He explained that it reminded him of the "*Venn diagram problem and that kind of whole picture* (draws Venn diagram with 4 overlapping circles) *just popped into my head*." Once again, it is clear that he is envisioning a Venn diagram for Union thinking although he did not draw it while counting.

The researcher reminded Al of Carrie's argument for task 14 (vi). Al said, "*So in this case, it would be* $5^4 - 4^4$." At this point, the researcher introduced the Venn diagram shown in Figure 4. She explained that the box represented the whole universe that they were concerned with. She then sketched the four circles representing subsets based on the location of E. The researcher asked Al what was actually being counted. Al said that the entire box was being counted and then everything that was not in the circles was being subtracted. As before, Al demonstrated that he could use Venn diagrams to represent his additive reasoning employed while engaging in Deletion or Union thinking.

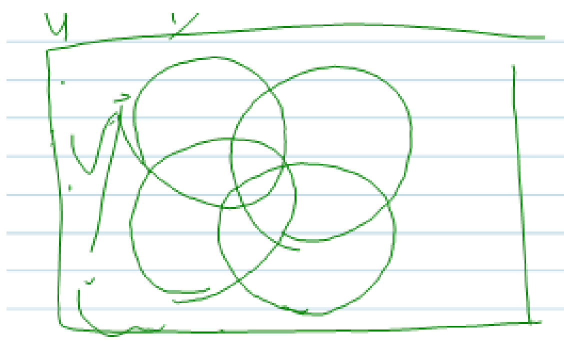

Figure 4: Venn diagram for Deletion thinking from session 8

Venn Diagrams to Represent Multiplicative Reasoning

Mapping diagrams were introduced as visual representations for Equivalence Classes thinking, yet Al never employed them himself. Instead, Al seemed to make a connection between Deletion and Equivalence Classes thinking and began used Venn diagrams with a universal set to represent his multiplicative reasoning in the latter case. This seems to be an example of actor-oriented transfer (Lobato & Siebert, 2002). In the last session of the study, the researcher asked Al to give some examples of visual representations. His response regarding Venn diagrams is below:

> "*There's been kind of Venn diagram style overlap* (draws the Venn diagram with a rectangle and three circles shown in Figure 5) *and then there's been kind of a way that you could also figure that out by taking the whole* (indicates entire rectangle) […] *and then you're dividing out* […] *this kind of bad area* (shades in the complement of the three

circles) [...] *Because when it comes to situations with* […] *a lot of different overlaps* […] *like if there's a fourth circle* (draws the fourth circle in the figure) […] *then it'll get kind of complicated and so it would almost be easier to kind of find the whole thing and then kind of take out the stuff you don't want* […] [*by dividing*] […] *You figure out the ratio.*"

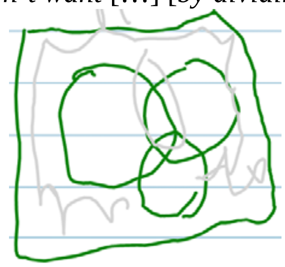

Figure 5: Venn Diagram for Equivalence Classes thinking

To Al, the universal set in Figure 5 is the solution set to a different problem, one which involves things that he wants represented as the union of the circles and things that he doesn't want. In the previous session, Al determined an additive relationship between the solution set of the original problem and that of the new problem. In this very general case, Al can imagine a multiplicative relationship existing and using this ratio to solve the problem.

Al then demonstrated his use of Venn diagrams for multiplicative reasoning. He had previously engaged in Equivalence Classes to solve task 31(i) from Table 2 in order to determine that there are $\frac{11!}{4!\cdot 4!\cdot 2!}$ permutations of the letters in MISSISSIPPI. He explained that there were 11! ways to permute 11 distinct objects and drew a rectangle to represent these 11! elements. He then drew an oval in this rectangle and wrote "*g*" for "good" inside it. He explained that for each "good" thing there were 4! ways to rearrange the Ss, 4! ways to rearrange the Is and 2! ways to rearrange the Ps, while shading in the complement of the set "*g*." He stated, "*I knew if I were to attempt to try to find what's inside the 'g' by itself, it's kind of hard. But I realized that if I were able to find everything* […], *it would be a bit easier*." Thus, he was visualizing a Venn diagram to explain the multiplicative reasoning he employed while engaging in Equivalence Classes.

Discussion

This study is a step towards better understanding the connection between student reasoning and visual representations as they solve counting problems. The data indicate that Venn diagrams were a powerful model for Al – he often stated he was envisioning them as he was counting and they helped him avoid and recognize over counting. He employed Venn diagrams to represent both his additive and multiplicative reasoning. In fact, he transferred the idea of a universal set to Equivalence Classes thinking to represent his multiplicative reasoning even though mapping diagrams, not Venn diagrams had been introduced for that way of thinking. Because of the similarities between Deletion and Equivalence Classes, it may be useful for students to see the same type of representation for both despite the difference in additive versus multiplicative reasoning.

The results of this case study support the inclusion of Venn diagrams in the combinatorics or basic probability curriculum as early as the use of arrangements with repetition. Indeed, it seems as if introducing Venn diagrams could push students to become more cognizant of over counting and recognize how to correct these types of errors. Further, it could be helpful for teachers to introduce the concept of a universal set when engaging in Deletion or Equivalence Classes in order to help students build connections between the solution set of the new problem with that of the original problem. Finally, Venn diagrams may help students explain their additive and multiplicative reasoning, just as they did for Al.

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Perspectives that some mathematicians bring to university course materials intended for prospective elementary teachers

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The undergraduate mathematical preparation of elementary teachers often occurs through mathematics departments. This study looks at the issue of teacher education as a site for collective work between mathematicians and mathematics educators. It works on a dual agenda of understanding what is involved, on the one hand, in using instructional support materials, and on the other hand, in creating usable instructional support materials. We analyzed three mathematicians' reviews of materials intended to teach *mathematical knowledge for teaching* to prospective elementary teachers. The analysis suggests that the mathematical issues most salient to these mathematicians concerned the coherence of the mathematical curriculum to be taught, the use and choice of mathematical representations, and what features of mathematical objects to make explicit. We discuss implications of this observation for development of materials for the mathematical preparation of teachers.

Key words: Mathematical Knowledge for Teaching, Teacher Education, Mathematicians, Mathematics Faculty

1. Introduction

The undergraduate mathematical preparation of elementary teachers often occurs through mathematics departments (Masingila, Olanoff, & Kwaka, 2012). There is wide agreement that content preparation of future teachers should connect to content demands of teaching (Conference Board of the Mathematical Sciences, 2012). Yet the educational work in which these mathematical demands arise – such as linking different representations of multiplication of fractions to each other, or leading a discussion on using the definition of fraction to place fractions on the number line – may involve activities on topics or with manipulatives not typically part of a mathematician's professional training. Preparing teachers to meet the mathematical needs of their work thus calls for collaboration between mathematics educators and mathematics faculty.

Mathematical knowledge for teaching (MKT) is defined as the knowledge required to meet the mathematical demands of teaching (Ball, Thames, & Phelps, 2008). The study reported in this proposal looks at the issue of teaching MKT, especially as a site for collective work between mathematicians and mathematics educators. It works on a dual agenda of understanding what is involved, on the one hand, in using instructional support materials for teaching MKT, and on the other hand, in creating usable instructional support materials for teaching MKT. The research questions we focus on are:

- What perspectives do mathematicians bring to interpreting and enacting instructional support materials intended for teaching MKT?
- What is salient to mathematicians regarding the work of teaching MKT and instructional goals for teaching MKT?

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We examined these questions in the context of the mathematical preparation of elementary teachers, in which mathematics courses are frequently based on materials by mathematics educators and taught by mathematics faculty. Our results shed insight on identifying key considerations for using and creating usable instructional support materials for teaching MKT.

To investigate the research questions, we analyzed reviews by 3 mathematicians of instructional support materials, for 10 hours of instruction to prospective elementary teachers in mathematics courses taught through mathematics departments, produced by mathematics educators. These materials were intended by their developers to teach MKT. We asked reviewers to comment on the quality of the materials in terms of their fit with the developers' stated instructional goals. We asked the reviewers to assess the fit of the materials with the goals as a way to position them as potential instructors interpreting these materials and their goals for what was entailed in enacting the instruction. This study focuses on insight yielded from the mathematicians' reviews. Though we also solicited reviews by 3 mathematics educators, these were primarily used as potential for contrast rather than as objects of examination.

2. Conceptual basis

Though mathematicians are experts in content, the way they typically interact with elementary content such as fractions likely differs from how they would need to interact with fractions if they were instructors for mathematical knowledge for teaching fractions. Interaction with content is only one part of the complex work of teaching; teaching includes interactions with students and their interactions with content. These interactions are captured by the conception of teaching as the management of interactions amongst teacher, students, content, and the environment (Cohen, Raudenbush, & Ball, 2002), constituting an "instructional triangle". In the context of teacher education that this study concerns, we use "instructor" to designate university instructors in place of "teacher", and "teachers" in place of "students" (Figure 1).

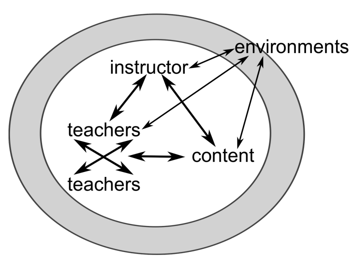

Figure 1. An instructional triangle for teacher education.

We take *mathematical knowledge for teaching* (MKT) to mean the "mathematics needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 399). Besides understanding student thinking based upon student productions, MKT also includes the mathematics for evaluating or creating precise yet accessible definitions for concepts such as fractions or polygons, selecting and sequencing which students' ideas, when shared in discussion, would support meeting a particular mathematical goal, or designing assessment problems to elicit particular student conceptions or misconceptions. As well, MKT encompasses pedagogical content knowledge such as being able to anticipate what students are likely to think and what they will find confusing, being facile with different representations of mathematical concepts, and evaluating the affordances and limitations of examples to illustrate concepts. MKT has been linked to quality of instruction and positive student outcomes (e.g., Hill, Rowan, & Ball, 2005; Kane & Staiger, 2012).

3. Design of analysis and data

We analyzed 3 mathematicians' reviews of instructional support materials intended for teaching MKT to prospective elementary teachers. The instructional support materials included detailed lesson plans for 10 sessions of instruction, instructional goals for each session, and mathematical and pedagogical commentary directed to the instructor. The materials also included records of practice such as video episodes of teaching and sample elementary student work.

The 3 mathematician reviewers each had experience with mathematics courses for prospective elementary teachers. We use the pseudonyms Abbott, Barrett, and Carter for the mathematicians. Additionally, we solicited reviews by 3 mathematics educators to provide contrasting perspectives.

The instructional goals for the materials, written by the materials developers, were based on the aim to develop instructors' understanding of the notion of MKT alongside prospective teachers' understanding of MKT for elementary grade teaching. The materials featured sessions on the concept of fraction, operations on fractions, and placement of the fraction on the number line. We asked reviewers to assess the quality and fit of the materials with the instructional goals regarding the:

- structure, clarity, and treatment of the content
- connection to the mathematical demands of teaching
- opportunities for teachers' learning of mathematical knowledge for teaching.

The focus on content and implementation with respect to goals was intended to position reviewers as potential instructors interpreting the materials and goals. We hypothesized that the focus on implementation would prompt reviewers to make sense of MKT as conveyed by the materials to the instructor and of the teaching of MKT to prospective teachers from the perspective of an instructor. The reviewers commented on the content as well as provided insight into other aspects of the management of instruction.

We parsed reviews into assertions made about the instructional support materials. In total, we parsed 130 assertions from the mathematicians' reviews and 73 assertions from the educators' reviews. (The educators' reviews were shorter than the mathematicians' reviews.)

To analyze assertions for their perspective on teaching, we used interactions of the instructional triangle as an initial coding scheme. We coded an assertion with an interaction if the assertion addressed that interaction. There were very few assertions specifically regarding the interaction between instructors and teachers or the mediation between teachers and content, and there were many assertions simultaneously discussing instructors, teachers, and content – frequently addressing dilemmas of teaching. Such assertions were coded together under the code, "work and dilemmas of teaching MKT". Additionally, there were many assertions about the content that teachers learned or ought to learn that did not discuss an instructional interaction. These were coded as corresponding to the content component of the instructional triangle. Ultimately, we used four codes, as shown in Table 1.

Table 1. Codes for reviewers' assertions.

Assertion code	Component(s) of instructional triangle
Work and dilemmas of teaching MKT	Instructor, teacher, and content
Views of mathematics and MKT	Instructor and content
Environmental/broader concerns	Environments
Mathematics that teachers should be learning	Content

Codes are shown in Figure 2 with their associated instructional component. Assertions could be associated with more than one code. For example, the mathematician Abbott described how he would discuss the concept of 3/4 with prospective teachers, and then wrote: "Though this is perhaps not the best way to introduce 3/4 to third-graders, it is, I believe, an important concept for all elementary teachers to grasp and come to grips with." This statement, along with the description of how he leads a discussion of 3/4, was coded as concerning the work of teaching (because it concerned his interaction with the teachers and the content of the concept of 3/4) as well as concerning content as Abbott asserts that this is a concept that teachers ought to learn.

This paper reports on assertions about the work and dilemmas of teaching MKT and views of mathematics and MKT. We analyzed assertions with these two codes for themes regarding the interpretation and enactment of materials, and analyzed the frequency of these themes in the mathematicians' and mathematics educators' reviews.

We emphasize again that this document focuses on reviews by mathematicians. The reviews by mathematics educators were primarily used as potential for contrast rather than as objects of examination.

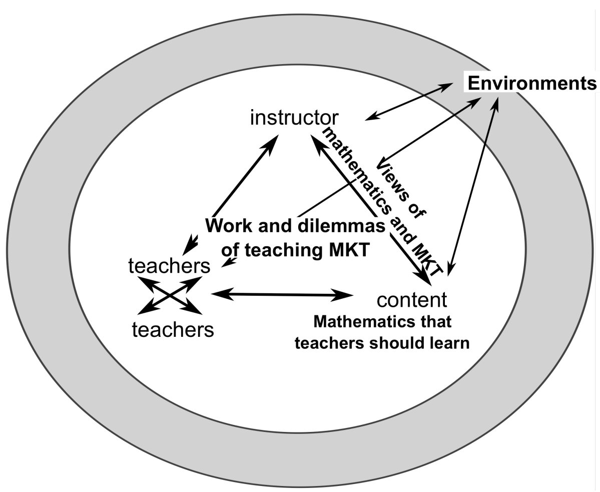

Figure 2. Codes and their corresponding components of the instructional triangle. Each review was parsed into assertions, and each assertion was coded with the component of the instructional triangle it concerned.

4. Results

Two themes regarding interpretation and enactment of materials that arose from our analysis of the reviews were: adapting or extending curricular content for prospective teachers' mathematical education, and pedagogical and mathematical issues influencing the way instruction is managed. We summarize and then illustrate the findings with an example from the reviews.

- **Adapting or extending curricular content.** Our analysis suggests that the mathematician reviewers would appraise the content of materials based on: prospective teachers' own mathematical knowledge and practice, the coherence and clarity of the mathematics, whether mathematical ideas were made sufficiently explicit, and how mathematical structure was used by the examples and exercises provided by the materials. These reasons for judging the merit of the content discussed by the materials, as representing the mathematics to be learned by future teachers, were most frequently cited in the assertions.

We note that the mathematical issues most salient to these mathematicians through these materials concerned the mathematical structure and coherence, as opposed to the ease or difficulty for teachers to learn a particular topic or the developmental needs of the teachers' future students, concerns more prominent to the mathematics educators who provided reviews. Both mathematician and educator reviewers were concerned with the prospective teachers' mathematical knowledge and practice.

- **Pedagogical and mathematical issues.** The issues most salient to the mathematicians – those they chose to comment on – were how representations were used to explain a mathematical concept, what ideas or features to discuss explicitly, allocation of instructional time, and how to sequence related but different concepts across a unit of instruction.

In contrast, some of the pedagogical issues most salient to the educators included choice of representations and how to connect different interpretations of one concept. Though mathematicians did also comment on choice of representations, their reviews contained more comments about the particular use of given representations than critiques of the collection of representations as a whole.

Illustration of some themes. We consider mathematicians' comments on the number line. These comments provide an instance of mathematicians' concern with the overarching mathematical structure of the materials, what ideas to make explicit, and how mathematical structure of an object is used. All three mathematicians commented on uses of number line to address perceived mathematical gaps or missed mathematical opportunities in the materials. For example, Abbott was concerned that the materials would leave prospective teachers without the understanding the 3/4 can be defined as the solution to the equation $4x = 3$. He chose to address this by developing an explanation using the number line, going into the meaning of multiplying and dividing by 4 as dilation symmetries of the number line. Barrett lamented the lack of emphasis on the number line, noting that the number line is a "key representational tool". Barrett stated several times that the relative emphasis on the part-whole definition of fraction as opposed to a number line conception is misplaced, and that he wished that discussion of the number line had come earlier. Although he does not elaborate further, he cites the IES Practice Guide (Siegler et al., 2010), which promotes the number line representation for its ability to illustrate connections between fractions, whole numbers, and percents; and describes student misconceptions arising from over-reliance on a part-whole conception. Carter asserted that the treatment of properties of the number line missed opportunities to articulate the mathematics fully, especially the informal description of the notion of infinitesimally close contained in the materials. As he noted, "It does not come close to the level of clarity, precision and accuracy that I think elementary teachers are capable of mastering and using in their classrooms."

5. Discussion

Abbott, Barrett, and Carter raised issues related to the number line and its mathematical properties and made fewer comments addressing how teachers might connect different conceptions of fraction to each other. Given the importance of elementary teachers' ability to help their students map representations to each other (e.g., Resnick and Omanson (1987)) mathematicians' sensitivity to the mathematical structure of particular representations could be leveraged in collective work between mathematicians and mathematics educators to yield curricular materials that would enable teachers to make more refined links between different representations. As well, though there are many comments about local mathematical issues (such as Carter's comments on the treatment of number line properties) and global mathematical issues (Barrett's comments about fraction conceptions), there are relatively few comments addressing connections between consecutive sessions, mathematically or pedagogically. Understanding the rationale behind the instructional design from session to session could as well be a site for collaboration between mathematicians and mathematics educators.

We cannot say that the perspectives in the reviews we examined are indicative of mathematics faculty in general, however the findings suggest that mathematics faculty who reviewed these materials were concerned with the mathematical structure of curricula to make sense of how to teach. We propose that it would be useful in advancing collaborations between mathematicians and mathematics educators to include discussions about how they approach making sense of the content preparation of teachers. It stands to reason that given the differences in their professional experiences working with teachers, different issues may be salient to them. Our experience listening to these reviews suggest that considerations for using and creating usable instructional support materials, for teaching MKT in mathematics courses taught in mathematics departments, could include elaborations of task enactments that connect in-the-moment decisions with how they support discussions about the mathematical structure of concepts, and that link the enactments with the overall structure of the materials.

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TRANSFER OF CRITICAL THINKING DISPOSITION FROM MATHEMATICS TO STATISTICS

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In this study we draw on the constructs of eagerness, flexibility and willingness to characterize the necessary disposition for critical thinking required in learning statistics in addition to specific content knowledge (Ennis, 1989). We investigated the challenges that students who are highly successful in mathematics might have in doing statistics and found that while a student might have an inquisitive disposition and good proficiency with the foundational mathematical concepts such as functions and function transformations, that same student might struggle in statistics. Even concepts that are seemingly related to their mathematical counterparts, such as what is a variable when considering population and sample, may cause problems as the statistical sense is distinct enough from the mathematical sense. We suggest that such students may experience greater than usual affective problems in a statistics class and may, therefore, give up easier and earlier than students who have been less successful mathematically.

Keywords: Statistics, Subject-specificity, Critical thinking disposition, Transition

Recently, statisticians have collectively voiced that statistics is a discipline distinct from mathematics in several aspects (Garfield & Ben-Zvi, 2010; Wild & Pfannkuch, 1999). One difference is in the nature of the goals for problem solving. While the major goal of doing mathematics problems is to improve student understanding of the structure of mathematical content, the goal of statistics problems is gaining insights from data (Moore & Notz, 2005; Rossman, Chance & Medina, 2006). In the process of making sense of data, statisticians draw on statistical procedures determined by the context. This matter of context is another characteristic of statistics distinct from mathematics (Cobb & Moore, 1997; Rossman, Chance & Medina, 2006). Although contexts are occasionally used in mathematics problems as examples to promote student understanding of mathematical concepts, the mathematical content still has meaning free of context. In addition, the context in mathematics problems can obscure the underlying structure (Cobb & Moore, 1997). Most statistics problems, however, originate from real situations and are naturally embedded in a context, which provides meaning for numbers in statistics (Cobb & Moore, 1997). Therefore, data cannot be meaningfully analyzed without careful consideration of the given context: how the data were collected and what they represent (Garfield & Ben-Zvi, 2010). The differences in the goals and the matter of context between the two disciplines are important because they give rise to another major dissimilarity, the subject-specific critical thinking dispositions, the main focus of this paper.

According to Ennis (1989), most cognitive psychologists hold the view that critical thinking is subject specific. Under this view, different types of critical thinking dispositions are required for mathematics and statistics due to their different natures; the critical thinking disposition specific to mathematics will have to promote understanding of the structure of mathematical contents and the critical thinking disposition specific to statistics will have to promote understanding of contextual characteristics, data analysis and result interpretations. Noting the difference in subject specific critical thinking dispositions needed for mathematics and statistics, it is not uncommon to observe that students who have been successful in mathematics, both at the introductory and advanced levels, experience difficulties in translating that success to their first statistics course. This phenomenon could be attributed to the different types of critical thinking dispositions between the two disciplines; according to Ennis, simple transfer of critical thinking dispositions and abilities from one discipline to another is unlikely. Especially when students take a first statistics course after twelve years (k-12) of mathematics courses, it may take students time and effort to adjust to the new culture of statistics after the familiar culture of mathematics. Taking up this issue, in the context of the differences in subject-specific critical thinking dispositions required for statistics versus mathematics, we raise the following questions:

- What challenges do mathematically successful students face in transferring their success in mathematics to statistics?
- How does the notion of subject-specific critical thinking dispositions explain the nature of these challenges?

In seeking answers to these questions, we discuss in the following section the issues cognitive psychologists have explored in the transfer of subject-specific critical thinking dispositions.

Literature Review

Subject-Specific Critical Thinking: There are some studies that have explored issues of general critical thinking disposition (Ennis, 1987; Sternberg, 1985). Also, as different problem-solving strategies are required in different subjects, some cognitive psychologists have explored critical thinking disposition with regard to subject matter knowledge (diSessa, 1987; Ennis, 1989; Even, 1993), which is referred to as *subject-specific critical thinking disposition*. Ennis (1989) set forth three principles to establish the relationship of subject matter knowledge to the development of subject-specific critical thinking. The first states that “knowledge about a topic is ordinarily a *necessary* condition for thinking critically in the topic” (p. 6). The implication is that it is imperative to acquire subject matter knowledge in order to be able to develop subject-specific critical thinking disposition. The second states that “simple transfer of critical thinking dispositions and abilities from one domain to another domain is unlikely” (p. 7). Under this principle, subject-specific critical thinking can be transferred to another subject only when there is sufficient practice in relevant domains and instruction that focuses on transfer. The last principle states that any general critical thinking instruction is not likely to be effective in developing subject-specific critical thinking. Unlike the first two, this principle is controversial and supported only by strong domain specificists (Ennis, 1989).

Subject-Specific Critical Thinking in Mathematics and Statistics: Aligning with Ennis’s (1989) views on subject-specific critical thinking disposition, a series of research studies has explored subject-specific critical thinking skills in mathematics and statistics. Aizikovitsh-Udi (2011) explored the effect of incorporating critical thinking training in a probability course. Her following study investigated if critical thinking skills depend on the content and the subject-specific concepts in that particular content (Aizikovitsh-Udi, 2012a). In this study, she claimed that “the construction and teaching of critical thinking skills are determined by specific contents and tasks the teacher uses” (p. 7). Recently, Aizikovitsh-Udi (2012b) extended the scope of her previous studies to explore how statistical literacy is linked with critical thinking skills.

Even though Aizikovitsh-Udi’s studies are valuable in that they incorporated critical thinking skills into the study of acquiring statistical knowledge, their focus was limited to comparing general critical thinking with statistical thinking. It still remains unknown what critical thinking skills are required in the study of statistics, how the development of the problem-solving skills in statistics advances the learning disposition specific to the discipline and the overall dynamics of statistics study, and how these processes in statistics contrast with those in mathematics. Keeping in mind Ennis's (1989) first principle, it is important to investigate how the different natures of the subject matter knowledge for mathematics and statistics require different problem-solving skills/strategies as the first step to answering the research questions. In the next section, we define critical thinking disposition and present our view on the transfer process between two different domains in relation with subject matter knowledge.

Theoretical Framework

In the first part of this section, we present the aspects of critical thinking skills that are relevant to the purpose of this study and then draw on these aspects to give our definition of critical thinking disposition. Then we explain our view on the process of how subject-specific critical thinking disposition transfers from one subject to another subject.

Conditions for Developing Critical Thinking Disposition: Halpern defines critical thinking disposition as deliberate use of skills/strategies that increase the probability of a desirable outcome (1998). Similarly, we define subject-specific critical thinking disposition as deliberate use of skills/strategies that increase the probability of a desirable outcome within a given subject. To develop this definition, one needs to satisfy certain conditions relevant to personal characteristics as a learner. Aligning with the views of Halpern (1998) and Facione, Sánchez, Facione & Gainen (1995) about the attitudes that a critical thinker exhibits, we consider that subject-specific critical thinking disposition is a habitual attitude towards the subject that can develop in the presence of the following three characteristics: (1) *eagerness* to immerse oneself in conceptually challenging tasks, (2) *flexibility* to apply problem-solving strategies developed within the study of a subject to problems that require the same strategies but in a new or different context for a different subject, and (3) *willingness* to discern the necessary critical thinking skills from the unnecessary ones. We believe that a learner's having developed a critical thinking disposition specific to mathematics transfers to his or her critical thinking disposition specific to statistics as he or she develops these characteristics.

Transfer of Subject-Specific Critical Thinking Disposition between Domains: Ennis (1989) categorized views on how to develop subject-specific critical thinking. For example, from the *general perspective*, critical thinking abilities and dispositions can be taught "separately from the presentation of the content of existing subject-matter offerings, with the purpose of teaching critical thinking" (p. 4). The *infusion perspective* holds that "deep, thoughtful, well understood subject matter instruction in which students are encouraged to think critically in the subject, and in which general principles of critical thinking dispositions and abilities are made explicit" (p. 5). We hold the general perspective, mixed with the infusion perspective in the sense that we advocate Ennis's first two principles.

As discussed earlier, mathematics and statistics are different in nature, and thus, demand different types of problem-solving skills and strategies. The implication of this, from our perspective, is that the subject-specific critical thinking dispositions developed in studying mathematics may not be automatically transferred to statistics. To explore this issue, it would be necessary to design instructions and practice that focus on transfer. Before we move to this study's methodology, we wish to note that we distinguish critical thinking disposition from the ability to think critically: "Some people may have excellent critical-thinking skills and may recognize when the skills are needed, but they also may choose not to engage in the effortful process of using them. This is the distinction between what people can do and what they actually do in real-world contexts" (Halpern, 1998, p. 452).

Method

The data for this case study with a single participant, Ian, are drawn from a larger study that explored student understanding of statistical concepts in two introductory statistics classes at a public research university. While the curricular organization of the courses conformed to those typically found in reform-oriented classrooms, the instruction itself was essentially traditional. The instructors had almost total responsibility for the classroom activities, and the content was delivered primarily via lecture.

We used a phenomenological approach to collect data, the process of which was conducted in two steps: a survey assessment and a follow-up interview. For the survey, we developed a fourteen-item assessment. Some of these items were modified from Assessment Resource Tools for Improving Statistical Thinking, developed by faculty members of the University of Minnesota in 2006. The rest of the items were developed by our research team. The entire survey is available by request from the first author. The assessment items sought to evaluate student understanding of what the symbols represented and their conceptual understanding primarily via their symbolic representations. The intent of the interview process was to identify how students' understanding of symbolic representations and their level of symbolic fluency potentially impacted their understanding of certain symbol-oriented concepts. The interviews were conducted immediately after the survey. Based on their work on the content survey, the eight students appear to range from low-achieving to high-achieving in statistics.

All interviews were audio-recorded and transcribed. For coding, each utterance was assessed to examine the information it gave about symbolic understandings. Within each transcript, we categorized and summarized the utterances that were deemed informative understandings by the type of concepts and connections they described with their symbolic understanding. We read within and across categories to develop conclusions. During the interviews, the grounded theory approach was blended in, to observe any interesting phenomenon with regard to the students' understanding of descriptive statistics. Both the survey and the interview were analyzed qualitatively. One thing we found was a stark contrast between Ian and other students as to how they understood the mathematical concepts that underlie statistical expressions and how they conceived of their statistics class. These findings motivated the authors to write this paper.

Result

The analysis of data informed us that Ian struggled in grasping some of the fundamental statistical concepts. We differentiate between those concepts that directly transfer from mathematics, such as symbolic manipulations and computations, and those that are statistical in nature and do not have exact analogs in mathematics. This was a surprising result because Ian had achieved A's in all his mathematics courses. The findings from the analysis of the data suggest what kinds of challenges mathematically strong students such as Ian may face by shedding light on how the critical thinking disposition favorably developed for learning mathematics could run counter to learning statistics. There were three specific main findings:

1. Strong Inquisitive Learning Disposition: During the interview, we found that Ian had a strong learning disposition for clarifying any confusion; he would not move on to solve the problems until he clarified the confusion. For example, Ian said, " I didn't really understand this. The highest governor's salary. Are they saying – … When they said the highest governor's salary, does that mean the highest ever reached?" Ian then asked questions to the interviewer to ensure he understood the symbols. Ian's inquisitive disposition was more explicitly revealed in the following claim about his statistics class, " … when you don't have that basic, basic stuff, it's, everything that comes after, you just struggle to try to put pieces together, all at the same time." This claim reveals Ian's view on how learning takes place as well as his inquisitive disposition.

2. Strong Understanding of Mathematical Concepts: The data also show evidence that Ian's understanding of underlying mathematical concepts of statistical expressions was exceptionally strong. For example, for the question, "Describe the distribution of x-mu in terms of the mean and the standard deviation as opposed to the distribution of x-μ where x follows a normal distribution," Ian claimed, "the center would still be zero. But the standard deviation would be sigma, because you forgot to divide. … our standard deviation would be sigma, instead of being one." For a subsequent question, "If you did x/σ-μ instead of (x-μ)/σ to obtain the z-score, would it matter?", Ian claimed, "Yes, it matters. Because you have to subtract μ divided by σ, because if you divided, if you do the shift first, by μ, you're centering it at zero." This claim shows that Ian understands the dynamics of the algebra that underlies the expression for the z-score. Among the eight students, Ian was the only student who provided the correct answers.

3. Failure of Transfer to Critical Thinking Disposition Specific to Statistics: During the interview, Ian showed evidence of successfully applying mathematical concepts in the context of probabilistic settings. To a question of how one can convince someone that something is wrong, Ian claimed, "If I had an example, I could show someone that it's definitely wrong." This claim shows that he knows a proof strategy often used in mathematics and is ready to use it in the given context. Ian further showed evidence of well-developed critical thinking strategies specific to mathematics. One question stated that in a university, 75% of the students are male and 25% are female; 5% of the males and 15% of the females own a car. The question asked whether we can conclude that 20% of the students in the university own a car. When the question was rephrased, without any instruction, as "Would you say that it's between 5 and 15, or would you say it's below 5, or would you say it's above 15?" Ian stated, "I would say it's between 5 and 15. Probably around 7%?" The claim implies that Ian grasps the mathematical concept of weighted average, which is primarily a concept that requires an a-contextual calculation.

In contrast, Ian showed weaknesses in transferring mathematical concepts to statistical contexts. For example, to the interview question, "What is a variable?" Ian replied, "I've always just seen variables as, like, things that could change, kind of, I'm thinking like algebra." This shows that he knows the definition of a variable in the mathematics context. But he was unable to transfer this mathematical concept to a statistical context until he was given instruction. The conversation went on as follows:

Interviewer: Yeah, that's a good point. So, what are the things there, then, that could change?

Ian: Based on what, though?

Interviewer: That, you have a population. OK? But the population is a population that's fixed.

Ian: OK. So, μ can't change, but x-bar and x could change … given different samples. And then, μ and σ can't change, because the population, overall, will always be the population.

Interview: But why did you pick x-bar and mu, in the beginning, there?

Ian: Because I didn't understand that at all. I didn't know what we were looking at as, what was changing and what wasn't changing.

It is important to note that when Ian was interrupted with an instruction, he was quickly able to transfer his understanding of a variable in a mathematical context to the statistical context.

Discussion

We speculate that the first finding, that Ian has a strong inquisitive learning disposition, explains both his success in mathematics and his struggle with statistics. This disposition led to his strong understanding of mathematical concepts and must have allowed him to be successful in his mathematics classes. However, this strong inquisitive learning disposition could have hindered his transfer of that success to his statistics course. Under the current statistics curricula that emphasize statistical literacy, reasoning and thinking (Ben-Zvi & Garfield, 2004), students are expected to accept certain statistical expressions without fully understanding the underlying mathematical concepts. In mathematics courses, students are often taught the underlying concepts (or at least given access to them) when a formula is given. The first and third findings together imply that an inquisitive disposition doesn't necessarily support learning in current statistics curricula.

We now shift gears to a brief discussion of the three characteristics needed to develop subject-specific critical thinking in statistics, and use them as a lens to analyze Ian's case. First, a strong inquisitive learning disposition could lay a favorable foundation for developing the *eagerness* to immerse oneself in conceptually challenging mathematical tasks, but could run counter to developing *eagerness* towards complicated statistical tasks because the focus of the statistical tasks is on understanding the context of the tasks, determining what statistical tests are appropriate, conducting the related computation, and interpreting the outcome, but not on understanding the mathematical concepts that underlie the statistical expressions. Second, students with a strong inquisitive learning disposition could have limited *flexibility* in applying problem-solving strategies developed within the study of a subject to problems that require the same strategies in the new or different context of a different subject. Third, it is possible to train a student's *willingness* to develop necessary subject-specific critical thinking disposition if he or she is provided with instruction, as we consider Ian's meta-cognitive comment, "I never grasped what variables were considered, in stats," as a first step to developing such willingness.

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A DIALOGIC METHOD OF PRESENTING PROOFS: FOCUS ON FERMAT'S LITTLE THEOREM

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Twelve participants were asked to decode a proof of Fermat's Little Theorem and present it in a form of a script for a dialogue between two characters of their choice. Our analysis of these scripts focuses on issues that the participants identified as 'problematic' in the proof and on how these issues were addressed. Affordances and limitations of this dialogic method of presenting proofs are exposed, by means of analyzing how the students' correct, partial or incorrect understanding of the elements of the proof are reflected in the dialogues. The difficulties identified by the participants are discussed in relation to past research on undergraduate students' difficulties in proving and in understanding number theory concepts.

Key words: proof as dialogue; students' difficulties with proving; number theory; equivalence by modulus; Fermat's Little Theorem

Introduction

The extensive professional literature on mathematical proof and proving tells us that virtually any aspect of understanding and producing mathematical proofs is a stumbling block for learners (cf. Knapp, 2005, Harel & Sowder, 2007, for comprehensive reviews). As a rule, students' difficulties with constructing and understanding proofs are exposed by means of documenting and interpreting their (often poor) performance when coping with various proving tasks. This research approach implies that students' understanding of proofs and their difficulties are mainly examined from an expert point of view. A complementary approach – inquiring what students themselves see as issues of difficulty – is still underrepresented in research on proof and proving. As such, the goals of our study were to inquire what students themselves perceive as problematic issues in a given non-trivial proof in number theory, to compare these with the expert view, and to describe how students cope with the identified difficulties.

Theoretical underpinnings

Our study is influenced by the idea of writing a fictional script of interaction as a part of a learning process. In particular, we refer to the dialogical approach for presenting proofs (Gholamazad, 2006, 2007), and to a "lesson play" (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, & Liljedahl, 2013). The roots of these methods are inspired by the style of Lakatos's (1976) evocative *Proofs and Refutations* and can be traced to a Socratic dialogue, a genre of prose in which a 'wise man' leads a discussion, often pointing to flaws in thinking of his interlocutor.

Gholamazad (2006, 2007) introduced the dialogical method of presenting proofs in her work with prospective elementary school teachers. This type of proof presentation consists of a script of a dialogue between characters that ask and answer questions about different steps in a proof. Gholamazad suggested that the dialogical method provided insights into the students' cognitive obstacles when creating and interpreting proofs. She developed the method based on Sfard's (2001, 2008) communicational framework, which conceptualizes thinking as a form of communication, specifically, as "individualized version of *interpersonal communication*" (Sfard, 2008, p. 81, italics in the original). The idea was that a request for a student to present a proof in a form of a dialogue makes his or her personal thinking salient. As such, in assigning the Task for our participants (see Figure 2) we expected to learn about their explanations of concepts and justifications of claims presented in the given proof that may not be apparent in a 'standard' form of presenting a proof.

Further, the idea of learning-via-scripting was implemented in teacher education in a different context, referred to as a "lesson play" (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, & Liljedahl, 2013). A lesson play is a novel construct in research and teachers' professional development in mathematics education. Using the theatrical meaning of the word 'play', lesson play refers to a lesson or part of a lesson written by a teacher or a prospective teacher in a script form, featuring imagined interactions between a teacher and her students. In teacher education, it provided a valuable tool for engaging prospective teachers in considering particular students' mistakes or difficulties, presented in prompts that serve as a starting point for the play. In research, it provided a window on how prospective teachers envision addressing students' difficulties, both mathematically and pedagogically. In particular, the prospective teachers' personal understanding of the mathematics involved became apparent in their attempts to guide students' solutions. As such, we wondered what mathematical understandings would surface when students decode proofs through script-writing.

Our Study

In light of the above considerations, our study addresses two interrelated research questions:

(1) What problematic issues do students identify in the given proof and how do they deal with these issues when decoding the proof into a script? In particular, which issues are treated as central?

(2)What can be learned from the dialogue method of presenting a proof about participants' understanding of particular concepts in number theory that appear in the given proof? In particular, how are students' correct, partial or incorrect understandings of the number theory concepts reflected in their scripts?

Twelve students participated in our study. Two of them were graduate students in mathematics education; the other 10 were working towards completion of a teaching certificate for secondary mathematics. At the time of the study the participants were enrolled in an elective course entitled "Proofs and proving", taught by the second author. An extensive mathematical background – an undergraduate degree in mathematics or in mathematics education – is required for teaching certification at the location of the study. Therefore, all the participants had broad exposure to undergraduate mathematics, having completed at least eight upper-division courses, including a course in Number Theory.

<u>Theorem:</u> For prime number p and natural number a, such that $(a, p) = 1$, $a^p \equiv a \pmod p$

<u>Proof:</u> $0, 1, 2, \ldots (p-1)$ is a list of all possible remainders in division by p.

When these numbers are multiplied by a, we get $0, a, 2a, 3a, \ldots (p-1)a$. When the numbers are reduced by p we get rearrangement of the original list.

Therefore, if we multiply together the numbers in each list (omit zero), the results must be congruent modulo p: $a \times 2a \times 3a \times \ldots \times (p\text{-}1)a \equiv 1 \times 2 \times 3 \times \ldots \times (p\text{-}1) \pmod p$

Collecting together the a terms yields $a^{p-1}(p\text{-}1)! \equiv (p\text{-}1)! \pmod p$

Dividing both sides of this equation by $(p-1)!$ we get

$a^{p-1} \equiv 1 \pmod p$ or $a^p \equiv a \pmod p$, QED.

Figure 1: Fermat's Little Theorem and its proof (adapted from Wikipedia)

The participants responded to the Task related to a theorem and its proof presented in Figure 1. Based on the mathematical background of the participants, we expected that they were familiar with all the concepts and symbols presented in the proof. The fact that the theorem is Fermat's Little Theorem was not announced, but we assumed that at least some of the students would recognize it from their prior studies. The Task is presented in Figure 2.

> Create a dialogue that introduces and explains the attached theorem [see Figure 1] and its proof. Highlight the problematic points in the proof with questions and answers. In your submission:
>
> - Describe the characters in your dialogue.
> - Write a paragraph on what you believe is a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.
> - Write a dialogue that shows how you address this hypothetical problem (THIS IS THE MAIN PART OF THE TASK)
> - Add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers, in connection to the characters, which may not be obvious for the reader.

Figure 2: The Task

Results and Analysis

In the initial stage of analysis we examined, first independently and then together, each student's work for problematic issues, which were explicitly identified as such in the students' comments or dealt with in the script. We considered a problematic issue to be "dealt with in the script" if there was an excerpt, in which the dialogue's characters explicitly addressed the issue with questions and answers. Further, a problematic issue was considered to be a 'central problematic issue' if it was explicitly identified by a participant as such, or if its discussion took significantly more space than discussions of the other issues. At the second stage of the analysis two kinds of problematic issues, for which the data seemed to have provided rich and solid evidence, were isolated: (1) gaps in the flow of the proof, and (2) the presumed lack of preliminary knowledge. The space allocation allows us only to exemplify briefly each kind.

Gaps in the flow of the proof

Focus on "rearrangement of the remainders". The proof in Figure 1 is not explicit about the reasons for why reducing the numbers $0, a, 2a, 3a, \ldots, a(p\text{-}1)$ by p results in the rearrangement of the list of remainders $0, 1, 2, 3, \ldots, p\text{-}1$. This point can be decoded, for instance, by assuming that there are two numbers on the list $0, a, 2a, 3a, \ldots, a(p\text{-}1)$, which give the same remainder when divided by p, and concluding that this assumption is wrong as it contradicts the conditions that p is a prime number and that a and p are co-primes.

Ten out of 12 students treated this issue as problematic, and six of the 12 treated this issue as a central one. Most students provided correct arguments based on the proof by contradiction. Only 2 mistakenly assumed that a reminder of ka is k (e,g., remainder of $3a$ is 3). To our surprise, which was informed by the literature about students' difficulties with the logic of indirect proofs (e.g., Brown, 2012; Leron, 1985; Tall, 1979; Koichu et al., 2012), most participants did not consider, at least not explicitly, the logical structure of proof by contradiction as a possible source of difficulty.

Focus on the properties of multiplication and division in equivalence relations. The "rearrangement of the remainders" step of the proof is needed in order to justify the multiplication in modular statements, that is, to justify the equivalence $a \times 2a \times 3a \times \ldots (p-1)a \equiv 1 \times 2 \times 3 \times \ldots (p-1) \pmod{p}$. Seven out of 12 students elaborated on why this congruence holds in their scripts, and 6 of them presented mathematically accurate explanations.

However, while the multiplication is easily explained based on the rearrangement of reminders, division in a congruence statement appeared to us as the second problematic point due to a gap in the proof. Why the equivalence remains when both sides of a modular congruence are divided by *(p-1)!* is not explained. Decoding this issue requires recalling the fact that dividing both sides of a congruence by a number does not always preserve the equivalence (e.g., $12 \equiv 16 \pmod 2$ is true, but the division of both sides by 4 – $3 \equiv 4 \pmod 2$ – results in a false statement). In the given proof, the division is possible because it is given that p is a prime number, and thus $(p-1)!$ and p are co-primes.

For some participants the treatment of division appeared to be similar to that of multiplication, without attending to the modulus. Apparently, the inevitable analogy between operations with regular equations and operations with modular arithmetic equations is responsible for these students' confusion. The analogy is particularly salient in the following excerpt taken from one of the scripts.

Teacher: Let's divide both sides by *(p-1)!* and get $a^{p-1} \equiv 1 (\text{mod } p)$.
Student: Is it allowed to divide like this?
Teacher: Yes. p is a prime number, so it is different from 1, therefore, *(p-1)* is different from 0 and so it is possible to divide by it.

As we see, the teacher-character argues that the division is possible just because it is not division by 0. Thus, she acts as if the same justification that applies to 'regular' algebraic and arithmetic expressions also applies in the modular case. Of the nine students that treated this issue in their scripts, four made this assumption. The (problematic) role of the analogy between familiar algebraic equations and modular equivalencies is further discussed below.

The presumed lack of preliminary knowledge

Focus on the meaning of equivalence relation. The formal mathematical definition of congruence, introduced by Gauss in his 1801 work *Disquisitiones Arithmeticae*, states the following: For $a, b \in N$ $c \equiv b \ (mod\ m)$ if and only if m divides $|c-b|$. In other words, natural numbers c and b are said to be congruent modulo m if they have the same remainder in division by m. In particular, with respect to the statement of the theorem discussed here, a^p and a have the same remainder in division by p. However, in the common usage of congruence, what appears on the right hand side of the equivalence statement is the remainder in division of the left hand side by the modulus. That is, while statements (1), (2) and (3) below are all correct according to the definition, (1) is the one that is usually used when working with congruence classes of integers.

(1) $13 \equiv 3\ (mod\ 5)$; (2) $3 \equiv 13\ (mod\ 5)$; (3) $13 \equiv 8\ (mod\ 5)$

This is likely what leads to a rather common view that the right hand side of the congruence statement indicates the remainder. Consider the following examples from two different scripts, where the first exemplifies the meaning of mod and the second defines it:

Student: I have never seen the word 'mod', what does it mean?
Teacher: Modulo means the remainder in division of whole numbers. For example, 7 modulo 6 equals 1 because the remainder in division [of 7 by 6] is 1. How much is 22 modulo 5?
Student: If we divide 22 by 5, we get 4 and remainder 2, therefore modulo it is 2.
***
Asker: What is the meaning of $a^p \equiv a (\text{mod } p)$?
Researcher: It means that when a^p is divided by p the remainder is a.

We found similar misinterpretations in 5 scripts. Actually, this claim about the remainder holds true only if a is smaller than p. For cases where a is larger than p, the remainder in division of a^p by p (or anything else by p) should be smaller than p (by the definition of a remainder), as such it cannot be a. Consider a simple example of a=3 and p=2. The remainder in division of 3^2 by 2 is 1, and not 3.

The misinterpretation of congruence relations is rather common and was noted in prior research. When the participants in Smith's (2002) study were asked in an interview to explain the meaning of the statement $a \equiv b(\bmod n)$, five out of six students gave the following interpretation: "*a* divided by *n* has a remainder of *b*." This is despite the fact that three appropriate equivalent definitions were provided by the professor teaching their course.

These responses are reminiscent of the extensive research literature on young children treating the equality sign as an instruction to find a solution, rather than an indication of equivalence (e.g., Behr, Erlwanger & Nichols; 1980, Booth, 1988; Kieran, 1981, Matthews et al., 2012). While the resemblance between the misconceptions in both cases can be explained by an inappropriate analogy, one of the dialogues offers another possible reason: the influence of programming experience. The command *mod* in Pascal, as well as in several other programming languages and mathematical programs, is a function of two variables that outputs a remainder.

Focus on the basic concepts. In some scripts we find extended attention to clarifying all the concepts that appear in the theorem. While we agree with the view that understanding of the underlying concepts is essential, we believe that at the stage of dealing with the given theorem most of the concepts should not be problematic for a learner. The following is an excerpt from one of such scripts.

Impatient:	The statement says that for 2 co-prime numbers *a* and *p*, where *p* is prime and a is natural, the remainder in division of a^p by *p* is *a*.
Clueless:	Wait a second, what are co-prime numbers?
Impatient:	This is when their greatest common divisor is 1.
Clueless:	And what is a common divisor?
Impatient:	This is some whole number, which divides the two numbers and gives whole quotients.
Clueless:	Can you give an example?
Impatient:	Yes indeed, 3 for example is the greatest common divisor of 3 and 6.
Clueless:	Why is this true?

The next 25 lines of the dialogue clarify and exemplify concepts of prime, co-prime, divisor, factorial and division with remainder. Only then the dialogue proceeds to the proof itself. Surprisingly, when all the concepts are clarified, the lines of the proof are presented with minimal explanation. However, the issues that most of the participants (as well as we) considered as problematic are simply restated without additional explanation. This corresponds to the participant's stated belief that complete understanding of all the concepts in the theorem paves the way for understanding the proof.

We noted that those students, who were less successful in the course in general, devoted in their dialogues unnecessary extended attention to details that could be considered 'trivial', or taken for granted at the expected level of mathematical sophistication. They then passed quickly through the statements that required clarification. Such extended attention to particular concepts appears to us as a 'shield' that protects the students from exposing their personal difficulties in understanding the 'real' problematic sections of the proof.

Conclusions and Contribution

Within a wide variety of research in mathematics education that attended to undergraduate students' ability to handle proofs, the tasks presented to students requested them to produce proofs (e.g., Smith, 2006) or to evaluate given proofs (e.g., Selden & Selden, 2003). The task of interpreting a given correct proof in the form of a script for a dialogue is a relatively novel approach that provides several methodological advantages.

The approach enabled us to reveal which issues the students chose to pause on and explain, how mathematical issues are treated in these explanations, and what is taken as shared understanding or assumed knowledge. A possibility to choose a focus of the dialogue and decide on time and space allocation of various issues can be considered both as an affordance and a limitation of the method. The issue of affordance is clear as the dialogue provides an opportunity of explaining what is not apparent in the dry formalism of mathematical proofs. However, it also provides an opportunity to avoid "real problematics" by directing the focus of attention to other issues.

We conclude that the task of working through a proof and presenting it in the form of a dialogue proved to be fruitful on several accounts: it provided a window into students' abilities to handle identified difficulties; it exposed misconceptions as well as personal strengths. Our contribution can be seen on several arenas: methodological innovation in task design and implementation, further insight on understanding proofs by students with strong mathematical backgrounds, and extension of research on understanding particular concepts in number theory.

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STUDENTS' CONCEPTIONS OF MATHEMATICS AS A DISCIPLINE

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Researchers have found that students' beliefs about mathematics impact the way in which they learn and approach mathematics in general. The purpose of this study is to categorize college students' various conceptions concerning mathematics as a discipline. Results from this study were used to create a preliminary framework for categorizing student conceptions. The results of this study indicate that the conceptions are numerous and range greatly in complexity. The results also suggest the need for further study to qualify the various student conceptions and the roles they play in students' understanding of and approach to performing mathematics.

Keywords: Students beliefs, mathematics,

Purpose and Background

College curriculums are designed to prepare students for specific careers. Part of this preparation is to enlighten the students about subjects that will play an integral part in their future. Thus, it is important to understand student perceptions of mathematics upon entering college. The purpose behind this investigation is to see how freshman calculus students thought of mathematics as a discipline. To clarify, this study is not focused on student beliefs pertaining to the nature of learning mathematics, but rather on what they feel mathematics is and the roles it plays.

Much research has been conducted on students' beliefs about mathematics concerning learning and motivation (e.g., Garafalo,1989; Hofer, 2001; Schoenfeld, 1985). These papers focused mainly on beliefs, which can be linked to academics. That is, the research focused on beliefs concerning the complexity of mathematics, ideas about learning and performing mathematics, and how these beliefs played a role in the learning process. Some student beliefs presented in these papers include math is hard, math is memorization of algorithms, and that there should be only one way to correctly answer a mathematics question.

Underhill (1988) summarized learners' beliefs about mathematics into four categories. These categories are *beliefs about mathematics as a discipline, beliefs about learning mathematics, beliefs about mathematics teaching,* and *beliefs about ones self within a social context in which mathematics teaching and learning can occur* (as cited by Op't Eynde, Corte & Verschaffel, 2002). Other research has shown that beliefs students have about mathematics play a role in the way the students approach learning and their motivations pertaining to mathematics (Hofer, 1999). Some beliefs are thought to be unhealthy and may even have a negative impact on learning mathematics (Spangler, 1992). The assessment of students' beliefs about mathematics can aid instructors in planning instruction and creating a classroom environment that can better help students develop a more enlightened system of beliefs about mathematics (NCTM, 1989).

Thompson (1984) investigated how teachers' perceptions of mathematics were related to their instructional behavior in the classroom. In her study, she aimed attention at how the teachers thought of mathematics as a field and how these views directed their instruction. As Thompson (1984) related teachers' beliefs about mathematics as a discipline and their beliefs about mathematics teaching, her research seems to be the closest, in the context of beliefs, to the current study. Two main differences are, that focus of this paper is on college students rather than instructors, and the beliefs being studied here pertain less to those about learning mathematics and more to what mathematics is and the roles it plays.

Participants and Methods

The data for this study come from semi-structured individual interviews (Bernard, 1988) conducted with undergraduate students at a large mid-atlantic university. Three of the four students interviewed were enrolled in integral calculus; the other was enrolled in differential calculus. All of the students were engineering majors. The students were selected on a volunteer basis. Three of the four students were expecting to attain an A- or better; the other was expecting a B. There were three parts to the interview. The first consisted of a sequence of short answer questions designed to determine the mathematical background of the interviewees. The second part of the interview consisted of three questions. The first asked the student to find a definite integral or evaluate the derivative (depending on the class in which the student was enrolled) of a polynomial. The second question engaged the students in a context novel to them, "fine functions" (Dahlberg, Housman;1997). The third question involved the students in an everyday situation where mathematics can be applied to solve the problem. The final part of the interview consisted of a series of questions asking the students to reflect on part two of the interview. The interview was designed to evoke the students' various beliefs of mathematics. Part two evoked these beliefs by directly engaging the students in mathematics and part three asked them to share their views of mathematics by reflecting on the questions from part two.

Grounded theory was used in completing the analysis for this study (Strauss & Corbin,1990). First, all of the responses were reviewed one question at a time. During this process similar conceptions emerged in the coding of the students' responses. After completing this iterative process, all of the similarly coded items were placed in respective groups and coded again. In this step, codes materialized with respect to how the students felt mathematics played a part in the various contexts (Layers). It was then determined the codes generalized across the various coded conceptions. The following section gives a background about the framework and some specifics are shared in the results section.

Development of the Framework

The model for this framework design was inspired by Zandieh's framework for analyzing students' understanding of derivative (2000). One must note that Zandieh's framework encompasses a broad range of understandings, which were developed by observing not only students at various levels but also by consulting textbooks, mathematicians, and mathematics education researchers. Here, however, the research is solely based on the student responses from this pilot study. No other additional sources played a role in the development of this framework. For this reason, there are no claims about this framework being exhaustive or in no need of refinement. However, much like Zandieh's framework, the framework presented here is not able to predict which beliefs mathematics students have and it does not place the various conceptions or layers in any sort of hierarchy. Thus, the framework provides no developmental nature for a student's beliefs of mathematics. As a result the framework is also not designed to make conjectures concerning what other beliefs may be a part of the student's overall view of mathematics based on the beliefs the student expressed. The framework is solely meant to arrange and format the various expressed student beliefs of mathematics.

The framework has been designed in the form of a matrix (Figure 1). The framework created for describing students' conceptions of mathematics as a discipline has two main elements: four concepts of mathematics and four layers of how these concepts can manifest themselves. A given student's conceptions of mathematics are organized as rows and the layers of each form columns. Each entry of the matrix represents the premise for that particular conception and layer combination.

Layer / Conception	Existence/Computation	Abstract	Justification
Numbers Operations Theorems	Mathematics is numbers, adding, subtracting and applying theorems to specific cases.	Each situation can be generalized, and math is applying theorems and computations to these situations.	Numbers, operations and theorems can be used to justify our solutions.
Tool for other sciences	Mathematics exists in other sciences. Mathematics is used by these other sciences.	Concepts from mathematics can be used as computations in general situations. For example the derivative can be used to explain velocity (but didn't give rise to it).	Use mathematics to verify that our answer from a certain science is correct.
As an interpretation	Mathematics appears in the world around us.	Specific mathematical ideas can be used to explain general ideas. Rate of change can be interpreted by derivatives (which gives rise to velocity for example).	Mathematics can be used to verify why these interpretations are correct.
Way of thinking	Mathematics is a way of thinking. Thinking through numbers and operations.	Mathematics can be applied to any situation.	The use of logical arguments results in a correct conclusion.

Figure 1: Framework for categorizing student conceptions about mathematics as a discipline.

Conceptions

Here the word *conception* is simply meant as something the student believes to be true. After analyzing the responses from all of the participants, four conceptions emerged from the data. These concepts include mathematics as: *numbers, operations and theorems; a tool for other sciences; a tool for interpretation;* and *a way of thinking.* All of these contexts were expressed verbally from the students except for most of *a way of thinking.* This context was interpreted based on the students' answers from part three and their written work from solving the various problems presented to them in part two.

The belief of *"Mathematics is about numbers, operations and theorems"* deals specifically with these three mathematical entities. Kloosterman (1996) noted that students often believe that math is simply computation. The students in this study presented this view; thus it is a conception in the framework. The category *mathematics is a tool for science* is based on the idea that mathematics is found in sciences such as physics and chemistry. The premise is that the science concepts existed before the mathematical concepts, which now serve to explain science. *Mathematics as a tool for interpretation* stems from the expressed belief that mathematics gives birth to the formulas used in different sciences. For example, because the derivative is used to calculate a rate of change (of anything), one can use the derivative to compute any rate of change that occurs in nature.

An effective example of *mathematics as a way of thinking* is demonstrated in the study by Carter and Norwood (1997) in which they found some teachers believe “to be good at mathematics you need a mathematical mind.” This mathematical mind is thought of as a way of thinking. The documented belief from their study is used here only because it adequately exemplifies the conception of the phrase *a way of thinking* as it is used in this paper, further is was not used as a conception a priori. For clarification, it should be noted that *way of thinking* does not refer to characteristics of understanding, but rather a characteristic of mathematics.

Layers

The term *layers* simply refers to the way in which the belief of mathematics manifests itself in the student. To be clear, the term *layers* is not meant to imply any form of a hierarchical structure. *Layers* is also not meant to imply that any manifestation of each concept is more or less desirable than any other. Given the nature of this study, making these conjectures is beyond the scope of the research performed here. The framework developed here accounts for three layers in each context. The layers are: *existence/computation*, *abstraction*, and *justification*.

“Numbers, operations and theorems can be found in physics” and “Performing mathematical operations with certain theorems is an idea also used to solve physics problems” are examples of student responses that fall under the *existence/computation* layer of the *numbers, operations and theorems* context. When a student expresses one of the above conceptions in the context of generalized situations, then the student is said to have expressed that conception in the *abstraction* layer. The distinction between the existence/computation and abstraction layers is significant. In the former only numbers and operations are discussed, whereas the latter concerns using a certain set of operations or theorems in generalized situations, such as derivatives being used to calculate a rate of change.

A large number of people believe that mathematics is irrefutable and infallible (Cooney & Wiegel 2003). For this reason people think that mathematics is a sufficient way to show their answer to a problem (academic or not) is correct. This was evident in the analysis of the interviews conducted. The students, in various contexts expressed mathematics being used to justify an answer, and thus *justification* emerged as a layer. For example, a student who says, “using numbers and theorems to attain your solution will ensure the solution is correct”, has expressed the conception of mathematics as *numbers, operations and theorems* in the *justification* layer.

Results

The following results are primarily centered around three students because their replies tend to best exemplify the concept-layer relationships in the framework. For brevity, only some of the analyses, which led to the framework are discussed in detail here.

Part two of the interview was designed to evoke the students’ different beliefs of mathematics. While completing the first question in this part, it was clear the students were following an algorithm for solving the question. E.g. integrating the polynomial, then evaluating at the end points and finally subtracting. Upon being asked how they knew their answer was correct, all referenced a “rule” learned in class. When asked how they would explain the idea behind their work to someone with no knowledge of calculus, interesting results emerged. All of the students initially answered they would teach the rule referenced prior. When asked if there was a way they could explain it not using the rule, two responded they could not explain it without the rule. One attempted to relate area under the curve to the “reverse power rule” but quickly gave up and stated he would just teach the rule. The differential calculus student stated he would teach the limit definition of derivatives and then explain how it would give the formula for the slope of the line at any point. Based on their reliance upon using numbers and rules to explain their work to others, it may be inferred that the students think of mathematics in terms of numbers, operations, rules and potentially theorems. Within this conception, Cameron, when asked how he knew his answer was correct, replied that a rule was followed appropriately and the numbers and operations were all correctly applied. Combined with his later statement “If you can get results from your model that fit a theorem, then you know your answer is correct”, gave rise to the *justification* layer of mathematics as numbers, operations and theorems.

When asked how the various questions from part two related to mathematics, Dan replied: “…that’s what mathematics is, taking actual things like dollars, physical objects and turning it into a number that I can then manipulate with rules and theorems and things that have made sense from numbers so henceforth apply to actual things...” Here Dan expressed *mathematics as numbers, operations and theorems* in the *abstraction* layer when he stated math was taking “…physical objects and turning it into a number that I can manipulate with rules and theorems…” The creation of *mathematics as a tool for other sciences* as well as *mathematics as an interpretation* was also driven by Dan’s responses. When asked if there was a benefit to learning mathematical concepts he went on to say that while attending high school, physics was only numbers and formulas to him and, “… where they would use calculus to explain something, it made me realize the calculus I was doing actually allowed me to understand it [physics].” Here Dan mentions the concept of a derivative is used to explain the ideas of velocity and acceleration from physics. It is important to note that Dan stated he was able to compute these values before but did not understand where they came from until explanation from calculus. He later added, “Algebra is plenty of numbers but calculus is making a form to this problem that works for all of nature which I find cool…” Here we also see Dan expressing *mathematics as interpretation* at the abstract layer; being able to take one mathematical idea and use it in numerous situations, “all of nature.”

To the question “What do you feel mathematics is, as a discipline” Beth stated, ”They [mathematicians] interpret other subject areas, for as an example physics… acceleration, velocity, you use calculus to interpret that stuff.” The fact that she expressed the idea that the other science concepts existed first, and the mathematician interprets them computationally, led to the development of the abstract layer for mathematics as a tool for other sciences. The distinction between the two concepts, *mathematics as a tool for other sciences* and *as an interpretation* is slight, but significant. The former expresses the idea that mathematics is used after the fact, the latter expresses the idea of mathematics leading to the discovery of these scientific concepts such as velocity and acceleration.

It should be noted that while the framework was derived directly from the students’ responses, two of the elements in the matrix are not accounted for in the results of the interview. These elements were placed in the framework for the sake of completeness and developing future work in this area. These elements are *mathematics as an interpretation* in the *justification layer* and *mathematics is a way of thinking* in the *abstract layer.*

Conclusion

The goal of this small study was to determine and then categorize the various student beliefs of mathematics as a discipline. Due to the size of this study, certainly not all student beliefs of mathematics have been discovered. However, there are significant implications based on the results. The results show that these concepts are numerous and range greatly in complexity.

Certainly the various beliefs affect how students approach, learn, are motivated by, and hence are interested in mathematics. For instance, students may be less motivated to indulge in mathematics based on their belief that mathematicians simply interpret the findings of other sciences. They may not see mathematics as being on the cutting edge of new science. On the other hand, if a student sees mathematics as yielding these scientific concepts and the scientists as simply applying them, the student may me more apt to joining the mathematical community. Also, the results here indicate that students do not reflect on mathematics as a problem solving tool, though their work indicates the use of these techniques. Often times we think of teaching mathematics as a tool for solving general problems. It is intriguing that, based on this study, students don't reflect on this. It is for these reasons, more work needs to be completed in this area of mathematics education.

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Using Disciplinary Practices to Organize Instruction of Mathematics Courses for Prospective Teachers

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One challenge of teaching content courses for prospective teachers is organizing instruction in ways that represent the discipline with integrity while serving the needs of future teachers—for example, choosing math problems that provide a logical development of a topic while also addressing mathematical knowledge for teaching. This paper examines the work entailed in structuring in-class work in mathematics courses for teachers. It argues that practices of teaching that are mathematical—such as representing ideas, grounding reasoning in mathematical observations available to the class, using definitions, or using mathematical language—can be used to negotiate mathematical and pedagogical aims, and therefore can be used to organize instruction of mathematical knowledge for teaching while simultaneously developing a disciplinary understanding.
Key words: Mathematics Teacher Education, Mathematical Quality of Instruction, Mathematics Task Framework

1 Introduction: Challenges of structuring courses for prospective teachers

Teachers must know subject matter for their work, and programs for prospective teachers often include courses on subject matter to address this need. This paper focuses on mathematics courses, often offered by mathematics departments, intended to be simultaneously relevant to mathematics as a discipline and to what is needed in the classroom.

Although there is limited examination of the relationship between mathematical knowledge for teaching, instructional quality, and student learning, existing studies reveal that measures of mathematical knowledge for teaching and instructional quality are significantly associated with positive student outcomes (Hill, Rowan, & Ball, 2004; Baumert et al., 2010; Rockoff, Jacob, Kane, & Staiger, 2011; Kane & Staiger, 2012). Thus mathematical knowledge for teaching is a natural candidate for what to teach to prospective teachers.

Yet, mathematical knowledge for teaching can be difficult to teach. When discussing mathematical knowledge for teaching, it can be easy to slide into mathematics not tied to teaching, or into instructional issues that are not mathematical. Because the elaboration of mathematical knowledge for teaching is the subject of current research, it is not codified in any standard form, making adoption as curricula difficult. Finally, the nature of mathematical knowledge for teaching, as mathematics not strictly for advancing the discipline, implies that pedagogical considerations for its teaching may differ from those for courses typically taught by mathematics instructors. Instruction of mathematical knowledge for teaching faces challenges in enactment and specification of content.

One aspect of these challenges is the course structure, from global curricular organization to in-class tasks and the accompanying demands for hearing and responding to ideas raised. Mathematical practices may offer a resource to meet the challenges. The structuring of courses with mathematical practices of teaching contrasts with, for example, using mathematical topics or techniques to organize coursework, as tables of contents of many textbooks would suggest. Most textbooks are organized by major theorems, computational techniques, or proof techniques. However, because mathematical knowledge for teaching intertwines mathematical and pedagogical considerations, a course structure based on disciplinary topics and techniques risks a slide into mathematics that is not as supportive of the work of future teachers. However, little is known about using mathematical practices of teaching to structure mathematics content courses for teaching.

In this paper, I examine the questions:

- What are affordances and challenges of using mathematical practices to organize instruction of mathematical knowledge for teaching?
- What is involved in effectively using mathematical practices to organize instruction in ways that engage mathematical knowledge for teaching?

Using the case of an instructor who used practices to organize his instruction, I argue that (a) mathematical practices provide a valuable resource for developing assignments that address mathematical knowledge for teaching; (b) challenges to engaging mathematical knowledge for teaching occur during task set up and implementation, especially in the absence of fine-grained knowledge of mathematical practices; and that (c) engaging mathematical knowledge for teaching requires the parsing of knowledge at two grain sizes: first, into a grain size that allows naming of entailed practices; and second, into a refined elaboration within practices. (Henceforth, I use *instructor* to refer to the "teacher", and *teachers* to refer to the prospective teachers who are the students of the instructor.)

2 Theoretical frameworks used

Shulman (1986) introduced the notion of pedagogical content knowledge as an amalgam of subject matter (such as mathematics) and pedagogy. Extending Shulman's work, Ball and her colleagues (e.g., Ball, Thames, & Phelps, 2008) conceptualized MKT as the knowledge of mathematics "needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399).

The recurrent work of teaching includes *using representations*; as Shulman (1986) observed, teaching requires knowing "the most useful ways of representing and formulating the subject that make it comprehensible to others" (p. 8). Representing mathematical ideas requires mathematical and pedagogical sensitivity. For instance, the most appropriate representation of a fraction—whether visual, verbal, symbolic, or another mode—depends on what students know, the purpose of the lesson, and what the representation suggests about the relationship between wholes and parts of fractions. The recurrent work of teaching also includes *explanations of mathematical reasoning*, in particular, *grounding the reasoning in evidence available to the class*. As Ball and Bass (2003) argue, the mathematical resources available to a class determine mathematical validity. For a statement to be valid in a community, its proof must be logically permissible according to the discipline and it must be grounded in observations accessible to the community.

Because representation and explanation occur across the teaching of many mathematical topics, I refer to them as examples of *mathematical practices of teaching*. I choose this phrase in analogue to "mathematical practices" entailed in doing mathematics. Mathematics is developed through practices (Kitcher, 1984); the recent Common Core Standards for Mathematical Practice (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) are one representation of disciplinary entailments. I also use the phrase "practices" in the sense of "practices of a profession", for example, as discussed by Grossman et al. (2009).

3 Methods of analysis

Teaching is the management of instruction, which consists of the interactions among the teacher, content, students, and environment (Cohen, Raudenbush, & Ball, 2002). An analysis of teaching is informed by analysis of interactions. This study focuses on interactions among teachers, the instructor, and the content of mathematical knowledge for teaching (MKT). I analyzed (a) records of practice for a mathematics course for prospective elementary teachers, and (b) interviews with the instructor for that course. I focus the analysis on two episodes featuring one task each, examining both the nature and role of the tasks used and the qualities of the interactions in the classroom.

In teaching, tasks serve as a medium for communicating the nature of mathematics to learners. Using the language of "instructor" and "teachers", a task focuses teachers' attention on a particular idea; it is characterized to include what teachers are expected to produce, how they are to produce it, and the resources available (Stein, Grover, & Henningsen, 1996). Stein et al. (1996) introduce the *mathematical task framework* (MTF) to understand the relationship between task set up, implementation, and resulting learning.

The deployment of practices influences the quality of instruction. For instance, making explicit links between representations and ideas may help students connect familiar and more technical representations, and asking for explanations of why and not just how procedures work may help students generalize thinking. These kinds of instances support the "richness of the mathematics" as measured by the Mathematical Quality of Instruction (MQI) instrument, which is associated significantly to positive student outcomes and to MKT (Hill et al., 2008; Kane & Staiger, 2012).

To understand the challenges of teaching mathematics useful for prospective teachers, I conducted separate analyses using MTF and MQI. I used MTF to analyze relationships between task characteristics and teachers' learning and to design interview questions addressing the instructors' goals, knowledge, and dispositions. I analyzed interviews for themes regarding his perceptions of the affordances and challenges of using mathematical practices to organize his instruction.

The second analysis of the data was based on MQI. I adapted the codes from MQI to analyze representations and explanations by the instructor and teachers in the course. I noted high-quality instances of representations and explanations, as well as instances where high-quality instruction might have occurred but did not. I wrote descriptions of these instances and analyzed them for themes regarding affordances and challenges of using representations and grounding reasoning. Finally, I compared the analyses for how the extracted themes contrasted and complemented each other in responding to the research questions.

4 Data

I used records of practice including video, course curricula, instructor notes, and interviews. The two analyzed episodes occurred in the first and third month of the course. In one episode, the instructor is facilitating discussion about representing 7/3 and the whole on the number line (*Task 7/3*). In the second episode, the instructor and teachers are explaining representations of $1/3 \div 1/5$ and connecting them to meanings of division (*Task 1/3÷1/5*).

5 Results

The analyses suggests that mathematical practices provide a resource for developing in-class, homework, and exam tasks that otherwise might not have been created. The data came from the first year the instructor used mathematical practices to organize instruction. To the instructor, these questions from this year represented MKT more genuinely than those in any previous year. This was also the first year in which he felt that he truly addressed mathematics serving the teachers' future work. He highlighted several assignments that illustrated the impact of using mathematical practices, including one asking teachers to "explain how to see the dividend, divisor, and quotient" in a variety of representations, followed by asking what interpretation of division was represented. This assignment has rich potential for teacher learning and high-quality instruction: it affords opportunities for high-demand processes such as justifying and interpreting, and high-quality interactions such as linking representations with mathematical ideas.

A key challenge is managing instruction to realize potential for learning. The discussion on the homework assignment described above, part of the setup for Task $1/3 \div 1/5$, exhibited substantive mathematical engagement by teachers and high-quality representation and explanation by both teachers and the instructor. In contrast, the instruction on Task 7/3 featured lackluster engagement and lesser quality of instruction—despite its rich potential. As Stein et al. (1996) observed, the potential for a task is only potential. The instructor had envisioned high cognitive demand for both tasks, but this was realized in only one. The analyses suggest that part of what it takes to organize instruction effectively is parsed knowledge of practices. During Task 7/3, the instructor asked teachers how they "liked" the representation; and he was unable to initiate substantive discussion about features. But suppose that explicit, finely parsed constituents of the practice of representation had been available—for instance, it was known that representation entails linking across different representations, linking to mathematical structures, or that to work with a representation one must first be able to discuss its salient features. Then perhaps the enactment of this task might have been sharper: instead of asking what appeals about a representation, the instructor might have asked, "What aspects of the representation make the whole less clear?" The proposed paper discusses potential elaborations of the mathematical practices of representing ideas and grounding reasoning.

6 Implications

The argument advanced here—that mathematical practices of teaching can be used to organize instruction of courses for teachers—may apply to other disciplines such as science and English language arts, and to other types of courses. In fact, decomposing practice has been used to structure methods courses (Kazemi, Lampert, & Ghousseini, 2007). I argue that decomposing the mathematical practice of teaching into its constituent practices could be used profitably for content courses; the practices should be disciplinary practices of teaching that support pedagogy rather than pedagogical practices of teaching that support the discipline. To examine to what extent this idea may cross subject matter boundaries, one would need to have an idea of the recurrent work of teaching a particular subject matter. This would involve identifying practices of an appropriate grain size for organizing instruction (for example, perhaps "explicit use of the scientific method" represents too large a grain size, whereas "organizing data to support hypothesis testing" represents a more appropriate size), selecting and sequencing domain topics and techniques that best support the learning of these domain practices of teaching, and then making empirical studies of how these practices inform task enactment. This approach may offer the advantage of promoting coherence between methods and content courses: if both types of courses were structured in a practice-based way, then practice could provide a commonplace for the improvement and understanding of the complex work of teaching, and how learning of teaching practice can build upon its constituent practices.

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DEVELOPING FACILITY WITH SETS OF OUTCOMES BY SOLVING SMALLER, SIMLER COUNTING PROBLEMS

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Combinatorial enumeration has a variety of important applications, but there is much evidence indicating that students struggle with solving counting problems. In this paper, the use of the problem-solving heuristic of solving smaller, similar problems is tied to students' facility with sets of outcomes. Drawing upon student data from clinical interviews in which post-secondary students solved counting problems, evidence is given for how numerical reduction of parameters can allow for a more concrete grasp of outcomes. The case is made that the strategy is particularly useful within the area of combinatorics, and avenues for further research are discussed.

Keywords: Counting, Combinatorics, Discrete Mathematics, Problem Solving

Combinatorial topics have relevant applications in areas such as computer science and probability (e.g., Jones, 2005; Polaki, 2005), and they provide worthwhile contexts in which students can engage in meaningful problem solving. In his undergraduate textbook *Applied Combinatorics*, Tucker (2002) says of his counting chapter, "We discuss counting problems for which no specific theory exists" (p. 169), emphasizing combinatorics as an ideal setting for fostering meaningful problem solving and rich mathematical thinking. Research indicates, however, that students face difficulties with combinatorial concepts, and this is certainly true at the undergraduate level (e.g., Lockwood, 2011a; Eizenberg & Zaslavsky, 2004; Godino et al., 2005; Hadar & Hadass, 1981). Given such difficulty, there is a genuine need for researchers to identify specific areas of struggle for students and to attend to potential ways in which students may improve in their combinatorial problem solving.

In this paper, the emphasis is focused on one particular aspect of combinatorial problem solving – the use of the problem solving heuristic of solving smaller, similar problems. The research goal addressed in this paper is to highlight the value of this heuristic in facilitating students' use of sets of outcomes. The case is made that work with smaller problems is a significant problem solving strategy for combinatorial tasks, particularly because such work can produce facility with sets of outcomes.

To clarify what is meant by "smaller, similar problems" in this paper, consider the following. In a counting problem, there are typically a number of conditions that specify what the problem is asking. These conditions refer to the rules or limitations that must be met in a given problem. Some of these might be numerical in nature (e.g., the specific number of letters in a password), but others might refer to non-numerical conditions (e.g., the fact that repetition of letters is allowed in a password). For clarity, numerical conditions are referred to as *parameters,* and non-numerical conditions are referred to as *constraints*. In relation to a given problem, any other problem that a student might attempt to solve is called "smaller" if it reduces one or more of the parameters in some way, and is called "similar" if it generally maintains the constraints of the original problem.

Literature Review and Theoretical Perspective

When talking about reducing problems to smaller, similar problems, an important issue is whether the similarity exists in the eyes of the student or the researcher. In an earlier paper, Lockwood (2011b) suggests *actor-oriented transfer* (AOT) as a valuable lens through which to examine students' combinatorial problem solving. Lobato (2003) introduced the notion of AOT, a methodological perspective in which the researcher focuses on student-generated connections between problems, and not on connections that the researcher may expect. In this study, in line with Lobato (2003), student-generated similarities are emphasized, not relationships that the researcher determined to be similar.

The specific strategy of solving smaller related problems has been alluded to by problem solving researchers like Schoenfeld (1979, 1980), Polya (1945), and Silver (1979, 1981). Polya discusses this strategy in terms of "discovering a *simpler analogous problem*" (p. 38, emphasis in original) and suggests that solving such a problem provides a model to follow when solving the original problem. Schoenfeld (1979) conducted a study examining the effectiveness of explicitly teaching problem solving heuristics, and considering "a smaller problem with fewer variables" (p. 178) was one such heuristic that he examined. Schoenfeld's and Polya's attention to such a strategy suggests that it could be valuable for problem solving across a variety of mathematical domains, but the heuristic itself has not been specifically targeted as an area of study.

Little has been investigated about the use of smaller problems in the domain of counting problems. Eizenberg and Zaslavsky (2004) allude to such a strategy in their work on undergraduate students' verification strategies on combinatorial tasks. One of those strategies they identified was "Verification by modeling some components of the solution" (p. 26), and one aspect of such verification involved applying "the same solution method by using smaller numbers" (p. 26). Eizenberg and Zaslavsky provide an example of an expert mathematician effectively using this strategy, but they note that while the strategy "could be very powerful…it requires deep structural consideration." They go on to say that "We speculate that although it may seem natural to students to employ this strategy (as indeed some tried to), applying it correctly needs direct and systematic learning" (p. 32). However, researchers have not yet tied the particular strategy of solving smaller, similar problems to meaningfully developing students' facility with sets of outcomes.

Additionally, the paper is framed within Lockwood's model of students' combinatorial thinking (Lockwood, 2012), which explores the relationships between formulas/expressions, counting processes, and sets of outcomes. Lockwood notes that different counting processes impose different structures on the set of outcomes and suggests that meaningful progress can be made for students as they gain facility with the set of outcomes. This perspective, emphasizing sets of outcomes, underlies the work presented in this paper.

Methods

Twenty-two post-secondary students participated in individual, videotaped, 60-90 minute semi-structured interviews in which they solved combinatorial tasks. All of them had taken at least one discrete mathematics course, and some had taken courses in combinatorics or graph theory. Semi-structured interviews typically involve "an interview guide as opposed to a fully scripted questionnaire" (Willis, 2005, p. 20), and this methodology allowed for flexibility that could allow the interviewer to adapt to students' responses. The structure of the interviews was first to give students five combinatorial problems to solve on their own, during which time they were encouraged to think aloud as they worked. After they had completed work on the five problems, the students subsequently returned to a subset of these problems, and they were presented with alternative answers to evaluate. The motivation for this design was based on a desire to put students in a situation in which they had to evaluate incorrect but seemingly reasonable answers. Further details of the study can be found in Lockwood (2011a).

Tasks

While students in the study were given five tasks, due to space limitations only one problem, the "Groups of Students Problem," is presented. Below, both a correct answer and an incorrect answer are provided to facilitate subsequent discussion.

<u>The Groups of Students Problem:</u> *In how many ways can you split a class of 20 into 4 groups of 5?*

A correct answer to this problem is

$$\frac{\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}}{4!}.$$

To arrive at this solution, five students can be chosen to be in a group, done in $\binom{20}{5}$ ways, then five of the remaining students are chosen to be in another group, $\binom{15}{5}$, then five more to be in a group, $\binom{10}{5}$, and then finally the last five to be in a group, $\binom{5}{5}$. However, the product must be divided by 4 factorial because the groups are not meant to be labeled or distinguished in any way – there is not a Group 1, Group 2, Group 3, and Group 4. Division by 4 factorial ensures that each solution gets counted once, as it should be. A typical incorrect solution neglects the division by 4 factorial.

Initial data analysis involved transcription of the videotape excerpts. Then, in line with grounded theory (Strauss & Corbin, 1998), the data was carefully coded for phenomena that could be organized into themes. What emerged were relevant instances of student work that highlighted students' uses of smaller, simpler problems, particularly as these related to students' work with sets of outcomes.

Results

The use of smaller, simpler problems arose a total of 15 times. Ten (six graduate and four undergraduate students) of the 22 students drew upon the strategy. While the strategy seems to have been underutilized among the students in the study, the strategy overwhelmingly helped the students who chose to implement it. Given students' overall difficulties on the problems, then, and given the fact that the use of smaller cases seemed to help some students, the strategy is worth examining as a potentially powerful aspect of combinatorial problem solving. The major result of this paper is that the use of smaller, simpler problems gave students greater access to sets of outcomes. That is, students were able to more clearly identify and manipulate outcomes when parameters were reduced, and this gave them leverage on the problems. For the sake of space, just two examples from the data are given here, both of which emphasize how the smaller case facilitated systematic listing of outcomes, which ultimately shed light on the original problem.

The first student, Mia, had initially arrived at an incorrect expression; she did not include division by 4 factorial. Upon revisiting the problem, she was asked to evaluate and compare her answer with the correct expression. Mia was thus in a position of comparing two different expressions to determine which was correct. Mia had some initial intuition about the role of 4 factorial, but she decided to attempt a smaller case in order to be sure. She worked through a smaller case of dividing six people into two groups of three. She first wrote down A, B, C, D, E, F to represent the people, and she wrote two circles with three dashes each in them. She noted that if she applied her initial method to the smaller case, she would get $\binom{6}{3}\cdot\binom{3}{3}=20$, and she stated that 20 "would not be too bad to write out." Mia then stated that if she applied the other expression to the situation, she would get 10. Next, Mia proceeded to write some examples out to see if she would get 20 or 10 as her answer for the small case. This is an instance in which Mia, in the context of the smaller situation, computed totals according to both possible expressions and compared the two. The smaller numbers allowed her to begin to write particular examples of outcomes, whereas within the original context this could not have been done feasibly.

Mia then wrote out divisions of six students into groups of two, and she wrote out ABC DEF, then CEF ABD, and then CDF ABE as possible divisions of the students. She paused and then wrote DEF ABC, and something significant happened: Mia noted that this was the same outcome as something she had already written – that is, ABC DEF was the same as DEF ABC. It seems that the smaller case (and specifically the smaller numbers) enabled Mia to write out some particular outcomes that she otherwise would not have been able to do (she had not written out such outcomes in her work on the original problem).

M: Um, alright A, B, C and then that forces DEF here. So that's one. ABD CEF. ABE and CDF. Hmm. <u>Let's see, oh right, because the first 3 could have been DEF, and then I would have been forced to put ABC in this group, but that's really the same, so these [referring to ABC DEF and DEF ABC] are really the same</u>...<u>Okay, yeah, I think that this double counts because if I just choose 3 people, it could have been A, B, and C, and then that forces DE and F in the second group. But, let's say the first three people were DEF, that forces ABC in the second group, and that's exactly the same, just, it doesn't matter, there's, there's, ABC are in a group and DEF are in a group.</u>

Mia explained the overcounting by referring to her initial solution; she identified two outcomes as being "exactly the same," as her language underlined above indicates. Mia was ultimately able to understand why division by 4 factorial in the original problem made sense and to identify the correct solution.

Our next student, Anderson, spent considerable time and energy listing out particular outcomes in the context of smaller cases. To scale back the original problem, Anderson decreased the number of groups and the number of options to make the problem more tractable. He started with dividing a class of four into two groups, and through listing found that there were three ways to do this. He then increased the problem to a class of six being split into two groups, and through careful systematic listing he found that there were 10 such possibilities. Anderson continued in this way, and at the heart of this work was pattern recognition – he was searching for a pattern in the numbers in order to generate the correct answer. While he ultimately ran short of time in the interview before Anderson could entirely finish this problem, he was on a very productive path toward making meaningful progress on the problem. When Anderson revisited the problem and was presented with the two common answers, he related them to the work and patterns he had generated initially. When it came to making sense of the solutions, then, it seemed as though his involved work of systematic listing and looking for patterns was instrumental in helping Anderson understand the problem generally.

Discussion

In prior work, Lockwood (2011a; 2012) indicated the importance of considering sets of outcomes for students as they count, suggesting that much benefit could be afforded by explicitly utilizing sets of outcomes in the activity of counting (see also Hadar & Hadass, 1981; Polaki, 2005). The findings in this paper build upon this notion, indicating that students' uses of smaller cases enabled them to engage with sets of outcomes through systematic listing.

In some instances, the use of a smaller problem allowed for work with the set of outcomes that might not otherwise have been attainable. The nature of counting problems makes them particularly appropriate for the strategy of using smaller, similar problems. Specifically, in counting problems, sets of outcomes are often so large that they can be difficult to conceive of and manage (for example, the answer to the Groups of Students problem is approximately 488 million). Smaller cases can reduce the magnitude of such sets and can make the problems and the solution sets more accessible.

Additionally, it is very important for a student to be able to articulate what he or she is trying to count, and smaller cases can facilitate such activity. In some counting problems the objects being counted can be quite difficult to articulate (in the Groups of Students problem, an outcome is one partition of 20 students into four groups of five; this necessitates coordinating a number of factors – 20 distinct students, what such a division might look like, and how they might be divided to create a desirable division). Smaller cases are particularly useful because they allow not only for the magnitude of the outcomes to be reduced, but often also for the outcomes themselves to be easier to identify. However, students must be aware of the fact that reducing a problem can introduce unexpected mathematical properties, and care must be taken when manipulating the original problem.

Conclusions and Avenues for Further Research

The results discussed above highlight the fact that solving smaller, simpler problems can allow students to work with sets of outcomes in meaningful ways. Overwhelmingly, the strategy helped the students who chose to implement it. Given student difficulties with counting problems, the use of smaller cases seems to be a promising domain-specific strategy that could be useful for students. One potential avenue for further research is to relate students' uses of smaller problems with their notions of what determines similarity among problems. That is, by identifying smaller, simpler problems, students can be thought of generating particular examples or instances of a given problem type. With increased attention on the role of examples in mathematics education literature (e.g., Bills & Watson, 2008), such an investigation could shed light on how students view examples of particular problem types. Another avenue to pursue is how to effectively generate this problem solving heuristic both among students and among pre-service and in-service teachers. This must be done with care, though, and students and teachers must be made aware of mathematical complications that can arise when reducing problems.

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MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY WHEN PROVING CONJECTURES

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Examples play a critical role in mathematical practice, particularly in the exploration of conjectures and in the subsequent development of proofs. Although proof has been an object of extensive study, the role that examples play in the process of exploring and proving conjectures has not received the same attention. In this paper, results are presented from interviews conducted with six mathematicians. In these interviews, the mathematicians explored and attempted to prove several mathematical conjectures and also reflected on their use of examples in their own mathematical practice. Their responses served to refine a framework for example-related activity and shed light on the ways that examples arise in mathematicians' work. Illustrative excerpts from the interviews are shared, and five themes that emerged from the interviews are presented. Educational implications of the results are also discussed.

Keywords: Examples, Proof, Mathematicians

Introduction

Much of the current literature on teaching proof in school mathematics underscores the goal of helping students understand the limits of example-based reasoning (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012) and typically characterizes example-based reasoning strategies as obstacles to overcome. However, given the essential role examples play in mathematicians' exploration of conjectures and subsequent proof attempts, example-based reasoning strategies should not be positioned only as barriers to negotiate. Indeed, the field may benefit from a greater understanding of the ways in which those who are adept at proof, such as mathematicians, critically analyze and leverage examples in order to support their proof-related thinking and activity. While the role of examples in learning mathematics more generally has received attention in the literature (cf., Bills & Watson, 2008), considerably less attention has been directed toward the specific roles examples play in exploring and proving conjectures. In this paper, we examine mathematicians' thinking as they explore and develop proofs of several conjectures. We report themes that arose during the interviews and discuss potential implications for the teaching and learning of proof.

Literature Review and Theoretical Framework

Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Clearly, examples play a critical role in mathematicians' development of and exploration of conjectures, and there is often a complex interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities (e.g., Alcock & Inglis, 2008). Several mathematics education researchers have accordingly examined various aspects of the relationship between example-based reasoning activities and deductive reasoning activities among both mathematicians and mathematics students (e.g., Antonini, 2006; Buchbinder & Zaslavsky, 2009; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Knuth, Choppin, &

Bieda, 2009;). The study presented in this paper builds directly upon a framework developed by Lockwood et al. (2012) that categorizes types of examples, uses of examples, and example-related strategies reported by mathematicians in a large-scale open-ended survey. Due to space, only part of the framework (Uses and Types of examples) is presented in Figures 1 and 2 below. This framework guided the coding of the interviews in this study and served to situate the themes presented below.

<Insert Figure 1> and <Insert Figure 2>

Methods

The data presented in this paper come from interviews that were conducted with six mathematicians as they explored and attempted to prove several mathematical conjectures (see Figure 3). Five of the mathematicians have a doctorate in mathematics, and one has a doctorate in mathematics education; all are currently faculty in university mathematics departments. All of the mathematicians were given Conjectures 1 and 2, and three each did Conjectures 3 and 4, which were randomly assigned. After working on each conjecture, the mathematicians were asked clarifying questions about their work. In addition, at the end of the interview they were asked reflective questions about their example-related activity, both that they had done during the interview, and more generally in their personal work. They were given approximately 15-20 minutes to explore each conjecture; although typically they were not able to complete proofs for each of the conjectures in the time allotted, they were able to make progress toward that end. (Note, that our interest was in their example-related activity while exploring and attempting to develop proofs, not in the proofs they may have produced given more time.)

<Insert Figure 3>

Conjectures 1-3 were taken from Putnam Exams, and Conjecture 4 was adapted from tasks in Alcock & Inglis (2008). We chose these problems for two primary reasons. First, the conjectures were accessible to the mathematicians (regardless of their area of expertise), but were not so clearly obvious that they could be proven immediately. Second, the conjectures were also accessible to the interviewer, allowing her to follow the mathematicians' work as well as to ask meaningful follow-up questions. While the choice of such conjectures may result in something of an inauthentic situation for the mathematicians, the choice did enable us to observe what mathematicians do as they actually explore and attempt to prove conjectures.

The interviews were transcribed, and a member of the research team analyzed them using the aforementioned framework (Lockwood et al., 2012). The process involved coding both mathematicians' observable example-related activity and their reflections about examples. The entire research group reviewed data excerpts that were difficult to code. These codes served to refine the initial framework, and the organizing of the codes in turn resulted in a number of themes about mathematicians' example-related activity in exploring and proving conjectures (Strauss & Corbin, 1998). We present these themes as the major results of this paper, as they shed light on how people who are adept at proof interact with examples as they consider conjectures.

Results

In this section, we share five main themes that arose from our analysis of the interview data. These themes not only illuminate the role examples play in the proving process for mathematicians, but also suggest implications regarding the role examples might play in classroom settings. Given the proposal's page limits, we do not go into great detail about these results; however, we do provide representative interview excerpts to illustrate the themes.

Theme 1 – There is a back and forth interaction between proving and disproving

All six mathematicians discussed the role of counterexamples in their proving process, noting that as they attempt to develop a proof, they engage in a back and forth process of formulating a proof and considering counterexamples. They described starting out by attempting to prove a conjecture, but then may get stuck, stop, and search for a counterexample. This search for (or inability to find) a counterexample might then provide insight into the development of the proof. An example of this is seen in Mathematician 2's reflection about his work with examples.

M2: You're trying to prove something and you go ahead and you try to prove it. And you realize that you're stuck at some point…Here's this gap. I start saying let's try, out of that gap, to build a counter example…Then you spend some time trying to build that object. And if you can't, then you try to sort of distill why can't you? And do the reasons why you can't build that, does that now fill in the gap in your proof? If it does, great. You've now pushed your proof further or maybe you've completed the proof entirely. And if it doesn't, then it refines what...the counter-example would have to look like…And so it's this sort of back and forth trying to use that. You know build a counter example and the failure or success of that to go back and look at what that says about your proof. And that dynamic back and forth can sometimes bear some fruit.

Theme 2 – Context and familiarity directly influence example choice

Four of the mathematicians also noted that context and familiarity have a direct impact on their selection of examples, often enabling them to make well-informed choices. Specifically, mathematicians indicated that if they were working in domain they knew well, they would regularly draw upon familiar, or "stock," examples. For instance, on the deficient number problem (Conjectures 4a and 4b), Mathematicians 4 and 6 clearly used their familiarity with the fact that 6 is a perfect number to make progress on that task, as seen in Mathematician 4's exchange below.

M4: Conjecture 4a: A number is abundant if and only if it is a multiple of six. Hmmm ok. So an example immediately comes to mind. Six is a perfect number and so that's going to be false if you are allowed to take a trivial multiple of six. So….
I: …Ok. And that you knew six was a perfect number from experience.
M4: Yeah, that one I just happen to know.

In other cases, if the domain was less familiar, the mathematician might rely on examples to make sense of the conjecture. This is exemplified in Mathematician 6's reflection below. Here we see that in a familiar domain he might simply launch into proving without having to consider examples, but that when he is "completely clueless" he tries to generate examples.

I: … Can you describe the role of examples in your work with mathematical, mathematical conjectures? How do you choose then? Do you have strategies for example-related activity? Like if you were to have to reflect on how you would use examples?
M6: So, well, first of all, it depends on the, the domain. I mean, there's some domains when I know, very familiar with all of the, like the more algebraic, formal techniques…and I can kind of recognize if it's a situation where I can actually get by without even really understanding…the problem, because I can just throw the tools at it…and it'll fall out… Other than that, I usually try to, I go in a couple different ways. Especially if I'm completely

clueless about what's going on, then I will usually use an example to try to figure out what's the conjecture is saying.

Theme 3 – Examples can lead to proof insights, both into whether the conjecture is true or false, and into how a proof might be developed

There were two ways in which mathematicians seemed to use examples to gain some insight into their proving process. First, examples served to inform whether or not a given conjecture might be true or false. At some point each mathematician used an example to decide whether he should go about trying to prove or disprove the conjecture. Second, examples served a richer purpose than simply shedding light on whether a statement was true or false. On several occasions mathematicians used specific features of an example in order to make significant steps toward a proof. In these instances, the mathematicians seemed to ground their thinking in a particular example, and by manipulating that example they developed an idea for how a more general proof might develop. As an example of this, we highlight Mathematician 6's work on Conjecture 4b as he tried to prove the contrapositive of the statement (that a number with factors that are not deficient must itself not be deficient). Mathematician 6 was examining what he called "test cases," in which he drew upon the perfectness of 6 to examine numbers in which 6 was a factor. His rationale for this is seen below.

M6: And then the real reason why I went after it with examples, not so much that I thought these would be counterexamples, as I thought they would be good test cases. And they'd maybe give me a feel for how, more information as to maybe why this is true.
I: Okay, and what do you mean by test case?
M6: Um, test case because the six, like I said before is perfect. So it's going to be, it's a, it's a pretty decent, uh, example of maybe, it's, so if anything has a chance to be a divisor that's not deficient inside of number that is deficient… I would guess it would be a perfect number.

Continuing to focus on 6, after trying to see if 6*2 and 6*3 would have to be abundant, he chose an example of 6*11. While working through this example, he had the following insight:

M6: It's almost like you get, like a duplication of the perfect-ness of six that shows up in this piece here.
I: Okay, how so?
M6: So, so, like this one, two, three adds up to six. Eleven, twenty-two, thirty-three actually adds up to sixty-six. So I'm feeling like I probably ought to be able to prove that this is a true statement.

His work with this example not only confirmed that he thought he could prove the conjecture, but work with the 6*11 example led him to make particular observations about the problem (in this case, the specific way in which certain factors added up). These observations pointed him to a more generic argument, and ultimately led him to a correct sketch of a proof.

Theme 4 – Knowledge of mathematical properties inform example choice

Another feature of the role of examples was that the mathematicians capitalized on their understanding of mathematical properties as they selected their examples. They took into account the domain to which the conjecture pertained (such as number theory or algebra), and they used that knowledge to pinpoint examples with certain properties. Their mathematical expertise came through as they spoke about mathematical features of their examples, such as choosing a number that is highly divisible or creating a set with no primes. This emphasis on properties came out most frequently with Conjecture 1, as the mathematicians tried to consider examples or counterexamples of the conjecture. In this case, the mathematicians clearly drew upon their knowledge of mathematical topics such as primeness, common divisors, the fundamental theorem of arithmetic, etc. As an example, Mathematician 3 constructed a set {4, 8, 12, 20} in an attempt to derive a counterexample. He had recognized that a counterexample must not have primes in it, and the excerpt below highlights his consideration of specific mathematical properties as he attempted to construct a possible counterexample and proceed with the problem.

> M3: The greatest common divisor between the two of them [looking at the statement of the conclusion] is not prime…Okay, it would have to be some set like 4 [writes {4, 8, 12, 20}]. That would be…their greatest common divisor is not prime. But, for every integer, the question is… There are some *n*s where the greatest common divisor is one or S…okay this one seems true, because if *n*, there are going to be integers which are multiples of S…But if the other ones here all have in common more than a prime [referring to the four numbers in his set]… So, so, if this were not true, that would mean that every two of these [referring to the four numbers in his set] have a composite number as a greatest common divisor.

In considering what might be needed to make a counterexample, Mathematician 3 displays knowledge of elementary number theory as he carefully selects four numbers that are not prime and that all have a composite number as a greatest common divisor. Facility with specific mathematical properties enabled him to make sophisticated decisions in constructing an example.

Theme 5 – Multiple examples can lead to meaningful patterns, resulting in conjecture generation and proof development

Five of the mathematicians demonstrated an explicit awareness of the relationship between examples and patterns in their work. As they worked through the conjectures, some mathematicians tried a series of examples that suggested they sought a pattern that could help them develop a proof. In his reflection on his own mathematical research, Mathematician 3 said that "he wouldn't come up with a conjecture without some examples" and suggested that looking for patterns through examples was the very activity that often led to conjectures. Mathematician 4 similarly noted that typically his work with conjectures is not externally motivated (such as solving the interview tasks), but that in his own work, examples that form a pattern tend to be the motivation for the conjectures that he formulates and ultimately wants to prove. Such statements from the mathematicians provide insights about how finding patterns in examples can ultimately lead to formulating, and perhaps eventually proving, conjectures.

Conclusion and Implications

Although the results presented here are based on a small set of interviews with mathematicians, the results are consistent with the results from our large-scale survey of mathematicians and their responses about their work with examples (Lockwood et al., 2012). The interview data highlight that examples play an important and meaningful role in the proof-related activity of mathematicians. Clearly, mathematicians possess an awareness of the powerful role examples can play in exploring, understanding, and proving conjectures, as well as the ability to implement example-related activity in meaningful ways. Yet, the role examples play in proof-related activities in mathematics classrooms, secondary school classrooms as well as undergraduate classrooms, often stands in stark contrast to the role examples play in the proof-related activities of mathematicians. Such a contrast between the role examples play in the work of mathematicians and in the work of students highlights the need for explicit instruction on how to strategically think about and analyze examples in exploring and proving conjectures—instruction students rarely, if ever, receive. Indeed, if students are to develop such awareness and ability, it is important to help them learn to think critically about how they can draw upon examples as they engage in exploring and proving conjectures.

Example Type	Definition
Simplicity	Expert appeals to an easy, simple or basic example. Includes "trivial" and "small."
Counterexample /Conjecture Breaking	Expert picks an example that might disprove the conjecture. The expert might explicitly say "a counterexample," but this can also be inferred.
Complex	Expert picks a complex example in order to test whether the conjecture holds for tricky ones; synonyms include "non-nice," "non-trivial," or "interesting."
Easy to Compute	Expert chooses an example that is easy to manipulate. The difference between this code and "Simple" is that the expert says something about working the example out.
Properties	Expert takes into account some specific mathematical property – he or she might reference a "property" or "features," or might mention particular properties.
General/Generic	Expert uses general or generic examples, or describes examples that are seen as representative of a general class of cases or otherwise lack special properties.
Boundary Case	Expert picks an extreme example or number, or a "special" case, such as the identity.
Familiar/Known case	Expert chooses an example with which he or she is familiar, or in which properties related to the conjecture are already known.
Unusual Examples	Expert picks an unusual number, which would be described as something that does not come up often. "Rare," "obscure," "strange," and "weird" are also synonyms.
Random	Expert describes the example as randomly chosen; this includes mathematical randomness, such examples chosen with a random number generator.
Exhaustive	Expert looks for "all" of the examples in an exhaustive manner. This can be by testing all possible examples or by using a computer.
Common	Expert describes the example as typical, common, or one many would choose.
Dissimilar Set	Expert indicates that he or she purposely selects a variety of types of examples.

Figure 1 – Types of Examples

Example Use	Definition
Check	Expert selects examples to make a judgment about the correctness of a conjecture; "test," "verify," and "check" are all synonyms.
Break the Conjecture	Expert tries examples to break the conjecture; this can include specifically looking for a counterexample.
Make Sense of the Situation	Expert uses an example to deepen his or her understanding of why the conjecture might be true or false, or to gain mathematical insight.
Proof Insight	Expert indicates that his or her production of examples (or counterexamples) might have a direct bearing on understanding how to prove the conjecture.
Generalize	Expert mentions using the example to generalize or to allow the expert to work in a more general situation.
Understand Statement of the Conjecture	Expert uses an example to better understand the statement of the conjecture.

Figure 2 – Uses of Examples

Conjecture 1
Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n)=1$ or $\gcd(s,n)=s$. Prove that there exist $s,t \in S$ such that $\gcd(s,t)$ is prime.

Conjecture 2
Let n be an even positive integer. Write the numbers 1, 2, …, n^2 in the squares of an n x n grid so that the kth row, from left to right, is $(k-1)n+1,(k-1)n+2,\ldots,(k-1)n+n$. Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black.
Prove or disprove: For each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Conjecture 3
Let S denote the set of rational numbers different from {-1, 0, 1}. Define $f:S \to S$ by $f(x)=x-1/x$.
Prove or disprove: $\bigcap_{n=1}^{\infty} f^{(n)}(S)=\varnothing$, where $f^{(n)}$ denotes f composed with itself n times.

Conjecture 4
All the numbers below should be assumed to be positive integers.
Definition. An abundant number is an integer n whose divisors add up to more than $2n$.
Definition. A perfect number is an integer n whose divisors add up to exactly $2n$.
Definition. A deficient number is an integer n whose divisors add up to less than $2n$.

Conjecture 4a. A number is abundant if and only if it is a multiple of 6.
Conjecture 4b. If n is deficient, then every divisor of n is deficient.

Figure 3 – The conjectures given to the mathematicians

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Using Cognitive Sciene with Active Learning in a Large Lecture College Algebra Course: Analyzing Motivation and Success

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ABSTRACT: At a research university near the east coast, a College Algebra class has been restructured into two large lectures a week, an active recitation size laboratory once a week, and an extra day of SP. SP was added as an extra day of class where the SP leader has students work in groups on a worksheet of examples and problems, based off of worked example research, that were covered in the previous week's class material. Two sections of the course were randomly chosen to be the experimental group and the other section was the control group. The experimental group was given the SP worksheets and the control group was given a question and answer session. The experimental group significantly outperformed the control on a variety of components in the course, particularly in light of the number of days students attended SP.

Key words: College Algebra, Cognitive Science, Interactive Compensatory Model of Learning (ICML), Worked Examples, Large Lecture, Supplemental Sessions.

Introduction

A Commitment to America's Future: Responding to the Crisis in Mathematics and Science Education states that "nationally 22% of all college freshman fail to meet the performance levels required for entry level mathematics courses and must begin their college experience in remedial courses" (2005, p. 6). The enrollment in College Algebra has grown recently to the point that nationally there are estimated 650,000 to 750,000 students per year (Haver, 2007) and has surpassed the enrollment in Calculus. Although there are almost three fourths of 1 million students enrolling in College Algebra, it is estimated conservatively that 45% of these students fail to receive a grade of A, B, or C and can reach percentages in the sixties at some colleges. To address this non-success of students at a large research university in the eastern part of the United States, faculty members teaching College Algebra have implemented a new structure in the course that emphasizes active learning through a day called Supplemental Practice, denoted SP.

Theoritical Framework

The Interactive, Compensatory Model of Learning (ICML) provides the framework for understanding and improving classroom learning (see Figure 1). Schraw and Brooks (1999) refer to a wide range of literature that reinforces ICML. There are five main components of ICML: cognitive ability, knowledge, metacognition, strategies, and motivation, which affect learning. Brief definitions of the five main components of ICML can be found in (Miller and Schraeder, 2011) and more detailed discussion in (Schraw and Brooks, 1999).

Figure 1 shows that knowledge, metacognition, and strategies are so closely connected that they are combined together into one area in the reproduced figure from (Schraw and Brooks, 1999). We will refer to this combined area as the knowledge-regulation component. The ICML captures the interactions between the four components that affect learning and describes how one component can compensate for deficiencies in others.

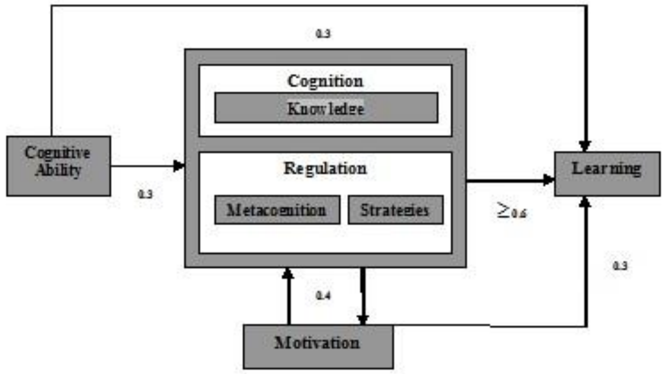


Figure 1: Interactive, Compensatory Model of Learning

A component can affect learning directly, as well as indirectly. For example, cognitive ability is related to learning directly, but also indirectly through knowledge-regulation. From Figure. 1, one can see that each component directly affects learning (the arrows from each component to learning) while only some components affect learning indirectly through another component. For example, motivation indirectly affects learning through knowledge-regulation, but not through cognitive ability. The numbers in the figure refer to the estimated correlation coefficient between two components. Each correlation coefficient is the estimated value of what has been measured in a number of empirical studies. Cognitive ability is correlated to learning with correlation coefficients ranging from 0.3 to 0.4 (Brody, 1992) and hence, the correlation coefficient of 0.3 relates these components. The other correlation coefficients are shown in Figure 1. Schraw and Brooks (1999) state that "most experts agree that knowledge and regulation exert a strong direct effect on learning that is greater than the effects of either ability or motivational beliefs" (p. 9).

The compensatory part of the model refers to how students can compensate for a weakness in one component with a strength in another component. For example, students who have weaker cognitive abilities (literature refers to this as intelligence) can compensate by having a stronger knowledge-regulation component. Through this iterative process, as they go from one topic to another topic in the course to gain knowledge, they more successfully compensate for their lower cognitive ability as compared to other students. The notion of compensatory processes is supported by many different theories (Gardner, 1983; Perkins, 1987: Steinberg, 1994). Schraw and Brooks (1999) state the following compensations can occur: (1) ability compensates in part for knowledge and regulation, (2) knowledge and regulation compensate for cognitive ability and motivation, and (3) motivation compensates for ability, knowledge and regulation.

Background and literature review

Supplemental Structure

SP was implemented during the Fall 2004 and was originally modeled after Supplemental Instruction (Arendale, 1994; SI Staff, 1997). The normal structure of the algebra class that consisted of three lectures a week morphed into a structure of two lectures a week in a large lecture room, and an active laboratory class once a week (at the same time as lecture) in two computer classrooms where students meet in smaller groups. The lab class was held on Tuesdays while the lecture class was held on Mondays and Fridays. The SP days on Wednesdays were originally added to the schedule to help lower-achieving students. This was done by requiring students that scored lower than an 80 on a placement exam, or scored lower than a 70 on any regular exam, to attend the SP sessions. Starting in the Fall 2006 semester, the SP sessions have since morphed into active problem-session days modeled after the cognitive science "worked-out example" research. The worked-out example research, henceforth denoted worked examples, asks students to study a worked example for a particular topic, ask questions about anything in the example that they do not understand, and finally work a similar example without reference to the worked example, nor other outside sources (Cooper and Sweller, 1985; Ward and Sweller, 1990; Zhu and Simon, 1987; Carroll, 1994, Tarmizi and Sweller, 1988). The SP sessions and worksheets have been developed based off of the worked example research for the following reasons: 1) helps students to be actively engaged with the material in a setting where they can get feedback and assistance as they solve problems, 2) assists students in transferring information from working (short-term) memory to long-term memory, 3) helps students to regulate their learning and build confidence that they can work the problems, 4) allows the instructor to work in the large lecture class to assist students as they learn the material, and 5) helps students as they work on homework, quizzes, and while they study for exams outside of class.

Worked Example Research

The discipline of cognitive science deals with the mental processes of learning, memory, and problem solving. Worked example research was developed based from Seller's cognitive load theory (1988). The total load on working memory at any moment in time is referred as the cognitive load. Most people can retain about seven "chunks" of information in their working memory and when they exceed that limit at any moment in time, there will be a loss of information in the working memory. In other words, there is an overflow of information in the working memory and cognitive overload. Cognitive overload can be thwarted if one limits information so that it does not exceed the student's working memory. One way this can be done is to transfer information from working memory to long-term memory as information is being processed (or soon after). According to Sweller (1988), optimum learning occurs in humans when one minimizes the load on working memory, which in turn facilitates changes in long-term memory.

Worked example research (Cooper and Sweller, 1985; Ward and Sweller, 1990; Zhu and Simon, 1987; Carroll, 1994; Tarmizi and Sweller, 1988) present students with a worked example on paper and tells them to study the example. Once the students are done studying the worked example and have asked any questions, the instructor asks the student to solve a similar problem without any help from the worked example. It has been suggested that worked examples reduce the cognitive load on a student and might optimize schema acquisition (Sweller and Owen, 1989**;** Sweller and Cooper, 1985). In addition, worked examples have been researched (and used) in a variety of subjects: mathematics (Cooper and Sweller, 1985; Zhu and Simon, 1987), engineering (Chi et al., 1989), physics (Ward and Sweller, 1990), computer science (Catrambone and Yuasa, 2006), chemistry (Crippen and Boyd, 2007), and education (Hilbert, Schworm, and Renkl, 2004**).**

Past research on worked examples in mathematics has been conducted in a laboratory setting. This study was conducted in a large lecture classroom setting and concentrated on determining if worked examples helped promote success in the course. In addition, past worked example research in mathematics has not dealt with college mathematics courses, classes in a large lecture setting, or implementing an extra day of class to focus on working with students to master material. The research could be valuable to other researchers that are working to promote student success in large lecture classes. The research question that will be addressed in this study is "Do students in the experimental group that attend the majority of supplemental sessions earn significantly different course grades/exam scores/quiz scores/etc… than other students in the control group?"

Methodology

Course and Worked Example Worksheets

The setting for the research was a large lecture 4-day College Algebra course with an annual enrollment of around 1000 students at a large research University near the east coast. Students place into the course (or other courses like Trigonometry and Calculus) by their score on an assessment exam given prior to enrolling into a math course. The weekly structure of the class is two lectures a week, one laboratory day, and one SP day.

During the SP days, worked example worksheets were handed out to the students to work on in groups. Since the class was still in the large lecture classroom setting with theatre-seating structure, students formed groups with other students near them as they saw fit. Usually students worked with 1 to 3 other students seated close to them. The worked example worksheets consisted of an expert solution of a College Algebra problem followed by a problem for the students to work out. An example of a worked example worksheet is shown in (Miller and Schraeder, 2011). The worksheet is always given to the students as one sheet (front and back) in a two column format with headings on all worked examples, followed by the section in the textbook (Sullivan and Sullivan, 2006) that can be referenced later outside of class. There are approximately 8 to 12 worked examples and problems on each worksheet. The material on the worksheets consisted of some of the material covered during the previous week in lecture (too much material to cover it all). No new material was ever covered and the worksheets comprised of problems directly from or derived from the problems in the textbook. The worksheets were never developed while referencing material from exams, quizzes, or labs. However, most of the questions from the exams and quizzes were similar to the homework in the book. Finally, the worksheets were modeled after worked example research since it presents an expert's solution to a problem followed by a problem for the student to work out. The only difference is that it is not plausible to ask the students to not reference the worked example while working another problem, so this was never done. Furthermore, most studies on worked examples state that the student should be given a similar problem (very similar in some cases), but in SP, the problems students were asked to do varied from very similar to somewhat different problems.

Experiment

The researcher randomly designated one of the course sections as the control group (n = 177) and the other two sections as the experimental group (n = 320). In the experimental group, the students were given a "worked-out example" worksheet at the beginning of each of the 13 SP days and asked to work in groups to complete the worksheet. Three to four class assistants circulated around the room to answer any student questions about the worksheet. In the control group, a graduate student organized a question-and-answer session during the extra day instead of giving a worksheet to the students. Students were able to get any question answered, but the graduate student only answered student questions and did not generate questions for the students, so the graduate student spent class time answering student-generated questions. Students in the experimental group who attended 8+ supplemental sessions were grouped into the 8+ experimental group denoted 8+E, and student in the experimental group who attended all 13 supplemental sessions were grouped into the 13 experimental group denoted 13E. Quantitative data (course scores on exams and quizzes, supplemental days attended, class attendance, total points,…) was collected for each student in both the control and 8+E and 13E groups and analyzed at the end of the semester.

Data

The 8+E and Control Groups

Data from the experimental and control groups were compared on a variety of levels by using t-test with equal and unequal variances depending on the data. The experimental and control groups had similar levels of retention (number of students that completed the course) at 80.5% and 84%, respectively. At the beginning of the semester, all students were given an old ACT math test that consisted of 60 questions. Students were given extra credit points for the ACT test on a sliding scale. This ensured that the better a student performed, the more extra credit, up to 10 points, they earned. The ACT test gave a good measure of students' prior mathematical knowledge. The control and 8+E groups mean scores of 28.40 and 26.88 with standard deviations of 6.41 and 6.91, respectively. The control group significantly outperformed the experimental group (p=0.01) on the pre-ACT test. At the end of the semester, students were given the same old ACT test to measure their post mathematical knowledge. The control and 8+E groups earned mean post-ACT scores of 32.81 and 32.35, with standard deviations of 6.46 and 7.13, respectively. There was no significant difference between the mean ACT scores of the two groups.

The data for total points in the course (Current Points), total points without attendance (CP w/o Attend), total points without attendance or labs (CP w/o Attend, Labs), and Current points for just exams (CP Exams Only), were compared between the two groups. Figure 2 shows the current points for the two groups and Table 1 shows the mean current points with standard deviations.

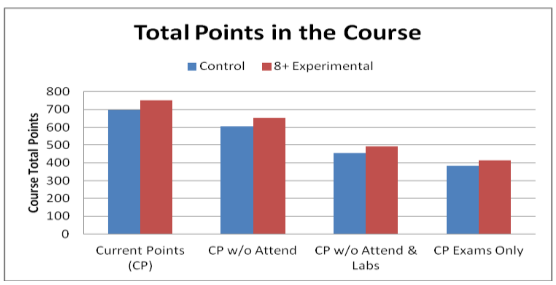


Figure 2: Total Points in the Course for Both Groups

	Current Points	CP w/o Attend	CP w/o Attend, Lab	CP Exams Only
Control Group (n=177)	698.81 (150.51)	605.88 (140.25)	453.24 (120.04)	381.72 (105.48)
8+E Group (n=279)	750.15 (117.00)	625.37 (113.88)	470.82 (104.10)	396.25 (91.69)

Table 1:Means for Total Points in the Course for Both Groups and Standard Deviations

There were strong significant differences between the mean scores of the control and 8+E groups with respect all Current Points (Total Points): Current Points ($p = 0.00007$), CP w/o Attend ($p = .00019$), CP w/o Attend & Lab ($p=0.00029$), and CP Exams Only ($p = 0.00059$). Current points did not include any extra credit (i.e. pre/post ACT exam).

The two groups were compared with respect to each exam, the final, and quizzes. Figure 3 shows the two groups mean scores on each exam, the final, and quizzes and Table 2 shows the exact scores and standard deviation in parentheses. The experimental group significantly outperformed the control group on every exam except exam 1: exam 2 ($p = 0.011$), exam 3 ($p =0.00079$), exam 4 ($p = 0.006$), quizzes ($p = 0.00012$), and final exam ($p = 0.00005$). There was no significant difference between the control and 8+E groups with respect to exam 1.

Course grade point average was calculated to compare the two groups on the average course grade earned. This was accomplished by assigned a quantitative score for the final grade that each student earned in the course (A = 4, B = 3, C = 2, D = 1, and F = 0). The course grade point averages for control group (1.97) and experimental group (2.34) had standard deviations of 1.17 and 1.16, respectively. The experimental group had a significantly better course grade point average than the control group (p = 0.00048).

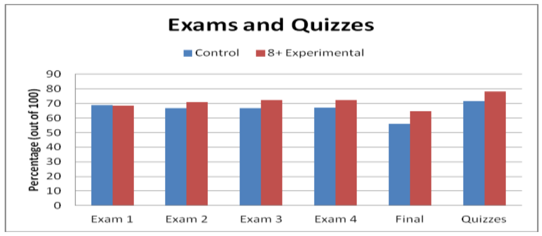


Figure 3: Exams and Quizzes for Control and 8+E

	Exam 1	Exam 2	Exam 3	Exam 4	Final	Quizzes
Control Group (n=177)	68.84 (16.21)	66.86 (19.93)	66.58 (19.61)	67.23 (22.67)	56.10 (24.02)	71.52 (18.89)
8+E Group (n=279)	68.62 (15.97)	70.86 (17.03)	72.24 (17.81)	72.35 (20.02)	64. 61(20.04)	78.09 (17.62)

Table 2: Means and Standard Deviations for Exam 1 through Exam 4, Final and Quizzes

The 8+E and Control Groups with Respect to Prior Mathematical Knowledge Level

When we compare the 8+E and control groups with respect to prior mathematical knowledge, there is some significance at the high and low prior knowledge levels (low group consisted of scores in the lower third, middle group in the middle third, and high group in the upper third on the pre-ACT test) that emerge. The control group, for all prior mathematical knowledge levels, significantly outperformed the 8+E group on the pre-ACT. The 8+ Exp group significantly outperformed the control group on the post-ACT at the high mathematical knowledge level. There was no significant difference in the post-ACT at the middle prior knowledge level and the control group significantly outperformed the 8+E group on the post-ACT at the low mathematical knowledge level.

With respect to the high prior mathematical knowledge level, the 8+E group significantly outperformed the control group on current points (no matter the format), exam 1, exam 2, and final exam. With respect to the middle prior mathematical knowledge level, the 8+E group significantly outperformed the control group on every course comparison except exam 1 where there was no significant difference. With respect to the low prior mathematical knowledge level, the 8+E group significantly outperformed the control group on every course comparison except exam 1, exam 4, and quizzes, where almost significance occurred on exam 4 and quizzes and no significance on exam 1.

The 13E Group versus the 8+E Group

The ICML purports that motivation is an important part of a students learning. In terms of this study, we examined the students that attended the majority of SP sessions versus the students that attend 8 or more of the supplemental sessions. We examine this in the study because of the ICML and because students voluntarily attended the SP sessions. Therefore, we are determining student success in the course when SP days are factored into the analysis. Instead of displaying graphs, we will state the means, standard deviations, and p-values in the table for the experimental group of students that attended 13 supplemental sessions, again denoted the 13E, and the 8+E Group.

Table 3 shows that the 13E Group significantly outperformed (usually, very strongly) the 8+E Group in every facet of the course. Therefore the students that were motivated to attend all the SP sessions significantly outperformed the students that were motivated to attend 8 or more SP sessions. The differences would be greater if we only grouped the students that attended 8 to 12 SP sessions.

	Mean 8+E (n=188)	Mean 13E (n=297)	Standard Deviation 8+E	Standard Deviation 13E	P – Value
Current Points	750.15	793.74	116.99	85.91	0.00003
CP w/o Attend	652.69	693.99	113.88	85.59	0.00005
CP w/o Attend & Lab	491.37	528.28	104.10	75.70	0.00005
CP Exams Only	413.28	443.21	91.69	69.71	0.00024
Exam 1	68.62	73.59	15.97	13.61	0.00093
Exam 2	70.86	74.36	17.03	15.39	0.02337
Exam 3	72.24	78.25	17.82	14.66	0.00029
Exam 4	72.35	79.49	20.02	13.36	0.00002
Final	129.29	137.52	40.15	32.72	0.01620
Quizzes	78.09	85.07	17.62	11.29	0.000002
Course GPA	2.344	2.752	1.16	0.955	0.00017

Table 3: Means, Standard Deviations, and p-value for 8+E and 13E Groups

Analyzing Prior Knowledge for the 8+E and 13E groups

Miller and Schrader (2011) found that only the middle prior knowledge experimental group (not the 8+E or 13E) significantly outperformed the middle prior knowledge control group with respect to every course comparison except for exam 1. The high prior knowledge experimental group significantly outperformed the high prior knowledge control group on exam 1 and quizzes, with almost significance on exam 3. A negative effect occurred with respect to the low prior knowledge group. The low prior knowledge control group significantly outperformed the low prior knowledge experimental group on each course component except for exam 2, exam 3, current points – exams only, and the final exam, with almost but not significance for current points – exams only and the final exam.

We will investigate the levels of prior knowledge further by investigating the 8+E and 13E groups based on their prior knowledge, by comparing the 13E and control groups and then comparing the 13E and 8+E groups. There was no significant difference in the Prior Mathematical Knowledge or Post Mathematical Knowledge between the control and experimental 13E group, except for the middle prior knowledge group where the control group significantly outperformed the experimental 13E group. We note that the control group, no matter the level of prior knowledge, started with a greater prior knowledge score, but finished the course with no significant difference.

The experimental 13E group significantly outperformed the control group on every course component no matter the level of prior knowledge, except for two comparisons: no significance between the high prior knowledge 13E group (p=0.073) to the high prior knowledge control group on exam 2 and between the low prior knowledge 13E and low prior knowledge control group with respect to exam 1 (see Table 4 below with boldfaced non-significance). The result on exam 1 could be attributed to the low level prior knowledge needing more help to overcome mathematical barriers and only having two SP session before exam 1. Also, low level prior knowledge students were struggling more so than the high and middle level prior knowledge students, as they acclimated themselves to the SP sessions.

The 13E group significantly outperformed the 8+E group on every component with respect to the high and medium prior knowledge levels except on the final exam for the middle prior knowledge level. Although the low prior knowledge 13E group significantly outperformed the low prior knowledge 8+E group on some of the components, there was no significance with respect to current points – exams only, exam 1, exam 2, final, and grade point average. It should be noted that the low prior knowledge 13E group did outperform the low prior knowledge 8+E group with respect to grade point average. Table 5 shows statistically comparison with non-significance, at the p = 0.05 level, in boldfaced.

	Control High	13E High	p-value	Control Middle	13E Middle	p-value	Control Low	13E Low	p-value
Current Points (CP)	774.51 (88.51)	842.62 (73.09)	0.00006	669.18 (175.71)	784.38 (85.75)	0.00003	669.57 (117.68)	756.09 (78.65)	0.00004
CP w/o attendance	678.77 (88.04)	742.75 (72.63)	0.0001	578.59 (160.41)	684.89 (84.90)	0.00003	574.42 (109.51)	656.27 (78.87)	0.00004
CP w/o attendance & labs	520.03 (78.58)	570.18 (61.25)	0.0004	431.33 (131.46)	523.19 (73.23)	0.00002	421.84 (93.94)	492.82 (73.13)	0.00006
CP exams only	442.31 (69.00)	483.29 (54.58)	0.001	362.12 (114.91)	435.59 (63.60)	0.0001	351.02 (82.84)	410.83 (87.26)	0.0001
Exam 1	75.83 (12.99)	81.18 (11.47)	0.02	67.88 (15.18)	73.09 (11.28)	0.032	64.80 (14.25)	67.02 (13.39)	**0.223**
Exam 2	77.69 (15.47)	81.97 (12.50)	**0.073**	64.41 (22.32)	74.41 (13.19)	0.004	59.08 (16.06)	67.14 (16.61)	0.011
Exam 3	74.17 (14.59)	81.97 (13.98)	0.006	66.19 (19.94)	79.26 (12.86)	0.0001	61.12 (19.02)	73.93 (16.02)	0.0004
Exam 4	75.37 (16.25)	83.95 (12.69)	0.003	63.47 (24.48)	78.82 (14.88)	0.0002	64.18 (20.52)	75.83 (12.19)	0.0006
Final	139.26 (31.85)	154.21 (26.27)	0.008	100.17 (54.63)	130.0 (34.20)	0.0008	101.84 938.98)	126.90 (31.660)	0.0005
Quizzes	77.72 (14.30)	86.89 (11.42)	0.0005	69.21 (20.09)	87.60 (9.06)	2E-08	70.82 (16.85)	81.98 (11.99)	0.0002
Grade Point Average	2.52 (0.99)	3.24 (0.73)	0.00002	1.85 (1.26)	2.62 (0.95)	0.0006	1.61 (0.91)	2.36 (0.93)	0.0001

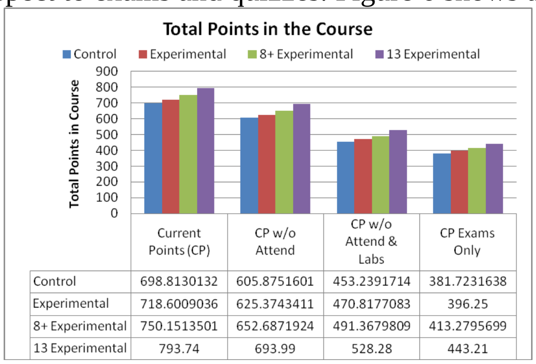
Table 4: Means, Standard Deviations (in parentheses), and p-value for control and the 13E Group with respect to prior knowledge levels

	8+ Exp High	13 Exp High	p-value	8+ Exp Middle	13 Exp Middle	p-value	8+ Exp Low	13 Exp Low	p-value
Current Points (CP)	807.93 (88.55)	842.62 (73.09)	< 0.001	755.23 (103.79)	784.38 (85.75)	0.0029	711.93 (128.55)	756.09 (78.65)	0.0233
CP w/o attendance	709.92 (86.15)	742.75 (72.63)	< 0.001	657.26 (101.60)	684.89 (84.90)	0.0039	614.83 (124.73)	656.27 (78.87)	0.0365
CP w/o attendance & labs	547.05 (71.05)	570.18 (61.25)	< 0.001	494.28 (92.98)	523.19 (73.23)	0.0019	454.74 (116.59)	492.82 (73.13)	**0.0516**
CP exams only	464.49 (64.20)	483.29 (54.58)	< 0.001	414.75 (81.20)	435.59 (63.60)	0.0082	379.18 (101.14)	410.83 (87.26)	**0.1096**
Exam 1	76.09 (12.55)	81.18 (11.47)	< 0.001	70.38 (14.20)	73.09 (11.28)	0.0026	62.16 (16.92)	67.02 (13.39)	**0.2748**
Exam 2	79.29 (12.58)	81.97 (12.50)	< 0.001	69.87 (15.23)	74.41 (13.19)	0.0220	65.58 (19.26)	67.14 (16.61)	**0.3739**
Exam 3	79.36 (14.65)	81.97 (13.98)	< 0.001	74.56 (15.36)	79.26 (12.86)	< 0.001	66.63 (19.16)	73.93 (16.02)	**0.0581**
Exam 4	79.87 (28.36)	83.95 (12.69)	< 0.001	73.73 (19.29)	78.82 (14.88)	0.0034	66.83 (22.17)	75.83 (12.19)	0.0025
Final	149.87 (28.76)	154.21 (26.27)	< 0.001	126.20 (36.70)	130.0 (34.20)	0.267	117.98 (43.50)	126.90 (31.660)	**0.3038**
Quizzes	82.56 (12.89)	86.89 (11.42)	< 0.001	79.54 (17.51)	87.60 (9.06)	< 0.001	75.56 (19.64)	81.98 (11.99)	0.0012
Grade Point Average	2.91 (0.97)	3.24 (0.73)	< 0.001	2.38 (1.11)	2.62 (0.95)	0.0221	1.98 (1.18)	2.36 (0.93)	0.072

Table 5: Means, Standard Deviations (in parentheses), and p-value for 8+E and 13E Groups with respect to prior knowledge levels (p < 0.001 denoted very small p-values)

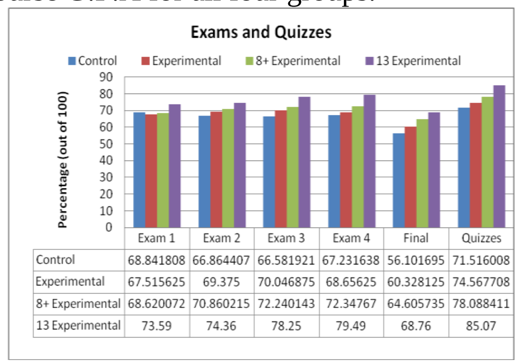
Comparing the Control, Experimental, 8+E, and 13E Groups

We have seen that the 13E group significantly outperformed the 8+E group, the 8+E group significantly outperformed the Experimental group, and the Experimental group outperformed the Control group. To see an overall picture of the data we will present graphs comparing all four groups together. Figure 4 shows the comparison of all four groups with respect to total points in the course. Figure 5 shows the comparison of all four groups with respect to exams and quizzes. Figure 6 shows the course G.P.A for all four groups.



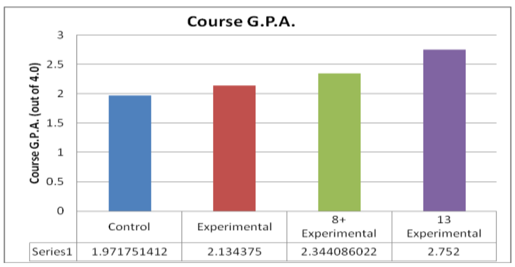
	Current Points (CP)	CP w/o Attend	CP w/o Attend & Labs	CP Exams Only
Control	698.8130132	605.8751601	453.2391714	381.7231638
Experimental	718.6009036	625.3743411	470.8177083	396.25
8+ Experimental	750.1513501	652.6871924	491.3679809	413.2795699
13 Experimental	793.74	693.99	528.28	443.21

Figure 4: Total Points for the 4 Groups



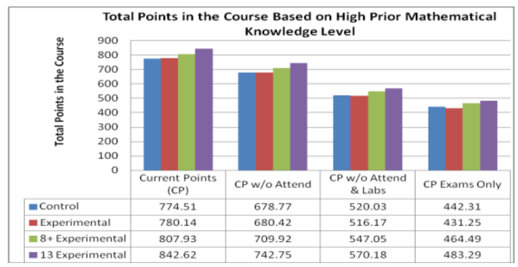
	Exam 1	Exam 2	Exam 3	Exam 4	Final	Quizzes
Control	68.841808	66.864407	66.581921	67.231638	56.101695	71.516008
Experimental	67.515625	69.375	70.046875	68.65625	60.328125	74.567708
8+ Experimental	68.620072	70.860215	72.240143	72.34767	64.605735	78.088411
13 Experimental	73.59	74.36	78.25	79.49	68.76	85.07

Figure 5: Exams and Quiz Performance for the 4 Groups



	Control	Experimental	8+ Experimental	13 Experimental
Series1	1.971751412	2.134375	2.344086022	2.752

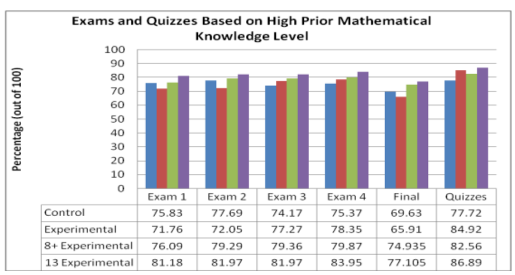
Figure 6: Course G.P.A for the 4 Groups



	Current Points (CP)	CP w/o Attend	CP w/o Attend & Labs	CP Exams Only
Control	774.51	678.77	520.03	442.31
Experimental	780.14	680.42	516.17	431.25
8+ Experimental	807.93	709.92	547.05	464.49
13 Experimental	842.62	742.75	570.18	483.29

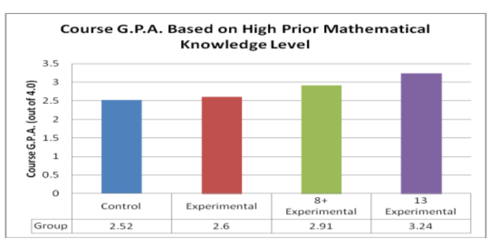
Figure 7: Total Points for the 4 High Prior Mathematical Knowledge Groups

Breaking things down in terms of the prior knowledge groups we can see more detail. Figure 7, 8, and 9 shows the comparison between the high knowledge prior knowledge group on total points, exams and quizzes, and course G.P.A.



	Exam 1	Exam 2	Exam 3	Exam 4	Final	Quizzes
Control	75.83	77.69	74.17	75.37	69.63	77.72
Experimental	71.76	72.05	77.27	78.35	65.91	84.92
8+ Experimental	76.09	79.29	79.36	79.87	74.935	82.56
13 Experimental	81.18	81.97	81.97	83.95	77.105	86.89

Figure 8: Exams and Quizzes for the 4 High Prior Mathematical Knowledge Groups



	Control	Experimental	8+ Experimental	13 Experimental
Group	2.52	2.6	2.91	3.24

Figure 9: Course G.P.A for the 4 High Prior Mathematical Knowledge Groups

Figures 10, 11, and 12 show the comparison between the middle knowledge prior knowledge group on total points, exams and quizzes, and course G.P.A. Figures 13, 14, and 15 show the comparison between the low knowledge prior knowledge group on total points, exams and quizzes, and course G.P.A.

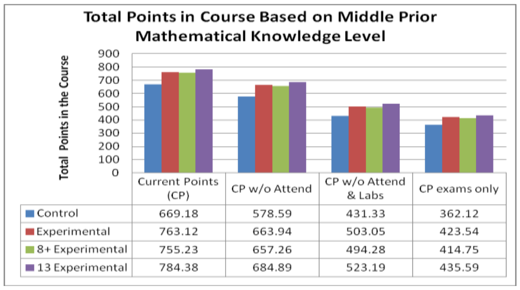


Figure 10: Total Points for the 4 Middle Prior Mathematical Knowledge Groups

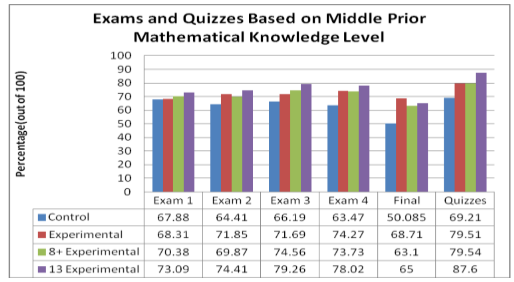


Figure 11: Exams and Quizzes for the 4 Middle Prior Mathematical Knowledge Groups

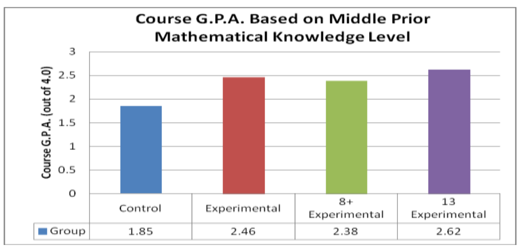


Figure 12: Course G.P.A. for the 4 Middle Prior Mathematical Knowledge Groups

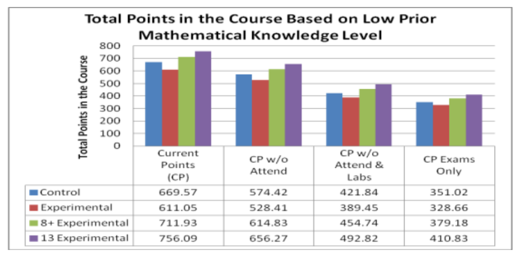


Figure 13: Total Points for the 4 Low Prior Mathematical Knowledge Groups

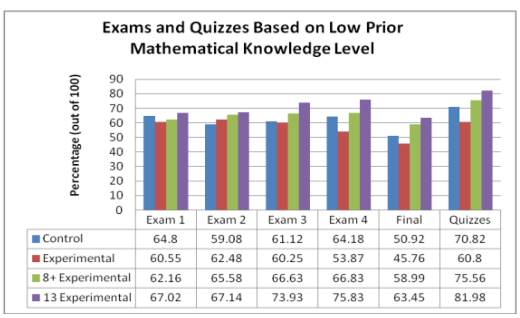


Figure 14: Exams and Quizzes for the 4 Low Prior Mathematical Knowledge Groups

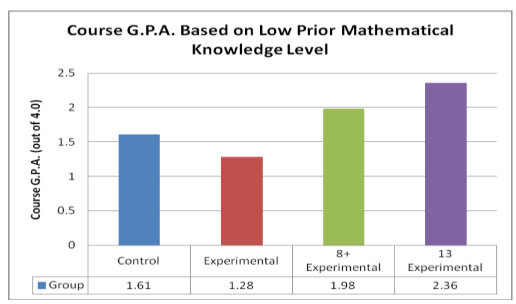


Figure 15: Course G.P.A. for the 4 Low Prior Mathematical Knowledge Group

Discussion and Results

The ICML states that (1) ability compensates in part for knowledge and regulation, (2) knowledge and regulation compensate for cognitive ability and motivation, and (3) motivation compensates for ability, knowledge and regulation. According to the ICML, motivation plays an important part in learning since a student can compensate for all other components of the ICML with motivation. We see that, although the 8+E group started the semester with significantly lower prior mathematical knowledge than the control group, they finished with no significant difference in post mathematical knowledge. The 8+E compensated for their lower prior mathematical knowledge to raise themselves to a comparable level as the control group. This compensation came from motivation and/or cognitive ability according to ICML. In terms of course components, the 8+E group significantly outperformed the control group on every course comparison. The authors compared the students that attended SP 8 or more days to the control group in order to capture how SP attendance affected students' performance in the class. We see that in (Miller and Schraeder , 2011), that the experimental group (without looking at attendance) significantly outperformed on current points – exams only, quizzes, and final exam, but there was no significance on any of the regular exams, current points with or without attendance and labs, and course G.P.A. Thus when examining students that attended SP 8 or more days, the non-significance that we had in the experimental group when compared to the control group is now significant. That is, attendance of 8 or more SP days improved a students' chance of becoming successful on each course component and in the course as a whole. The format of the extra day was important. Students in the control group were able to attend class an extra day in a question/answer format. This session was a more passive format where students asked questions and the Teaching Assistant answered the questions. Again, students did not receive attendance points for coming to the question/answer session. Overwhelmingly, 279 out of 320 attended 8 or more SP days; however, students in the control group choose overwhelmingly (6 out of 177) to not attend 8 or more of the question and answer sessions. Maybe the passive environment or the feeling that they weren't helpful led to this dismal attendance. Ultimately, the authors wanted to determine that the extra SP day helped students to be more successful in the course so that they could develop better interventions for students on the SP days. The data shows that adding the extra day of SP helps students to be more successful in the course. In particular, it helps the students that attend the SP sessions on a regularly basis.

Examining things further, the data shows that students in the middle prior mathematical knowledge group benefit the most from the SP days. Miller and Schraeder (2011) showed that the middle prior mathematical knowledge experimental group significantly outperformed the middle prior mathematical knowledge control group on every course comparison except for exam 2 and 4, where there were notable differences, and exam1 where there was no significance. There was no significance between the experimental and control group with respect to the high prior mathematical knowledge except on exam 1 where the control significantly outperformed the experimental, exam 3, where the experimental outperformed the control, and on quizzes, where the experimental significantly outperformed the control. When considering prior mathematical knowledge level in comparing the 8+E and control group, statistical significance at the high and low prior mathematical knowledge level emerge to a greater extent than what was seen when comparing the experimental and control groups. The statistical significance remains the same with smaller p-values for the middle prior mathematical knowledge level when comparing 8+E and control group to what it was when comparing the experimental and control groups.

The data shows that the students that attended all the SP days significantly outperformed the students that attend 8 or more SP days in all course comparisons. The significance ranged from $p = 0.000002$ on quizzes to $p = 0.02337$ on exam 2. It is clear that students’ performance in the course rises dramatically when they are motivated (intrinsically) to attend all the SP days. Furthermore, it does not matter whether they started the semester out with a lower prior knowledge. Students, no matter whether they have a low, medium, or high prior knowledge of algebra when they start College Algebra, that attend 13 SP days are very successful in the course and outperform students in the same prior knowledge groups that attend 8 or more SP days. The students that attend all SP days compensate for their prior knowledge level through motivation, regulation, and, as the semester progresses, strengthening their knowledge component. Motivation plays a key role with these students in the beginning of the semester, continues compensating throughout the semester, and is a factor as students build knowledge and regulation throughout the semester.

When comparing all four groups, control, experimental, 8+E, and 13E, we see that students performance in the course improves usually as the number of SP days increase. The 8+ E and 13E groups outperform the control on current points (out of 1000) nearly by 50 points and 100 points, respectively. On the exams and the final, the 8+E and 13E groups outperform the control group from a half of letter grade up to as much almost a letter grade and a half. When comparing the 8+E and 13E groups to just the experimental group the differences on exams vary from a half of a letter grade to a letter grade. As a result of this better performance, we see that students earn better grades in the course (course G.P.A.) as they attend more SP days. Similar results occurred when prior knowledge level was examined except between the experimental and control group at the low level.

SP was implemented as an extra day where students in a large lecture College Algebra class could actively engage in their own learning using worked examples from cognitive science. Students attended the SP days without getting any attendance points or extra points. It is the authors’ viewpoint that students were motivated to attend the SP days since they were actively working on problems and were given (expert) examples to help them. In addition, there was an opportunity to get help with problems they had questions on. As a consequence, students built a better understanding that helped them as they worked homework and quizzes, helped as they studied outside of class, and strengthened their knowledge-regulation component which in turn increased their learning and compensated for other weaknesses.

This study has shown that not only does the experimental group significantly outperform the control group (Miller and Schraeder, 2011) in many of the measured course categories, but the students that were motivated to attend 8 or more SP sessions (8+E group) significantly outperformed the control group in a very convincing fashion in the majority of these categories. In addition, the highly motivated students (13E group) significantly outperformed the 8+E group in a very strong fashion in every facet of the course. This provides strong evidence that motivation is a very important part in a student’s learning of College Algebra. In fact, the students in the 13E group had varying levels of prior knowledge measured by the pre-ACT (low, middle, and high), but become very successful learners. That is, students compensated for their weak prior knowledge with a stronger motivation component (along with the knowledge-regulation component that they continued to increase throughout the semester) to become very successful in the course. In the 8+E group, the middle prior knowledge group was the only group that significantly outperformed the control group, similar to what (Miller and Schraeder, 2011) found for the experimental group versus the control group.

Chi et al. (1989) state that “good” students have sufficient self-regulation skills and that “bad” students do not. We propose as a whole that the students in the low prior knowledge experimental group (“bad” students) started the course with such a low prior knowledge-base that they are not able to compensate for this lack of knowledge through knowledge-regulation or motivation. Furthermore, the students in the high prior knowledge group (“good” students) would do well in the course no matter what took place because they had sufficient knowledge-regulation and motivation. The students in the middle prior knowledge group were the ones that were helped by the worked example worksheets and active group SP sessions. In terms of the Chi et al. (1989) study, the worked example worksheets and active group SP days helped the middle prior knowledge group learn to self-regulate their learning. Furthermore, for the low prior knowledge group, attending all of the SP days helped the students become more successful. Through active group sessions and using worked example worksheets, students in this group became more aware and improved their self-regulation of their learning.

Implications to Teaching

Many college instructors teach large lecture sections of introductory mathematics classes and struggle with high percentages of students that earn grades of D or F, or simply withdraw (DFW rate) from the class. It takes resources to offer recitation sessions, out-of-class sessions, or tutoring. These are familiar modes of intervention that colleges use to lower the DFW rate and, in general, help students be more successful in learning material in a course. This study shows that carefully designed worksheets modeled after worked examples coupled with active group sessions can be very beneficial in helping students become more successful despite lower prior knowledge in a course. The SP day each week allowed extra active session where students could work on comprehending material in groups. These SP days are like an extra day of class which emphasizes only the material that has been covered during prior lectures. The obligation of the instructor is to have a group of class assistants ready to help students with the material. This extra day of class per week is very important because, for the most part, students will show up for the extra day of class to actively participate compared to an out-of-class session. Moreover, students are usually less likely to visit the instructor in his office during office hours, or go to a mathematical tutoring center. For the instructor, SP days are the most efficient way to help many students at the same time and can be thought of as an office hour with the whole class. There would be no way for the instructor to help this many students during office visits and reduces the task of explaining material many different times to different students during office hours. In addition, students review the worked example sheets outside of class and the worksheets act like a tutor by presenting students with a number of examples and problems to practice. Furthermore, students that attended all the extra day of active class were more successful in the class no matter their level of prior knowledge. Instructors are often concerned that interventions only help certain groups of students, but this study has shown that all prior knowledge levels were affected when they attended all the SP days compared to only the middle prior knowledge group for the students that attended 8 or more days of SP. Instructors would have to contemplate mandating that all students attend all SP days. It is in the opinion of the authors that you would see similar results but not as profound because students that voluntarily attend all the SP days are motivated to do so.

This study can provide some insight to teaching using worked example worksheets (or other similar interventions grounded in research) imbedded into a class as active learning to help students become more successful learners. In addition, the authors believe that the worksheets provide a basis that helps students when studying and working homework problems and quizzes which is backed up by student's comments on an end of the semester survey. Furthermore, working to design components of the class that motivate students to learn is very important in a course. Since the low prior knowledge students, in general, struggle to be successful in the course, instructors could work on providing additional intervention (perhaps mandatory) so that students could compensate for this weak area along with strengthening their knowledge. The authors plan to work on developing an additional intervention for the low prior knowledge students so that they can become more successful in the course.

Finally, the authors believe that the worked example worksheets could be expanded into worksheets over each section covered in the class and used in a large lecture inquiry based learning class. Instructors might have to teach in a mini-lecture and use the majority of the time asking students to use the worked example worksheets to build understanding. In fact, instructors plan to "flip" the classroom to make it totally active learning where the students watch short video lectures (mini-lectures) outside of class and work actively on worked example worksheets in class and therefore, make a large lecture college algebra class as active as possible.

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COVARIATIONAL REASONING AND GRAPHING IN POLAR COORDINATES

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An extensive body of research exists on students' function concept in the context of graphing in the Cartesian coordinate system (CCS). In contrast, research on student thinking in the context of the polar coordinate system (PCS) is sparse. In this report, we discuss the findings of a teaching experiment that sought to characterize two undergraduate students' thinking when graphing in the PCS. As the study progressed, the students' capacity to engage in covariational reasoning emerged as critical for their ability to graph relationships in the PCS. Additionally, such reasoning enabled the students to understand graphs in the CCS and PCS as representative of the same relationship despite differences in appearance. Collectively, our findings illustrate the importance of covariational reasoning for conceiving graphs as relationships between quantities' values and that graphing in the PCS might create one opportunity to promote such reasoning when combined with graphing in the CCS.

Key words: Polar coordinates, Covariational reasoning, Graphing, Function, Teaching experiment

Introduction

First introduced at the elementary grade levels, graphs are essential representations for the study of numerous mathematics topics including modeling relationships between quantities, exploring characteristics of functions, solving for unknown values, and investigating geometric transformations. Highlighting the central role of graphing in mathematics education, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) contains some form of the term *graph* on more than a third of the document's pages. Building off of the emphasis on graphing at the K-12 level, graphing is central to the study of several undergraduate mathematics courses, including calculus, differential equations, and analysis.

Reflecting the heavy focus on graphing in school mathematics, mathematics education research gives significant attention to graphing, with a number of studies (e.g., Carlson, 1998; Monk, 1992; Oehrtman, Carlson, & Thompson, 2008) having investigated student thinking in the context of the function concept. Although graphing receives a significant focus in mathematics education research, little of this focus includes graphing in the *polar coordinate system* (PCS). The PCS is critical to the study of calculus, complex numbers, and modeling any system that entails radial symmetry or motion about a center point. Despite the important role of the PCS in undergraduate mathematics, available research (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007) suggests students hold limiting understandings of the PCS, where some of their issues stem from problematic connections with the *Cartesian coordinate system* (CCS).

In the present study, we explore connections between student thinking when graphing in the PCS and existing research on student thinking in the context of graphing and function. Specifically, we discuss two undergraduate students' reasoning when graphing functions in the PCS. To graph functions in the PCS, they engaged in several ways of reasoning that ranged from plotting discrete points to reasoning about how quantities continuously vary in tandem. The former way of thinking enabled them to gain a sense of more basic (e.g., constant rate of change) functions, but was not sufficient in and of itself to graph more complex (e.g., trigonometric) functions and connect these graphs to their counterparts in the CCS. By engaging in covariational reasoning, students were more flexibly able to graph relationships in the PCS. Additionally, thinking about functions in terms of covariational relationships enabled the students to conceive graphs in the PCS and CCS as representing the same relationship despite the graphs' visual differences.

Background

The function concept and graphing are widespread in the teaching of mathematics. Yet, research (e.g., Carlson, 1998; Monk, 1992; Oehrtman et al., 2008; Thompson, 1994b) has revealed that students often develop understandings of the function concept, particularly in the context of graphing, that restrict their future learning. For example, students construct function and rate of change understandings that form an impoverished foundation for the study of calculus (Oehrtman et al., 2008; Thompson, 1994b). As Carlson (1998) noted, even high performing calculus students can lack robust understandings of rate of change and exhibit difficulty when interpreting graphs. These difficulties can stem from students' images of function being rooted in visual objects and not two quantities' values varying in tandem (Thompson, 1994b).

Specific to the PCS, Montiel et al. (2008) identified that students' function concept can inhibit their interpretation of graphs in the PCS. Additionally, the authors identified that the (sometimes incorrect) connections students create between the CCS and PCS are often tied to their meanings for functions and graphing in the CCS. For instance, students applied "the vertical line test" to determine if a graphed relationship in the PCS is a function. Similarly, Sayre and Wittman (2007) found that some students rely on the CCS when solving problems better suited for the PCS. Collectively, the studies highlighted that students' ways of thinking about function and graphing often do not support robust connections between the PCS and CCS, nor do their ways of thinking about the CCS support their PCS concept.

Covariational Reasoning and Connecting Coordinate Systems

In exploring students' sense making in the context of graphing and function, Carlson et al. (2002) illustrated the central role of students' capacity to engage in covariational reasoning. *Covariational reasoning*–defined as the "cognitive activities [of an individual] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354)–is central to students' understanding of numerous precalculus and calculus topics, including exponential functions (Castillo-Garsow, 2010; Confrey & Smith, 1995), trigonometric functions (Moore, 2012), rate of change (Carlson et al., 2002; Thompson, 1994a), function (Oehrtman et al., 2008), and the fundamental theorem of calculus (Thompson, 1994b). While these studies illustrate the importance of covariational reasoning in conceiving relationships between two quantities' values and using the CCS to reflect these relationships, these studies have been limited to the CCS. Due to the importance of covariational reasoning in graphing relationships, we hypothesized that covariational reasoning is critical to students' PCS graphing capabilities.

The mental actions[1] associated with covariational reasoning are not specific to the coordinate system in which one is graphing (nor are they specific to the act of graphing), and thus covariational reasoning characterizes ways of thinking that might support students in connecting relationships represented in multiple coordinate systems. To illustrate, consider

[1] See Carlson et al. (2002) for an elaborate description of the mental actions associated with covariational reasoning.

graphing the relationship defined by $f(x) = x^2$. For the function *f*, $x > 0$, as the *input* increases, the *output* increases with an increasing rate. It follows that the *output* increases such that the *change of output* also increases for successive equal *changes of input*. These *change of output* values increase by a constant amount for successive equal *changes of input*; as the *input changes* from 0 to 1 to 2 to 3 and so on, the *output increases by* 1, 3, 5, and so on; hence, the *change of output* increases by 2 for each successive *change of input* of 1.

The aforementioned covariational relationship can be represented in the CCS ($y = x^2$, Figure 1, left) and PCS ($r = \theta^2$, Figure 1, right). Although changing coordinate systems results in a different visual object, covariational reasoning enables conceiving the graphs in the same way; changing the coordinate system changes the shape of the curve, but the relationship remains invariant. Graphs in different coordinate systems form different visual objects that represent the same relationships because the shape of the graph matters only in that it represents how two quantities' values change in tandem.

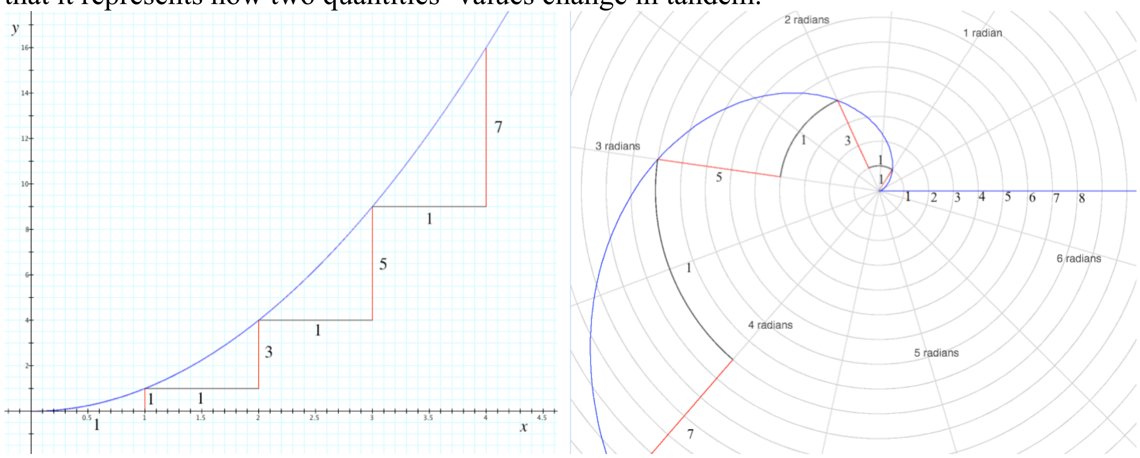

Figure 1. Graphing the same covariational relationship: $y = x^2$ (left) and $r = \theta^2$ (right).

Methodology

Stemming from radical constructivism (Glasersfeld, 1995) underpinnings, which takes the stance that an individual's knowledge is fundamentally unknowable to any other individual, we used qualitative methods to develop *models* of students' thinking (Steffe & Thompson, 2000) that explain their observable behaviors. Specifically, we conducted a teaching experiment (Steffe & Thompson, 2000) to investigate the following research questions:

1. What ways of reasoning do students engage in when graphing functions in the PCS?
2. How do the ways of reasoning identified in the first research question relate to the students' thinking when graphing functions in the CCS?

Subjects and Setting

The subjects of this study (John and Katie) were two undergraduate students enrolled in a pre-service secondary mathematics education program at a large public university in the southeast United States. At the time of data collection, the students were third year (in credits taken) students taking the first pair of courses (one methods and one content) in a pre-service secondary mathematics education program. We chose the students on a voluntary basis from the content course, in which the lead author was the instructor.

The content course engaged the students in quantitative reasoning (Thompson, 1990) and covariational reasoning to explore topics central to secondary mathematics (e.g., trigonometry, exponential functions, linear functions, rate of change, and accumulation). Prior to graphing in the PCS, the course explored ideas of angle measure and trigonometric functions. The approach to these topics was grounded in previous research (Moore, 2012) on students' learning of angle measure and trigonometric functions, and included a significant focus on covariational and quantitative reasoning. We expected the students to be familiar with covariational reasoning when entering the study, but we questioned whether they would or would not spontaneously engage in said reasoning when graphing in the PCS.

Data Collection and Analysis

The teaching experiment (Steffe & Thompson, 2000) consisted of five 75-minute teaching sessions with the pair of students. The first teaching session developed conventions of the PCS (e.g., coordinate pairs representing the distance from a fixed point and the measure of an arc) and supported students' spatial reasoning in the PCS (e.g., considering the location of a point that has a varying arc measure and a constant distance measure, and vice versa). The subsequent teaching experiment sessions, which are the focus of the present report, involved graphing functions of the form $r = f(\theta)$ or $\theta = g(r)$ in the PCS.

The teaching sessions and all student work were videotaped and digitized. Also, fellow researchers observed each teaching session, taking notes of the interactions between the researcher and students. We debriefed immediately after each session in order to discuss the students' thinking and document all instructional decisions. We analyzed the data using an open and axial coding approach (Strauss & Corbin, 1998). The data was first transcribed and instances offering insights into the students' thinking were identified. We then performed a conceptual analysis (Thompson, 2000) of these instances in order to generate and test models of the students' thinking so that these models provided viable explanations of their behaviors. We particularly sought to characterize the students' reasoning when graphing in the PCS and CCS.

Results

After exploring[2] the meaning of coordinates (e.g., a radial distance and an angle measure in radians) and various conventions of the PCS during the first teaching session, the teaching sessions explored representing relationships in the PCS. These relationships included linear functions, quadratic functions, and trigonometric functions. As John and Katie completed the proposed tasks, their solutions offered insights into ways of thinking that support graphing in the PCS and connecting graphs in the PCS and CCS.

Covariational Reasoning and Graphing Relationships

We transitioned into graphing relationships by tasking John and Katie with graphing the function $f(\theta) = 2\theta + 1$. To begin, they graphed the relationship in the CCS (e.g., $y = 2x + 1$) by identifying both the x- and y-intercepts and connecting these points with a line. The students then graphed the function in the PCS by plotting points for θ values of 0, 1, 2, 3, and 4, and connecting these points (Figure 2). They then continued describing their graphs (Table 1).

Table 1

1	Katie:	They went out by two, like you know here (*pointing at the two in the formula*
2		$r = 2\theta + 1$) the slope is like two (*tapping along the CCS graph*).
3	Int.:	This has no slope (*pointing to the PCS graph*)…
4	Katie:	No, I'm relating the slope here (*pointing to the CCS graph*), to the difference
5		in the radius of two each time (*tapping along the PCS graph*). Like [the
6		radius is] one, three, five, seven, nine, eleven *(pointing to the corresponding*
7		*points on the polar graph)*, [the radius] increases by two.

This interaction illustrates Katie reasoning about the amount of change in the distance

[2] The results of these explorations will be reported elsewhere, but we note that during the construction of the PCS, the students showed little familiarity with the PCS.

from the pole, which she referred to as the "radius," for successive changes of angle measure and connecting this relationship with the "slope" of the line in the CCS. After this interaction, Katie and John claimed that both graphs convey a "constant rate of change" between the input and output values. Katie then added, "That's cool…because you'd never see this (*referring to the PCS graph*) and be like, that's a linear function," suggesting that by conceiving the graphed relationships in terms of covarying quantities' values, they conceived both graphs, which are perceptually different, as representative of the same relationship.

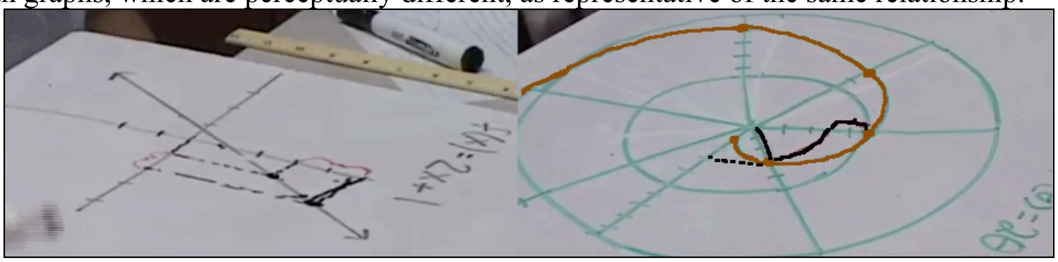

Figure 2. Students' graphs of a constant rate of change relationship.

Based on the students' approach to graphing a linear relationship, we conjectured that similar reasoning would enable them to graph a relationship with a non-constant rate of change. We asked the students to graph the relationship $r = \theta^2$, which they spontaneously compared to the relationship $r = \theta$. Like their solution to graphing the linear function, the students first plotted points and connected the points. They then compared their graphs (see Figure 3 for their written products) (Table 2).

Table 2

Katie:	r of theta, but compared to r of theta squared it's like expanded (*Katie points to the two graphs and then spreads her hands apart*). Like, like, this one's like much more tighter swirled (*moving her hands in a circular motion*) but then this one (*referring to quadratic*) is just like looser I guess.
John:	Yeah, we can see better, with both of them, both graphs, that the change in radius (*referring to quadratic*) for every radian further that the angle is increasing (*rotating his hand in successive movements while spreading his index and middle finger apart*)…Um, the radius, every time is increasing at an increasing rate (*referring to quadratic*).
Int.:	Okay now what's that mean in terms of amounts of change?
John:	We could do equal changes in theta and then…
Katie:	Like, if we looked at first these two then these two points (*indicating the points (9, 3) to (16, 4), and then (16, 4) to (25, 5)*), the change of theta here would be this, that length (*drawing an arc from (9, 3) to (9, 4)*). But then the change is radius would be up that line (*drawing a segment from the point (9, 4) to (16, 4)*).
John:	Which is seven.
Katie:	And then we have the same thing (*draws an arc from (16, 4) to (16, 5) and a segment from (16, 5) to (25, 5)*)…so you can see these black lines, the [change in radius] is increasing.
John:	So that's like nine to sixteen (*pointing to the segment connecting the points (9, 4) and (16, 4)*), which is seven, and this one is sixteen to twenty-five (*pointing to the segment connecting the points (16, 5) and (25, 5)*), which is nine, which we can see there too (*pointing to the Cartesian graph*).

Katie first compared the perceptual shapes of the graphs (lines 1-4). Following this, the students reasoned about amounts of change and rates of change between the two quantities to compare and make sense of each graph's shape. Specifically, the students reasoned that the graph of $r = \theta^2$ is "looser" or moves away from the pole "faster" because *r* increases at an increasing rate with respect to an increasing θ, which they confirmed by identifying specific changes in the quantities' values.

Immediately following John's last statement (line 24), the students denoted amounts of change on a graph in the CCS (Figure 3), while Katie claimed, "Like our change input here (*referring to CCS graph*) would represent the change in this angle measure (*indicating the corresponding change on the PCS graph*), and then our output, change of radius length, and that's increasing for equal changes." Thus, compatible with the students' actions when graphing the linear function, the students constructed a structure of covarying quantities that enabled them to see a graph in the PCS and a graph in the CCS as one in the same.

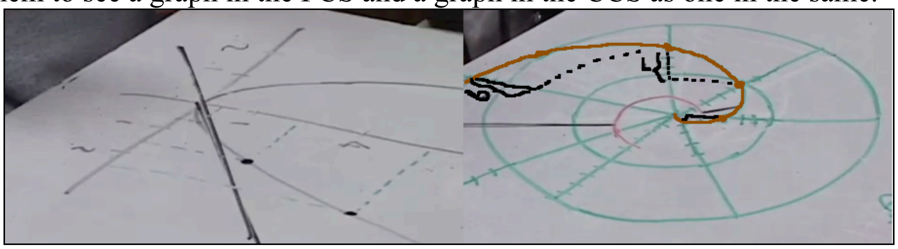

Figure 3. Students' graphs of the quadratic function.

Covariational Reasoning and Conceiving a Trigonometric Function

Over the course of the study, Katie and John leveraged covariational reasoning to construct and connect graphs in both coordinate systems. We thought the students might engage in similar reasoning to interpret a *given* graph in the PCS, and thus we tasked the students with determining a formula for a given graph ($r = \sin(\theta)$, Figure 4). After identifying *r* values corresponding to θ values of 0, π/2, π, and 3π/2, the students conjectured that $r = \sin(\theta)$ is the appropriate formula for the given graph. The students then drew a graph of the sine function in the CCS and explained their solution (Table 3).

Table 3

Katie:	So we start here *(pointing to the pole in the PCS).*
John:	Ya, and we're sweeping around *(making a circular motion with his hands).* As theta's increasing, distance away from the origin is increasing *(Katie traces along the polar graph from 0 radians to π/2 radians)* and then decreases again…it increases until pi-over-two and then it starts decreasing.
Int.:	And then what happens from like pi to two-pi?
Katie:	It's the same.
John:	Same idea except, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping (*using marker to sweep out a ray from π to 3π/2 radians – see Figure 3*) until three-pi-over-two, where it's negative one away.
Katie:	This is the biggest in magnitude, so it's the furthest away (*placing fingers at (1, 3π/2) and (1, π/2)*), and then [the distance] gets smaller in magnitude (*tracing one index finger along an arc from (1, 3π/2) to (1, 2π) and the other index finger along the graph – see Figure 3*).

When making sense of the graph, and compatible with the previous interaction (Tables 1-2), the students used a combination of identifying points and covariational reasoning. Specifically, the students reasoned about the distance from the pole as increasing or decreasing for an increasing angle measure to make sense of the relationship conveyed by the graph. For instance, Katie reasoned that as the angle measure increases from 3π/2 radians to 2π radians, the distance from the pole decreases from a magnitude of one to zero, which corresponds to the value increasing from -1 to 0 (Figure 4). Following this interaction, the students continued justifying their formula by describing that a graph of $y = \sin(x)$ in the CCS conveys the same covariational relationship with identical critical points as the PCS graph. Thus, like the previous tasks, by identifying various points and engaging in covariational reasoning, the students identified the formula for the given graph and concluded that both the CCS and PCS representations of the formula convey the same relationship.

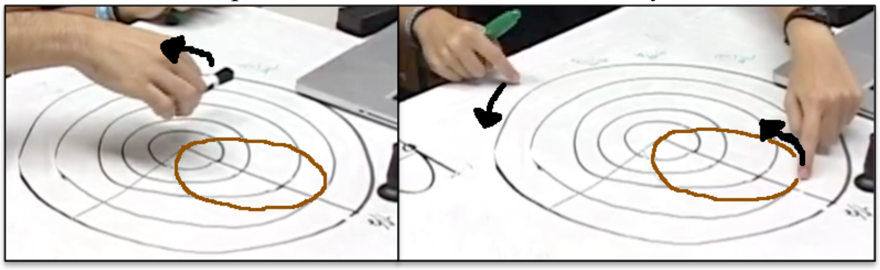

Figure 4. Students covarying quantities.

Discussion and Implications

In the present study, the students spontaneously reasoned about quantities' values varying in tandem when graphing and interpreting functions in both the CCS and PCS. The students' capacity to reason about graphs as conveying covariational relationships appeared to support connections among graphing functions in both systems. That is, as the students moved from coordinate system to coordinate system, covariational reasoning enabled the students to understand each representation (including the formula) as conveying the same relationship despite differences in the visual features of the graphs (and formula). Such reasoning, in combination with understanding the conventions of the PCS, might support avoiding the difficulties that Montiel et al. (2008) found when working with calculus students.

The findings of this study suggest that a *potential* benefit of incorporating the PCS in the study of mathematics is increasing an emphasis on reasoning that enables a student to approach graphs in both systems in compatible ways. Investigating graphing only in the CCS has the possible consequence of reinforcing common student conceptions of the function concept and graphing that do not entail reasoning about covarying quantities (e.g., conceiving graphs as pictures). By prompting students to transition from coordinate system to coordinate system, a need can be established for ways of thinking (e.g., covariational reasoning) that enable conceiving graphs in each system as conveying the same relationship. As such, graphing in both systems might foster abstractions stemming from various operations involved in covariational reasoning (e.g., rate of change reasoning and coordinating amounts of change). For instance, for Katie, graphing in both coordinate systems seemed to foreground the "constant rate of change" of a linear relationship (Table 1), as opposed to the slope of a line. At this time, this potential use of the PCS in secondary mathematics remains merely a hypothesis, and future studies should investigate promoting covariational reasoning through the use of the PCS in combination with the CCS.

We also note that the students appeared to engage in a combination of *smooth* and *chunky* images of change, as defined by Castillo-Garsow, Johnson, and Moore (submitted), when graphing relationships in the PCS. Throughout the instructional sequence, the students relied on first graphing discrete points and comparing discrete amounts of change between these points, which is suggestive of chunky images of change. The students also exhibited behaviors consistent with smooth images of change. For instance, the students reasoned about relationships in terms of one quantity increasing or decreasing for a continuous increase in the other quantity to make sense of graphs' behaviors between identified points (e.g., Table 3). The students' actions highlight the importance of both images of change for graphing relationships and constructing connections from one coordinate system to another, suggesting that instruction should emphasize both ways of reasoning so that they work in tandem. As the authors (Castillo-Garsow et al., submitted) described, these ways of thinking have different mathematical roots and consequences, and additional research is needed to investigate relationships between these ways of thinking in the context of student learning.

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ON THE EMERGENCE OF MATHEMATICAL OBJECTS: THE CASE OF e^{az}

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In this report we propose an alternate account of mathematical reification as compared to Sfard's (1991) description, which is characterized as an "instantaneous quantum leap", a mental process, and a static structure. Our perspective is based on two in-service teachers' exploration of the function $f(z) = e^{az}$, using Geometer's Sketchpad. Using microethnographic analysis techniques we found that the long road to beginning to reify the function entailed interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the "same" thing, cultural and emotional background of life with things-to-be, and categorical intuitions. Our results suggest that perceptuomotor activities involving technology may serve as an instrument in facilitating reification of abstract mathematical objects such as complex-valued functions.

Key words: Complex variables, Intuition, Perceptuomotor, Reification, Technology

Background

Sfard (1991, 1994) is one of the leading contributors to the theory, which describes how one navigates from an operational (process) conception to a structural (object) conception of the same mathematical notion and identified three stages of this development, which are *interiorization*, *condensation*, and *reification*. During the *interiorization* stage one is able to perform a process on a familiar object and thus, the process becomes a mental entity. As an example, Sfard portrays students who are proficient at taking square roots to be in the interiorization stage for conceptualizing complex numbers. *Condensation* is the second stage and it occurs when one is able to view a process as a whole and compact entity. For example, students who perceive $5+2i$ as a shorthand for certain procedures would still be able to use it in multifaceted algorithms. The third stage, *reification*, is present when one identifies a novel entity as an object-like whole. Learners who are at this stage would recognize $5+2i$ as a legitimate object and member of a well-defined set. According to Sfard,

> Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification*, therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. (1991, 19-20)

While Sfard's theoretical framework is frequently adopted for investigating mathematical reasoning, in this preliminary report we propose an alternate account of mathematical reification, which is neither "an instantaneous quantum leap," nor a matter of mental processing, nor a static structure. We describe reification as a long, never-ending, embodied process that grows organically out of perceptual and motor practices. We draw a parallel between the common experience of an object or type of object coming to be and a mathematical object coming to be. Our perspective is based on results from the research question: How do in-service teachers determine the behavior of the function e^{az+b}, after working with the function e^{z}?

Theoretical Perspective

The question of how a thing comes to be in the course of human experiences is a very old one in philosophy. All perspectives in philosophy have elaborated on ideas that relate to it in different ways, across ontology (i.e. the investigation into what kinds of entities exist, and how they are organized in classes) and epistemology (i.e. the investigation into how we come to know that which exists, and what different forms of knowable existence are). When the entities in question are mathematical (e.g. exponential functions), questions of existence and "thinghood" lay at the center of the philosophy of mathematics. A review of the relevant literature falls necessarily outside of the scope of this paper. Thus, we limit ourselves to asserting that we adopt those crucial experiential aspects of how physical objects come to be, as a guide into the emergence of mathematical objects. Specifically, we identify three major interwoven aspects that participate in the advent of a physical thing: (1) perceptuomotor activity, (2) cultural and emotional background of life with things-to-be, and (3) categorical intuitions, or the evolving intuitive sense for categories of objects.

We cite certain works of Husserl as inspiring investigations on these three aspects. Regarding perceptuomotor activity, Husserl (1907/1997) discussed the interplay between visual and tactile "spaces" in the constitution of a thing, and how these spaces get traversed and meshed by motor activity and kinesthesia (i.e. sensations of movement and of muscular activity). In this regard, Husserl elaborated on two issues that we will use for our present study: a) the interplay between body-generated motion and object self-motion (e.g. to catch a falling ball we compose our bodily running to a certain target location, while we visually track the ball as it falls "on its own"), and b) the kinesthetic continuity between different sides of the "same" thing (e.g. two sides are of the "same thing" to the extent that we can transition visually and tactually *continuously* from one to the other). Cultural and emotional background of life with things-to-be encompass our life-histories with objects, endowing them with emotional-cultural values (e.g. a certain atlas is the one my father gave me as a gift when I was a child). Husserl (1970a) proposed that these emotional-cultural histories, which attach themselves to objects, symbols, and others, form the "lifeworld" in which we live. Finally, Husserl (1970b) proposed that we perceive not only individual entities but also categories that subsume them. In other words, that the notion of a class is not derived from an intellectual grouping of individual instances — the latter presumed to be the only ones perceptually accessible — but of direct intuitive grasp. A useful example to better understand this idea is the situation of looking for something of a certain color; we are told, that a missing book "is blue" and in our efforts to find it we identify all the blue things in sight. This quality of blueness is not of a specific tone of blue unique to the missing book; rather, we survey our surroundings from which a certain color-based class of books becomes prominent and segregated in perceptuomotor ways. Capturing such interplay and continuity requires meticulous analysis, which we discuss in the methods section.

Methods

The participants in this study were enrolled in a technology course designed for pre- and in-service secondary mathematics teachers. The instructor taught the course in a situated learning environment, with a focus on complex numbers. Using *Geometer's Sketchpad (GSP)*, the students made connections between the Cartesian and polar form of complex numbers, explored the dynamic aspects of the arithmetic of complex numbers, discovered the behavior of the functions $f(z)=az+b$, $f(z)=z^2$, and $f(z)=e^z$, and examined the geometric aspects of the complex roots of a quadratic equation. A hallmark of the course was that the students' questions and discoveries served as a source for further activities. Our data for this report is a result of such a discovery made by Brent and Rick (both pseudonyms) as they explored the behavior of the function $f(z)=e^{az+b}$ using *GSP*, during an interview. We incorporated microethnographic analysis techniques (Erickson, 1996) in our research, which demands detailing moment-by-moment, audible, and visible human interactions as they occur naturally within a given context. In *Elan*, we synchronized the video with the *GSP* screenshots and documented the moment-by-moment gestures, facial expressions, utterances, and reactions, as the students interacted with one another and their *GSP* production.

In exploring the function $f(z)=e^{az+b}$, Brent and Rick simplified the task by focusing on e^{az} and created a motion controller for the parameter *a*. In our selected episode, the participants attempt to determine why the curve for e^{az} collapses to a circle. From this episode we garnered 2.33:417 minutes of data. As part of our results we provide detailed descriptions of selected moments accompanied with video clips and commentary supporting our hypothesis. We used the following coding system in our analysis: numbers in parenthesis indicate the length of a pause in seconds, bold indicates overlapping speech, and double parenthesis and italics indicates a gesture description. The B signals Brent's utterances and the R signals Rick's utterances.

Results

We exemplify four moments from the interview that implicate the constitution of e^{az} as an object. The four moments entail: interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the "same" thing, cultural and emotional background of life with things-to-be, and categorical intuitions. Before describing these moments, we provide a backdrop for the reader. The motion controller, created by Rick, enabled the participants to compare e^z and e^{az} where $z=-.5+.96i$ and $a=1.68+yi$ with $-1.5\le y\le 2.2$. Prior to our selected moments, the participants are surprised by the various behaviors of e^{az}. Rick finds it humorous that as the imaginary component of *a* approaches -1.2 the curve "flattens", and is intrigued that the curve gets "tighter" as the imaginary component gets larger and creates more spirals going counter clockwise. Both participants are mesmerized as the spiral transforms into a circle and suddenly the circle unwraps into a spiral going in the clockwise direction. They describe this instant as "cool", "crazy" and "the twilight zone".

In an effort to better understand the "something special" that occurs at this instant, Rick stops the motion controller in order to drag *a* manually and determine the exact value of *a*, that produces the circle. This is the moment where we observe interplay between body-generated motion (Rick dragging the point *a*) and object self-motion (spiral becoming a circle). With both hands on the mouse pad, Rick looks at the screen as he toggles the pointer around the point *a*. As Rick drags the point *a* down the vertical line, the spirals begin to collapse onto one another to form a tighter circle. Below is the dialogue that occurred, with descriptions of gestures, along with Figure 1, which contains video-clips and *GSP* screenshots from this exchange.

R: There's something special about this value right here ((*Rick has both hands on mouse pad as he looks at the screen and toggles the pointer around the point a. The point a moves down the vertical line and the circle become tighter*.))

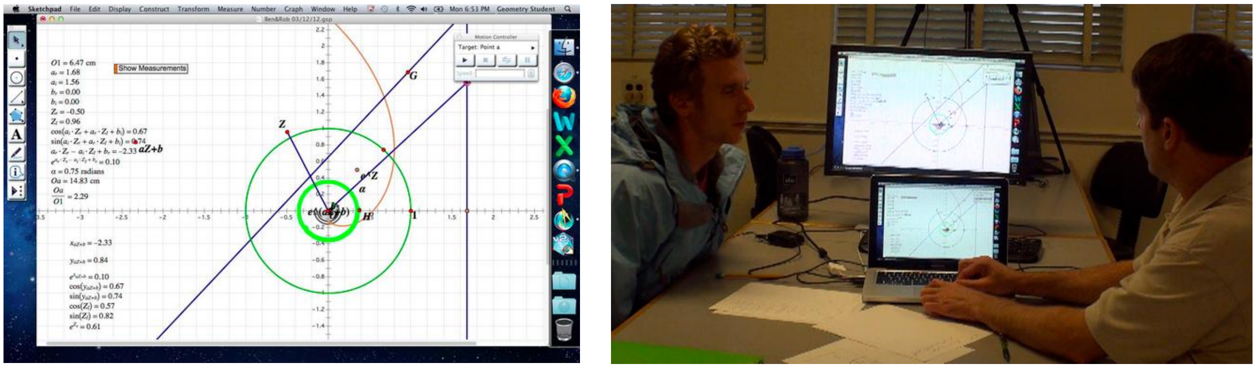


Fig. 1a. Circle circumference thicker

Fig. 1b. When it comes back on itself

Fig. 1. Something Special

B: Which value you looking at? ((*Brent brings arms down and leans slightly forward to look at screen. Rick continues to work on the mouse pad with gaze towards the screen and toggles the point a so the circumference of the circle is thicker*)) (See Fig. 1a)

B: oh when it comes kind of back on **itself** ((Brent *leans in forward to where his chest touches the edge of table as he looks at Rick who begins to lean back. Rick continues with same hand position on mouse pad causing the circumference of the circle becomes thinner*)) (See Fig. 1b)

R: **Ya** it's ya

R: **and it's** that's just ((Rick *continues to slightly toggle the pointer around the point a so that the circumference gets thicker and thinner while he looks towards the screen. At end of episode he slightly lifts his right hand off the mouse pad*))

B: **and then springs back out** ((Brent *leans back slightly, puts hands on chair and lightly lifts himself up as he turns toward the screen*))

This instant when the spiral "comes kind of back on itself … and then springs back out" prompted the participants to explore the possible origins of such behavior. During this exploration the participants discover that the curve of e^{az} becomes a circle if $a = x + yi$ and $\frac{x}{y} = 1.09$. This finding provoked the participants to investigate whether they obtain the same result when $x < 0$, which required dragging their vertical line so that it intersected the negative real axis. This moment could be considered as interplay between body-generated motion and object self-motion, but we also interpret it as kinesthetic continuity between different sides of the "same" thing, because the participants are investigating the same thing but from a different angle. As Rick drags the vertical line towards the negative real axis, he exclaims, "it's doing the same thing but in reverse". During this utterance, both participants look towards the screen as the camera zooms to the screen.

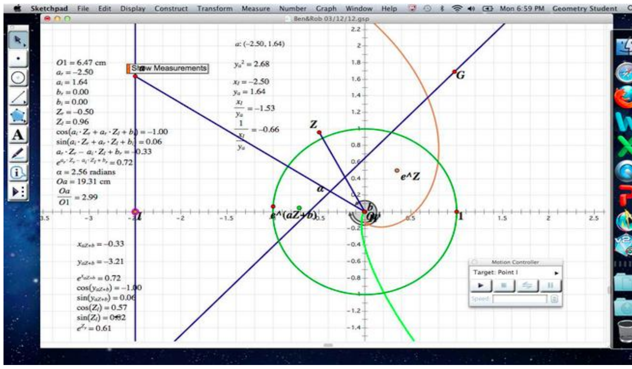


Fig. 2a. *x*-intercept of line is -2.5

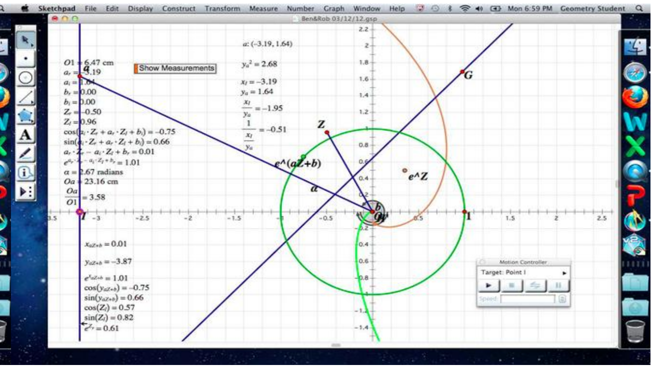


Fig. 2b. *x*-intercept of line is -3.9

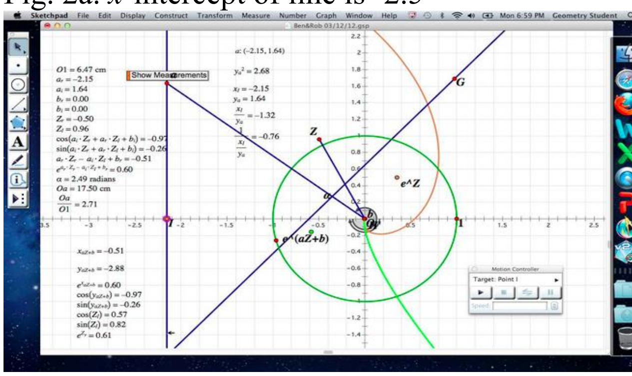


Fig. 2c. *x*-intercept of line is -2.15

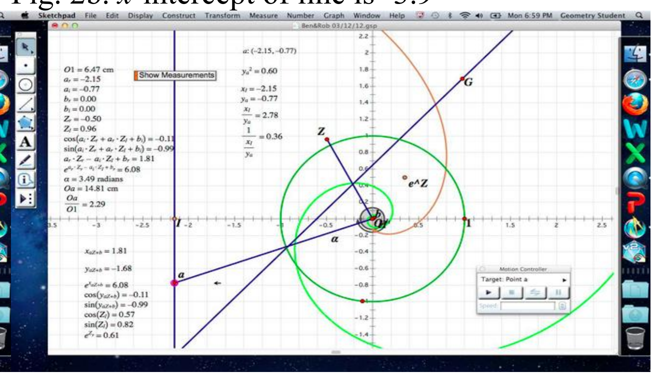


Fig. 2d. Point a = (-2.15, -0.77)

Fig. 2. Same thing but in reverse.

The *GSP* screenshot started at x=-2.50 (See Fig. 2a) and then Rick drags the vertical line using the point I (x-intercept of the line) to the point x=-3.19 (See Fig. 2b) and then brings it back to the point x = -2.15 (See Fig. 2c) where the curve of e^{az} is in the 4th quadrant but looks a bit flat. At this point, the curve of the function does not look like a spiral, which might have motivated Rick to drag the point a to (-2.15, -0.77), where the curve begins to take on the shape of a spiral (See Fig. 2d) and Brent is eager to see when the spiral will collapse onto itself. The dialogue and behavior below further illustrates that the participants fully anticipate obtaining the "same" thing for negative x-values.

B: see when you can collapse it in on itself ((*Brent has hands to his side, sitting back and looking at screen. Rick has his hands on the mouse pad and is looking at the screen. Camera starts to zoom into the screen at the point that Brent says 'when'. Initially the screenshot contains a spiral going counterclockwise and a =(-2.15,-1.84). Rick drags the point a up the vertical line to the point (-2.15,-.76) so that less of the spiral is visible.*))

(1.553) ((*Rick drags the vertical line to x=-2.05*))

B: like when we were doing ((*Rick continues dragging the vertical line to x=-1.35 and more spirals start to emerge going counterclockwise.*))

(3.900) ((*Camera is still zoomed in on the screen and starts to zoom out as Rick drags the point a down the vertical line and quickly bypasses the point where the circle appears and gets the spiral going clockwise. Both participants maintain their stance of looking towards the screen.*))

The participants' interactions with *GSP* did not fail them in generating a circle where the real component of a was negative, but it was not the circle they expected. The verbiage below highlights a moment where the participants' cultural and emotional background of life with things-to-be is accentuated. Although both Brent and Rick anticipated obtaining a circle, because of their experience with e^{az}, where the real component of a was positive, they were not prepared to see a circle of radius greater than one. Their familiarity with the behavior of e^{az} did not prepare them for the pleasant surprise that they observed on the screen – it seemed that they were anticipating a circle with radius less than one. They were also amazed to find that the circle appeared for the same ratio of $\frac{x}{y} = 1.09$.

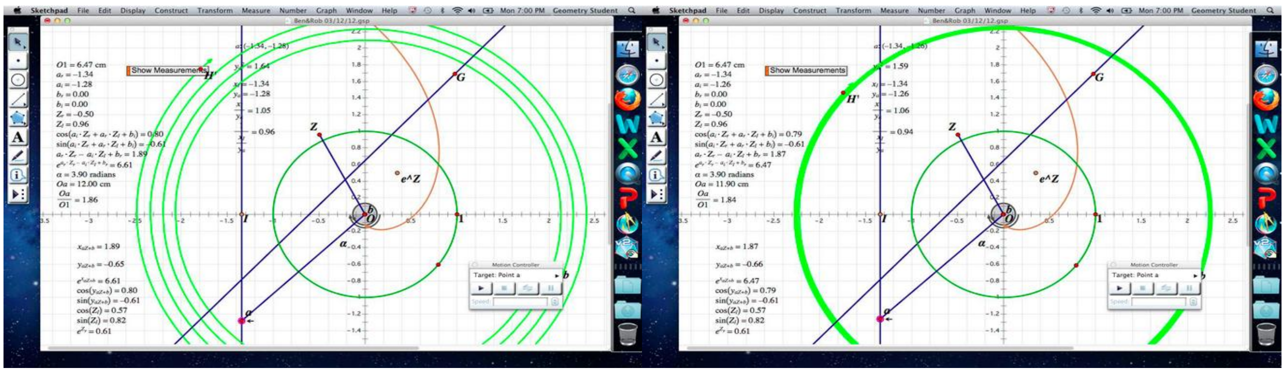


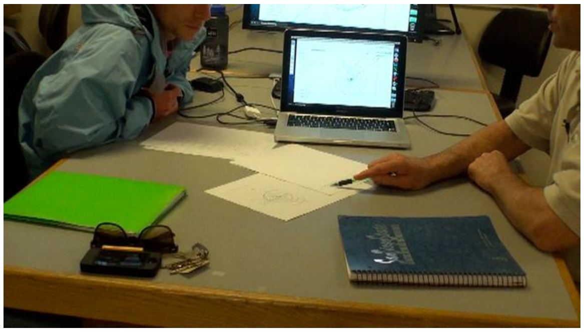
Fig. 3a. Huge Fig. 3b. Wow

Fig. 3. Circle with $x < 0$.

R: Hu**ge** ((*Rick drags the point a and the concentric circles appear to be getting closer to one another and their radii are bigger than one*)) (See Fig. 3a)
B: **Wow** ((*Rick toggles the point a so that the concentric circles become one circle at the point (-1.34,-1.26). Both participants maintain their stance*)) (See Fig. 3b)
R: Huuuge ((*GSP screen shot is the same and both participants are still looking towards the screen as the camera starts to zoom out*))
(1.356) ((*camera zooms out*))
R: oh and ((Rick *toggles the point a up and down to get a slightly tighter circle for the value a = (-1.34,-1.26)*)) **the ratio we have**
B: **and it's** still the same ((*looks towards Rick*)) ratio ((*turns back to look at screen*))

The three moments described thus far, center on the senses i.e. seeing the behavior, dragging points, and feeling anticipation and excitement. For the most part these three moments are intertwined – that is, it is possible to observe interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the "same" thing, and cultural and emotional background of life with things-to-be as the participants reason about the function e^{az}. Our fourth moment related to categorical intuitions, is slightly distinct from the others in that it centers on the algebraic notation used to treat the curve on the screen as a particular class of functions characterized by an algebraic form: e^{az}. This occurrence was prominent when the participants attempted to explain why the curve became a circle.

R: that's just this ((Rick *points to the expression on his paper with his pencil, his right pointer finger is stretched out on the pencil as his left forearm rests on the table slightly parallel to the table and his knuckles touch his right forearm which is outstretched so that the two make about a 45 degree angle. Brent leans forward with his hands clasped together and his arms lying on the edge of the table as he gazes towards Rob's paper. On GSP the point a is initially not visible because it is under the motion controller frame. The spiral looks like* *concentric circles spiraling clockwise. As a appears, the spiral becomes a circle and starts spiraling counter clockwise*)) (See Fig. 4a)



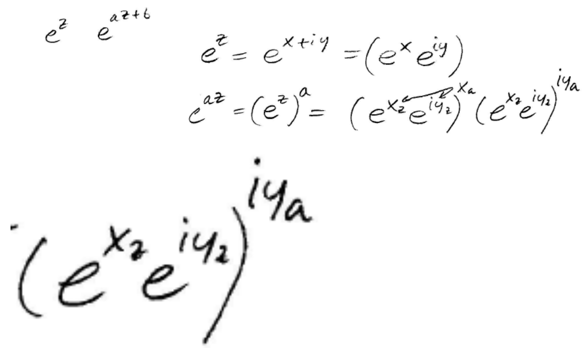


Fig. 4a. Connecting screen output to algebra Fig. 4b. Rick's algebra

Fig. 4. Algebraic notation

R: guy right here ((*Rick starts to pull his pencil away and the camera zooms into his work. On GSP the point a continues to move down the vertical line and less spirals appear on the screen*))

R: and that's ((Rick *points to the same piece of the expression using his index finger, while the point a continues to move down the vertical line*))

R: because it's taking thee ((*taps the last factor of his expression* (bold part of algebra work shown in Fig. 4b) *twice with his left index finger as he gazes forward. Brent is still leaning forward looking at Rick's paper and then turns his head left towards the computer as Rick says "thee". The point a traverses the vertical line down and gets below the x-axis.*))

R: cosine and sine of ((*While leaving his left index finger on his paper he turns towards the screen and then towards Brent as he says the word of. The point a is almost off the screen at the bottom of the vertical line and the green curve is close to intersecting the point G.*))

R: the y-value ((*looks back down at paper while Brent stays in same body position but turns his head right to look at Rick's work. On GSP the point a is traveling up the vertical line.*))

The participants' attempt to reason about the curve by thinking about the algebraic pieces that make-up the function and the role that each piece plays may not be surprising given, Sfard's (1991) stages to reification begin by performing a process on a familiar object. What is more telling is that the participants did not do this in isolation – thinking about the process entailed comparing their algebra in conjunction with the dynamic aspects of the function itself.

Discussion

In the end the participants were not able to explain why the curve of the function e^{az} collapsed to a circle, but they were on a road to reifying the function e^z. Our evidence suggests that this road did not consist solely of mental processes or working with static structures. Instead it entailed an embodied process resulting from their prior experiences as they engaged in perceptual and motor practices with *GSP*. These results may imply that perceptuomotor activities are a vital component in developing an object-view, a "thinghood" perspective of abstract notions. There are various ways how such activities can be integrated into the classroom, but it appears technology is an excellent instrument for reifying complex-valued functions.

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PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF GREATEST COMMON FACTOR STORY PROBLEMS

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Little is known about preservice elementary teachers' mathematical knowledge for teaching number theory concepts, especially greatest common factor. As part of a larger case study investigating preservice elementary teachers' understanding of topics in number theory, both content knowledge and pedagogical content knowledge (Shulman, 1986), a theoretical model for how preservice elementary teachers understand GCF story problems was developed. An emergent perspective (Cobb & Yackel, 1996) was used to collect and analyze data in the form of field notes, student coursework, and responses to task-based one-on-one interviews. The model resulted from six participants' responses to three sets of interview tasks where participants discussed concrete, visual, and story problem representations of GCF. In addition to discussing the model and relevant empirical evidence, I suggest language with which to discuss GCF representations.

Key words: Preservice Elementary Teachers, Story Problems, Number Theory

Many preservice elementary teachers have a limited understanding of the mathematics that they will teach, including many topics in number theory (e.g., Zazkis & Liljedahl, 2004), which suggests that they may not be prepared to teach mathematics for understanding. The research also suggests that pedagogical content knowledge (PCK) and specialized content knowledge (SCK) are important for teaching (e.g., Ball, Thames, & Phelps, 2008; Shulman, 1986), but little is known about preservice elementary teachers' PCK and SCK in number theory. While the literature partially addresses preservice teachers' understandings of number theory content such as evens and odds (Zazkis, 1998), multiplicative structure (Zazkis & Campbell, 1996), primes (Zazkis & Liljedahl, 2004), and least common multiple (Brown, Thomas, & Tolias, 2002), preservice elementary teachers' understanding of greatest common factor (GCF) has yet to be investigated.

My overarching research question was: What is the nature of preservice elementary teachers' understanding of topics in number theory? 'Understanding' pertains to both understanding content (primarily SCK) and PCK (Shulman, 1986; Ball, Thames, & Phelps, 2008) of number theory topics such as GCF, least common multiple (LCM), prime numbers, prime factorization, and congruences. As little is known about preservice elementary teachers' understanding of GCF in particular, this report will focus on this topic.

Methodology

I conducted an interpretive case study (Merriam, 1998) of preservice elementary teachers with a mathematics concentration enrolled in a number theory course. Data for this study came from multiple sources: classroom observational notes, student coursework, as well as responses from two sets of one-on-one, task-based interviews. This report focuses on the development of a theoretical model that emerged from the six interview participants' (Brit, Cara, Eden, Gwen, Isla, and Lucy) responses to three sets of tasks pertaining to GCF representations.

The first set of tasks, posed during the first interview, asked participants to (1) create a GCF story problem that would require one to determine the GCF of 28 and 32, (2) create or describe representations for the GCF of 28 and 32 using manipulatives or pictures, and (3) identify GCF story problems from a list of given story problems. Participants' responses to this set of tasks led me to pose two follow-up sets of tasks during the second interview. First, I asked participants to create concrete, visual, and story problem representations for division and to identify valid division story problems. Finally, I asked participants to validate story problems created by hypothetical students.

An emergent perspective (Cobb & Yackel, 1996) served as my lens for collecting and analyzing data. I primarily used the psychological lens to analyze the data described in this report, since my model is based on individual conceptions about GCF representations. Constant-comparative coding (Corbin & Strauss, 2008) was used as part of the coding process. After determining emergent themes, I found that quite a few of these codes contributed to participants' understanding of GCF story problems, a model for which I describe here.

Results

Participants did not have an opportunity to create, or even answer, GCF story problems in their number theory class. They were, however, given the opportunity to briefly explore LCM visual, concrete, and story problem representations while working on an assignment. Some participants designed tasks that required students to use Cuisenaire rods to find the LCM of two numbers while others wrote story problems involving LCM in some way. Their limited experience with GCF story problems presented me with a unique opportunity during the interviews to observe their processes for trying to understand this novel concept. In the next few sections, I generalize this process using specific examples from the interviews.

Creating Concrete and Visual Representations of GCF

The first GCF task I posed to participants was to create a GCF story problem. Aside from Brit, who immediately attempted to create a story problem, participants engaged in different activities to help them respond to this task. Eden and Isla verbally recalled the basic definition of GCF; others used numerical methods to find the GCF of 28 and 32. For instance, Cara and Eden found the GCF by selecting the largest common factor after listing all of the factors of 28 and 32. Isla and Lucy discussed how to use factor trees to find the GCF of two numbers. Participants switched strategies quickly, as these did not appear sufficient for creating story problems. Most spontaneously created visual or concrete representations of GCF with which to inspire their story problems.For instance, after finding the GCF of 28 and 32, Cara described how to break up 28 and 32 objects to show the GCF.

Cara: So you would end up with 4 groups of a certain number in it. So for 28, you would have 4 groups of 7, and with 32 you would have 4 groups of 8. So the number in your groups would be different, but the amount of groups is the same, showing that that represents the [greatest] common divisor.

Gwen had a different approach for representing GCF, and she even began discussing how she might use this representation to create a story problem.

Gwen: So maybe we have 28 objects and 32 objects… make them into equal groups with the same amount in each group for… So there's going to be 8… this is going to have 7… I'm thinking that this will show that there's 4 in each one. But I would have to word it in a way that would make sense that there are equal groups in each one for the numbers 28 and 32 to have the same amount in each group...

While Cara and Gwen both accurately represented the GCF of 28 and 32, neither one described how you might *find* the GCF. Instead, they first found the GCF, then broke up the groups of 28 and 32 objects into smaller groups using the GCF. This process de-emphasizes the importance of maximizing the common factor, and it proved to be problematic when they attempted to create story problems from their representations. Brit, Isla, and Lucy described how they might use their representations to find the GCF, and they all attempted to account for maximizing the common factor in their story problems.

As exemplified by Cara's and Gwen's responses, participants created two types of GCF representations, each one drawing from a different meaning of division. While the literature frequently refers to these two meanings or models of division as *partititve* and *measurement* (e.g., Ball, 1990), participants were familiar with Beckmann (2008), so the language I suggest here draws from her terminology. Beckmann refers to these meanings as the "How many groups?" meaning of division, where the quotient is represented by the number of groups you can make and the divisor determines the size of the groups, and the "How many in each group?" meaning of division, when the quotient is represented by the number of objects in each group and the divisor determines the number of groups. While intuitively it makes sense that there should also be two meanings of GCF, a review of relevant textbook materials and research did not reveal language with which to discuss them.

We determine the GCF of two numbers, *A* and *B*, by breaking down *A* objects and *B* objects into equal groups (either an equal number of groups or equal sized groups). However, participants frequently referred to the *A* objects and the *B* objects as "groups" as well. To avoid confusion, I reserve the term "group" for the groups of *A* or *B* objects. I refer to the smaller groups that amount to *A* or *B* objects as "subgroups". Thus, for the purposes of this study and drawing from Beckmann's (2008) phrasing, I refer to the two meanings of GCF as "How many subgroups?" and "How many in each subgroup?". Besides Cara, Lucy was the only other participant to create a "How many subgroups?" representation of GCF. Gwen, Brit, Cara, Eden, and Isla created "How many in each subgroup?" representations of GCF.

I suspected participants' understanding of division to be connected to their understanding of GCF due to the connection between the meanings of division and GCF. For instance, I suspected that Lucy, who demonstrated a strong understanding of and a strong inclination towards the "How many subgroups?" meaning of GCF, would be more inclined towards the "How many groups?" meaning of division. To investigate this, I posed tasks pertaining to various representations of division during the second interview. Surprisingly, Isla and Lucy demonstrated an inclination towards the other meaning of division.

Creating GCF Story Problems

Aside from Eden, all of the interview participants attempted to create GCF story problems, with varying degrees of success. A GCF story problem should maintain a GCF structure (groups of objects broken into subgroups, where the number of subgroups or the number of objects in each subgroup is maximized), but there are other things to consider; the narrative of a story problem contextualizes the structure of a mathematical concept, and a story problem poses a question related to this concept for students to answer. For GCF story problems, this question should be precise enough that the only answer is the GCF. Additionally, to ensure that the story problem is as authentic to real life as possible, the context should necessitate the conditions of the mathematical structure somehow. With GCF story problems, it is not enough to describe breaking up groups of objects into smaller groups; the context of more authentic story problems presents a reason for doing so.

Unsurprisingly, participants' GCF story problems drew from the meaning of GCF they used to create their visual or concrete representations of GCF. Lucy created a "How many subgroups?" GCF story problem, while Brit, Gwen, and Isla created "How many in each subgroup?" GCF story problems. Cara was the only participant to create both types of story problems. It is important for the factoring structure of the story problem to specify that all objects are used by the subgroups, but Brit and Isla were the only participants to explicitly mention this. All of the interview participants sufficiently established that they were looking for common factors by stating that the number of subgroups or the size of the subgroups between groups should be the same. However, Cara and Gwen neglected to include a statement maximizing the common factor. Recall that both Cara and Gwen deemphasized maximizing the common factor in their visual or concrete representations of GCF. Brit, Isla, and Lucy used the words "greatest", "largest number", and "highest number", respectively, in their story problems to indicate that they were looking for the GCF. However their phrasing was choppy or unclear, which may indicate difficulty in contextualizing this condition.

While Brit's story problem was somewhat unclear and required clarification, it was perhaps the most contextualized of the story problems.

Brit: I have 28 dinosaur stickers and 32 flower stickers and I want to group the dinosaur stickers and the flower stickers together… and I want to give them to individual students. So I want to know what is the greatest… how many, how many dinosaur stickers and flower stickers am I going to need in each group? I want to use all of them in an equal amount of groups. So I want to know how many stickers are going to be in each group.

Not only were the numbers 28 and 32 put into a context, but most of her conditions were also phrased consistently with this context. Maximizing the common factors was the only condition that she neglected to phrase in context. Cara's story problems were similarly contextualized, but it lacks reasoning for grouping objects the way Cara suggested. Gwen, Isla, and Lucy, however, posed story problems that were contrived and barely contextualized. As an example, consider Lucy's story problem below.

Lucy: Someone has 32 marbles and then student B has 28 marbles. Can they divide them into the same amount of groups, like the highest number, the same amount of groups?

The only discernable difference between Gwen, Isla, and Lucy visual or concrete representations and their story problems was that they posed a question for students to answer. Lucy's question, however, was vague and would not result in the GCF. Isla and Cara posed similarly vague questions. While the other participants posed more specific questions, they would result in common factors rather than the GCF, specifically.

Validating GCF Story Problems

Shortly after I asked participants to create GCF story problems, I asked them to identify valid GCF story problems from a list of four story problems, three of which were valid. The first of these three story problems was structured using the "How many subgroups?" meaning of GCF, while the other two used the "How many in each subgroup?" meaning of GCF. Cara and Lucy, the participants that created "How many subgroups?" representations of GCF, were the quickest to recognize the validity of the first story problem. Gwen eventually solved the story problem to validate it, Isla thought it might be valid but could not explain why, and Eden was not sure. Brit, however, was convinced that it was not a valid story problem because it was "asking the wrong question", i.e., a "How many subgroups?" question. Similarly, Cara and Lucy determined that the "How many in each subgroup?" story problems were asking the wrong questions. This was surprising considering Cara's success in modeling GCF representations using both meanings. The other participants were more likely to correctly identify the "How many in each subgroup?" story problems, as they created similar representations themselves.

While this task helped me to identify participants' predilection to a certain meaning of GCF, it did not provide participants with a sufficient opportunity for discussing the various minutiae involved with GCF story problem design. Thus, in the second interview, I asked participants to critique hypothetical student story problems with various issues, inspired by the participants themselves. One story problem did not maximize the common factor, while the other maximized the wrong factor, not all of the objects were used, and posed an incorrect question. Both of these story problems drew from the "How many in each subgroup?" meaning of GCF, as more participants demonstrated success in this type of representation. Surprisingly, in spite of this success, participants incorrectly determined that these story problems were, for the most part, valid. Brit was the only participant to accurately identify more than one issue. Lucy incorrectly determined the second story problem to be valid since it posed a "How many subgroups?" question, indicative of the "How many subgroups?" structure she was oriented towards. In general, the results of the validation tasks implied that success in creating GCF story problems was not indicative of success in validating them, or vice versa, since Eden felt unable to create a GCF story problem but demonstrated some success in validating them.

Theoretical Model for Understanding GCF Story Problems

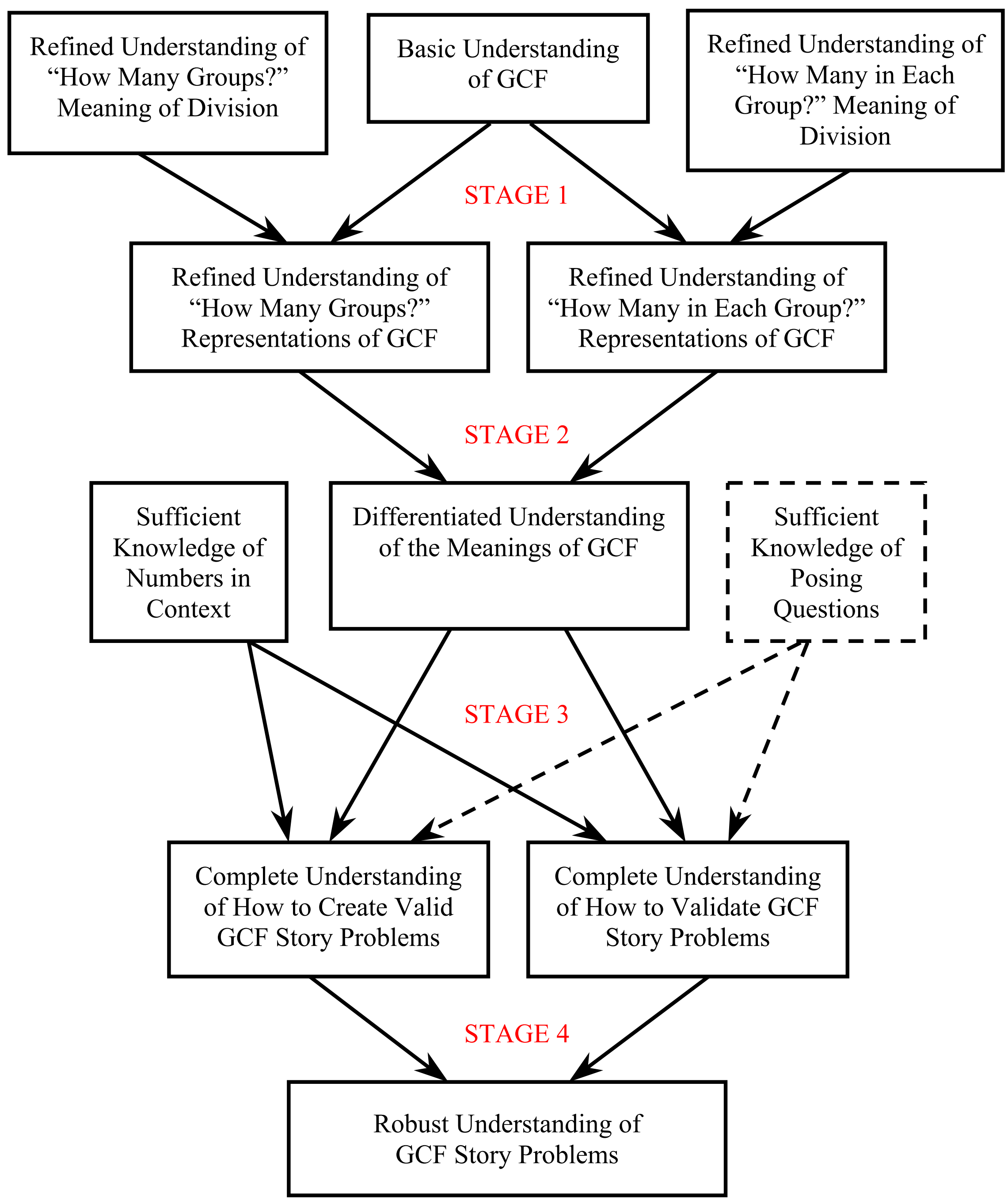

Figure 1. Preservice elementary teachers' process for developing a robust understanding of GCF story problems

The figure above is a proposed model for preservice elementary teachers' process for developing a robust understanding of GCF story problems. Due to the similarities in their representations, it is clear that representing GCF is connected to and understanding of the meanings of division. However, as evidenced by Isla and Lucy specifically, this connection is not always made, which can hinder preservice elementary teachers' understanding of GCF. This suggests the importance of making clear connections between the meanings of division and the basic definition of GCF, Stage 1 in the model, to help students develop a clear understanding of the different representations of GCF and their structures. Participants who demonstrated finding the GCF using their representation had a better understanding of the representation than the participants who used the GCF value to visually divide the original two numbers. This distinction proved to be important when participants created story problems, so using their representations to find GCF should be emphasized in scaffolding preservice elementary teachers' understanding of GCF representations.

Most participants did not represent GCF using the two different meanings, and thus did not reconcile the differences and similarities between the two before attempting to create or validate a GCF story problem. As a result, these participants partially conflated the two meanings of GCF in their story problems due to ambiguous wording, they struggled to validate story problems dissimilar from their own, and they struggled to identify students' mistakes in creating GCF story problems. This suggests that comparing and contrasting the two types of GCF representations, Stage 2, thus encouraging the development of a differentiate understanding of GCF, may facilitate keeping track of the various minutiae involved with GCF story problems.

Even though Cara demonstrated a relatively differentiated understanding of GCF, having successfully created and compared both types of representations, she had limited success in validating GCF story problems because she struggled understanding the GCF structure in context. Furthermore, Eden, Gwen, Isla, and Lucy all struggled, to varying degrees, contextualizing their GCF representations. This suggests that a differentiated understanding alone is insufficient, and that perhaps participants require some understanding of numbers in context to create and validate GCF story problems. Half of the participants also posed vague questions in their story problems or incorrectly validated the questions posed in given story problems. It is unclear if this is due to a weak understanding of the meanings of GCF, or if preservice elementary teachers' understanding of GCF story problems might benefit from a general understanding of how to pose questions. Regardless, in Stage 3 of the model, it is important that preservice elementary teachers negotiate their understanding of GCF with their understanding of how numbers behave in context to gain more complete understandings of how to create and validate GCF story problems.

While participants simultaneously developed understandings pertaining to validating and creating GCF story problems in Stage 3, I propose that it is not until they have successfully done both of these things and reconciled the two types of experiences, Stage 4, that they will have a robust understanding of GCF story problems. It is even possible that an interplay between the two concept images is necessary before either one is robust.

Discussion and Implications

The research pertaining to how preservice elementary teachers understand story problems and how to create them is limited. However, the researchers who have investigated this phenomenon in part (e.g., Ball, 1990; Crespo, 2003; Goodson-Espy, 2009) have found that preservice elementary teachers tend to struggle to represent mathematics contextually through story problems. Understanding story problems, creating and critiquing them, lies within the realm of specialized content knowledge or SCK (Ball, Thames, & Phelps, 2008). Creating and critiquing story problems requires content knowledge specific to teachers, but as many of my participants acknowledged, story problems are also useful in "helping kids understand" concepts, suggesting that their use in the classroom may demonstrate pedagogical content knowledge or PCK (Shulman, 1986). As story problems can be an important pedagogical tool, especially in elementary school, the proposed model has implications in elementary teacher education. While this model specifically informs teacher educators on how to best scaffold preservice elementary teachers' understanding of GCF story problems, it is likely that a similar process might be useful for understanding number and operations story problems in general.

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STUDENTS' EMERGING UNDERSTANDINGS OF THE POLAR COORDINATE SYSTEM

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The Polar Coordinate System (PCS) arises in a multitude of contexts in undergraduate mathematics. Yet, there is a limited body of research investigating students' understandings of the PCS. In this report, we discuss findings from a teaching experiment concerned with exploring four pre-service teachers' developing understandings of the PCS. We illustrate ways students' meanings for angle measure influenced their construction of the PCS. Specifically, students with a stronger understanding of radian angle measure more fluently constructed the PCS than their counterparts. Also, we found that various aspects of the students' understandings of the Cartesian coordinate system (CCS) became problematic as they transitioned to the PCS. For instance, mathematical differences between the polar pole and Cartesian origin presented the students difficulties. Collectively, our findings highlight important understandings that can support or prevent students from developing a robust conception of the PCS.

Key Words: Polar Coordinate System, Cartesian Coordinate System, Pre-service Secondary Teachers, Teaching Experiment, Quantitative Reasoning

Introduction

In addition to having many mathematical applications, polar coordinates have a multitude of real-world applications, some found in physics and engineering (Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007). Often, students are introduced to polar coordinates in pre-calculus, where the emphasis tends to be on geometric interpretations and conversions between Cartesian and polar coordinates (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009). Polar coordinates later arise in single-variable calculus where students are asked to determine areas contained by polar curves using integration and again in multi-variable calculus where they are used to model three-dimensional objects using spherical and cylindrical coordinates.

Available research, which is sparse, on students' understandings of polar coordinates mainly focuses on student conceptions and misconceptions about the *polar coordinate system* (*PCS*). However, there is little research investigating problems that arise as students *construct* the PCS, which has inherently different features than the *Cartesian coordinate system* (*CCS*). In order to better understand students' conceptions of the PCS, we explored four secondary pre-service teachers' (who we refer to as students) thinking using a teaching experiment methodology (Steffe & Thompson, 2000). We used this methodology to investigate the questions: (a) What ways of thinking do students engage in when constructing the PCS? (b) What issues arise during students' construction of the PCS? (c) How do students utilize their understanding of the CCS while learning the PCS? and (d) How do conventions of the CCS influence students' understanding of the PCS?

In the present work, we focus on issues that arose when students were developing meanings of the PCS. For example, students' meanings for angle measure influenced their conception of the PCS. Further, consistent with previous research, we found some students relied on rules and conventions from the CCS when plotting points or interpreting graphs in the PCS. Against the backdrop of such issues, we highlight ways of reasoning that emerged as important for the PCS.

Background

Research has shown collegiate students often struggle with the PCS, both in mathematical and real-world application problems (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007). Sayre and Wittman (2007) identified that engineering students with a stronger understanding of the PCS used the PCS to simplify real-world problems while students with a weaker understanding of the PCS relied on the CCS, often to their own detriment, in similar situations. This research points to the importance of students developing a robust understanding of the PCS in order to interpret and solve problems that arise in real-world contexts, particularly in engineering and applied fields.

Montiel et al. (2008) investigated second semester calculus students' understanding of the PCS by focusing on students' function concept and how they adapted their definition within the PCS. They found many students equating the vertical line test with the definition of function, which became problematic in the PCS. For instance, students applied the vertical line test to the polar graph of $r = 2$ (forming a circle) to conclude the graph was not a function. Others first converted the equation to its CCS form before applying the vertical line test; again concluding the graph did not represent a function. Even students who reasoned that a function is a relation where each input is mapped to one output struggled when attempting to interpret relations in the PCS. Montiel et al. (2008) also noted that students struggled with the notational convention of the input appearing after the output in ordered pairs when asked to graph functions of the form $r(\theta)$. Summarizing their findings, they claimed, "some misconceptions about functions in rectangular coordinates transfer to polar coordinates, and that some new misconceptions related specifically to the polar coordinate system arise" (p. 62 Montiel et al., 2008).

The aforementioned study (Montiel et al., 2008) was a precursor to a larger study exploring multi-variable calculus students' conceptions of the three-dimensional rectangular, polar, and cylindrical space (Montiel, et al., 2009). Both studies (Montiel et al., 2008; Montiel et al., 2009) emphasized the importance of students developing a robust understanding of the various representations of functions in different systems, as well as the ways in which conventions played a role in students' mathematical understandings. Both relied on comparing their research to current calculus textbooks as a way to compare and contrast their findings with practices supported by these text, and both articles were mainly concerned with students' conception of function in interaction with multiple coordinate systems. We extend this work by looking at issues that arose when students construct the PCS. We conjectured that aspects of the PCS could be problematic because of the way in which the PCS is defined. For instance, whereas the CCS is based on two directional lengths, the PCS includes a coordinate that represents an angle measure, and thus how a student conceives the PCS will be reliant on their angle measure meanings.

Quantitative Reasoning and the Polar Coordinate System

In addition to leveraging available research on the PCS, we also rely on theories of *quantitative reasoning* (QR) (Smith III & Thompson, 2008; Thompson, 2011) to inform the study. QR takes the stance that quantities are cognitive constructions and therefore should not be taken as a given (Thompson, 2011). As an illustration of QR, consider the role of angle measure in the PCS. Previous research (Moore, 2012) has illustrated that QR plays an important role in students' meanings for angle measure, including supporting students in coming to understand angle measure as an equivalence class of arcs. Because the PCS entails a coordinate that conveys an angle measure, we conjectured that students' meanings for angle measure will contribute to their capacity to construct and apply the PCS. For instance, consider the set of points defined by $(r,2)$ in the PCS; the radial component defines a set of points *r* units from the pole and the angular component defines a set of points forming a ray at an angle of measure 2 radians counter-clockwise from the polar axis. When viewing the set of points defined by $(r,2), r>0$, with an understanding of angle measure rooted in arcs, one can conceive *every* arc subtended by the formed angle as having a length of 2 radii when measured in a magnitude equivalent to that circle's radius (Figure 1); the ray defines an equivalence class of arcs upon which the coordinate system is based. On the other hand, as we later illustrate, without understanding the ray as defining an equivalence class of arcs tied to measuring in radii, one is left with arcs of different lengths being used to represent what is supposed to be a constant value. This image can lead to interpreting points along a ray as having different angular (arc) values.

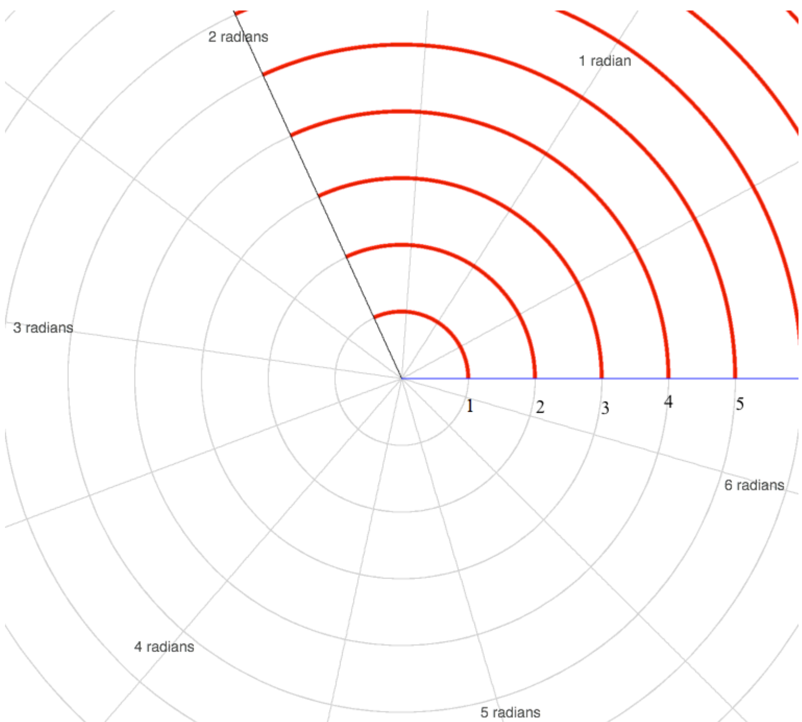

Figure 1. Polar coordinates and the angular component defining an equivalence class of arcs.

Methods and Subjects

The four participants (Katie, Jenna, John, and Steve) of the study were third-year (in credits taken) undergraduate students at a large state university in southeast United States, enrolled in a pre-service secondary mathematics education program. Prior to the study, all students had taken mathematics courses through at least Calculus II. We chose the four students from a pool who volunteered based on results from a pre-assessment given to students at the beginning of a content course and homework evaluations for the first month of this class. Throughout the study, the students worked in pairs (Steve & Jenna / Katie & John). Based on their homework evaluations and pre-assessment, we paired two higher performing students (Katie & John) and two lower performing students (Steve & Jenna). We conjectured pairing in this manner would help us identify different ways of thinking about the PCS.

Our intention in the study, based on a radical constructivist epistemology (Glasersfeld, 1995), was to build and test models of the students' thinking. To accomplish this goal we conducted a teaching experiment (Steffe & Thompson, 2000) with each pair of students working together with one teacher-researcher while an observer-researcher videotaped each of the sessions. Over the course of three weeks, the teaching experiment spanned five sessions, each approximately one hour and fifteen minutes long.

All of the videos were transcribed and then reviewed individually by the research team. When analyzing the data, we used a combination of an open and axial coding approach (Strauss & Corbin, 1998) and conceptual analysis (Thompson, 2000). We first analyzed each pair of students' words and actions in order to characterize their thinking. We further analyzed common and often unexpected themes which arose in our analyses. Upon our characterization of these themes, we searched the data for instances confirming or conflicting with our models of the students' thinking. We sought to build more viable models (Steffe & Thompson, 2000) of the students' thinking, including shifts in their thinking, through this iterative process.

Results

Through our conceptual analysis, many themes in the students' activity emerged. We focus our results around two themes. The first theme centers on the importance of students' radian angle measure meanings in the PCS. The second theme deals with characteristics of the polar and Cartesian systems, and how the students handled various conventions of the systems.

The Importance of Radian Angle Measure

The first activity in the teaching experiment asked the students to determine what information a surface radar system on a ship, with the ship defining a center point, might give in order to determine the exact location of another object at sea. We posed that the system provides how far a detected object is from the ship and asked the students to determine a second quantity that would define an exact location of the object. Our goal was for students to realize a distance from the fixed point and an angle measure (with a defined reference ray and rotation direction) suffices to provide an exact location. At the point in which the students determined these quantities, our goal was to task them with constructing a coordinate system based on these quantities (e.g., the PCS).

When comparing the two pairs of students' construction of the PCS over the course of the task, it became apparent that a robust understanding of angle measure was critical to their work on the task. Prior to the teaching experiment, students received instruction intended to develop a strong understanding of radian angle measure using curricula based in research on the topic (Moore, 2012). Unsurprisingly, the students' work on the radar task established that the students had developed different understandings during the angle measure lessons.

Both Katie and John exhibited an understanding of radian measure consistent with that described above (e.g., equivalence of arcs based in measuring in radii) when constructing the PCS. As such, Katie and John experienced little difficulty constructing a coordinate system that involved an angle measure and a radial distance, and often used measuring along arcs to give meaning to angle measure coordinates. Contrarily, early in the teaching experiment, Steve and Jenna experienced difficulty constructing the angular component in the PCS.

Beginning the task, Steve and Jenna raised the idea of using angle measure to provide a second quantity for the radar system. The pair chose a radial unit, which represented a fixed magnitude, and drew several concentric circles based on this unit. They experienced a perturbation when attempting to reason about points along a single ray as being an equivalent arc from the reference ray. With a fixed unit length in mind, Steve and Jenna interpreted arcs on different circles but subtended by the same angle as producing different coordinate measures (Figure 2). Thus, the students questioned using angle measure, as they had conceived, as an appropriate coordinate.

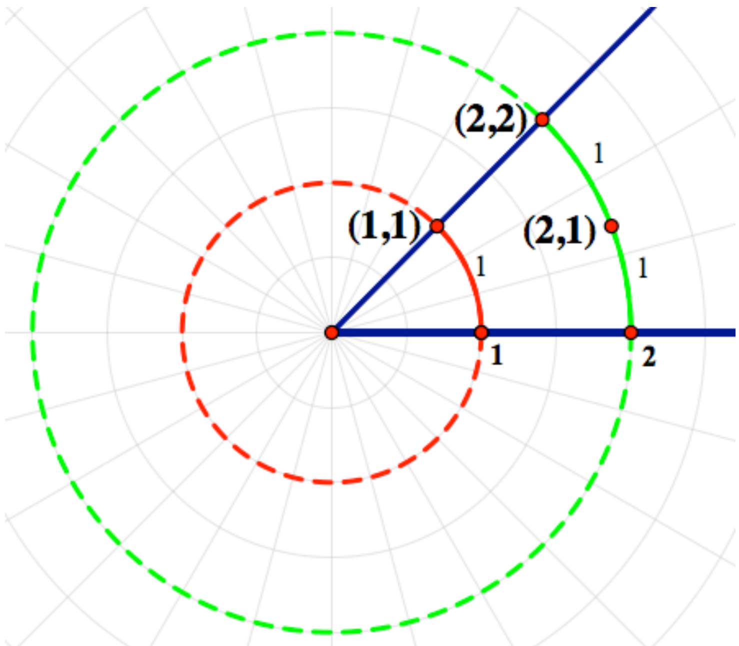

Figure 2. Using a fixed magnitude to measure arcs.

As the teaching experiment progressed, we worked with the students to develop radian angle measure as defining an equivalence class of arcs. As the pair constructed such an understanding of angle measure, they both made strides in their PCS concept. However, this progress was not trivial. For instance, Jenna continued to struggle coordinating angle measures and the measures of subtended arcs. At one point she stated, “See that’s what confuses me, because how can we say that the length, I mean it’s, I understand that [the arc length] is the same length as one radius, one radian. But the angle measure is also one radian, so they’re all (*referring to several subtended arcs*) one radian?” After a discussion that entailed identifying that “one radian” refers to *all* of the arcs subtended by the angle, she explained that she had difficulty reasoning about an angle measure in terms of measuring arcs. Collectively, the students’ difficulties with angle measure corroborate Moore’s findings (2012) about students’ images of angle measure, while highlighting the importance of students’ meanings for angle measure when learning the PCS.

Polar and Cartesian Characteristics

We also noted the students encountering perturbations when considering outcomes from the PCS with the CCS, especially when trying to make a connection between the origin in the CCS and the pole in the PCS. For example, an issue arose when we gave them a graph of $r(\theta)=2\theta-0.5$ and asked them to determine a function rule to define the graph.

After finding the rate of change of the relation using points on the graph, Steve used the fact that the graph passed through the "origin" (e.g., pole) to determine the rule of $r(\theta)=2\theta+0$, with 0 representing that the graph passes through the "origin." However, when the students tested their rule with another point, they found that the constant is -0.5 rather than zero (e.g., $r(\theta)=2\theta-0.5$). This led the students to assume that the given graph was misleading and never actually passed through the "origin." Furthermore, when asked if the graph did or did not pass through the "origin," Steve replied, "It doesn't go through the origin. 'Cause that wouldn't match our function (*referring to the rule*)." When probed as to why it "wouldn't match our function," Steve argued, "If you use the origin zero-zero, then if your radius is zero and your theta is zero, there's no way you can get zero unless b is zero." This interaction suggests that Steve encountered a perturbation with his interpretation of the "origin" in the PCS as being the same as the origin in the CCS, namely the ordered pair (0,0). Steve, not realizing the pole in the PCS is represented by all pairs of the form $(0,\theta)$, believed passing through the "origin" in the PCS required that the function represent the pair (0,0). Thus obtaining a rule that did not entail this pair led him to conclude that the given graph could not pass through the pole as he had conceived it.

Later in the conversation and after the researcher followed up with a discussion of the points represented by the pole in the PCS, Steve explicitly stated, "It's so hard to grasp that it goes through the origin but it's not zero-zero. I feel like that's ingrained in our mind," further indicating Steve's experience with the CCS influenced his understandings of the PCS. Specifically, Steve's concept of the origin from the CCS was a problematic influence on his understanding of the pole in the PCS, providing an example of an issue that might arise when students apply conventions from the CCS with little consideration of the differences between the two systems that stem from their underlying structure.

Another issue that arose was that the students were irked by the mathematical convention of writing the coordinate point in the PCS as (r,θ), which they interpreted as (output, input), due to initially working only with functions of the form $r(\theta)$. To the students, this contradicted the CCS convention of writing pairs in the form (input, output). After adopting the (r,θ) convention, Jenna and Steve then exhibited discomfort when equations were given with θ as a function of *r*, instead of *r* as a function of θ. For instance, when asked to graph $\theta = r^2$, both chose values of θ and found corresponding values of *r* using the rule $r=\theta^2$. After some questioning, Steve caught their error: "It's theta equals r squared, so wouldn't r be the square root of theta, right? So for every theta you would get the square root of that would give you your radius." In spite of his awareness of their error, he desired to rewrite the equation so that θ formed the input of the function ($r(\theta)=\sqrt{\theta}$). After substituting in the point (4,2) to $\theta = r^2$ to convince Jenna of their error, Jenna claimed, "We just did the math backwards." Steve confirmed that the roles of the variables had switched in the given rule (e.g., the given rule implied that θ is a function of *r*), which conflicted with their prior experiences that defined (r,θ) as (output, input). Steve was later able to identify such conventional practices as being arbitrary decisions on the part of the practitioner, but his actions indicated that he remained fixated on a convention (e.g., (r,θ) as (output, input)) he adopted from early experiences with functions in the PCS.

Conclusions and Implications

Research suggests students are often given a superficial presentation of the PCS at the precalculus and single-variable calculus level (Montiel et al., 2008; Montiel et al., 2009). This approach to the PCS likely leaves students with impoverished PCS concepts, some of which stem from problematic connections to the CCS. Our results support previous findings concerning the influence the CCS has on student understandings of function in the PCS (Montiel et al., 2008; Montiel et al., 2009). While previous research identified students' (mis-)use of the vertical line test in the context of the PCS, our results indicate that other issues stem from certain features of the PCS. For instance, it was important that the students understand the pole of the PCS as represented by an infinite number of coordinate pairs in the PCS. In the case that the students conceived the pole as represented by a unique pair (e.g., (0,0), the CCS origin), they had difficulty reconciling the graph of a function in the PCS and the function rule. As another example, students' understandings of angle measure were critical to their PCS concept. In the CCS, directed lengths form both coordinate quantities. However, in the PCS, an angular component forms one of the quantities. It follows that ideas of angle measure, including equivalence of arcs, emerged as central to the students' progress.

Future researchers may be interested in exploring how to support students in overcoming various conventions of the CCS and PCS when attempting to make connections among these systems. Our findings, in combination with previous research (Montiel et al., 2008; Montiel et al., 2009), suggest that if students' experiences are predominantly within one coordinate system, then their understandings for related concepts (e.g., function) become inherently tied to that coordinate system. Future research that investigates using a multitude of coordinate systems in the teaching of mathematics might prove useful for determining critical ways of reasoning about function and how to promote these ways of reasoning.

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Entity versus Process Conceptions of Error Bounds in Students' Reinvention of Limit Definitions

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We report results from a guided reinvention of the definition of sequence convergence conducted in three second-semester calculus classes. This report contributes to the growing body of research on how students come to understand and reason with formal limit definitions, focusing on the emergence of students' understanding of the epsilon quantity, conceived in terms of error bounds. Using Sfard's framework of the condensation of processes to entities, we mapped the possible conceptual trajectories followed by the students in the study. In this report, we detail our map, these trajectories and students' reasoning about other aspects of the formal definition, and the influence of reasoning about approximations and error analyses in students' progression.

Key words: error bound, formal limit definition, guided reinvention, sequence convergence

Introduction

Limit concepts are central to the study of calculus, but students often develop only a weak understanding of limits that does not support their learning or reasoning about other central concepts in calculus (Davis, 1986; Oehrtman, 2009; Sierpinska, 1987; Tall, 1980; Tall, 1981). Recently, a series of guided reinventions where pairs of students reinvented formal definition for concepts related to limits have provided multiple insights into how students come to understand limits (Hart-Weber et al., 2011; Martin & Oehrtman, 2010; Martin et al., 2011; Martin et al., 2012; Oehrtman et al., 2011; Swinyard, 2011). We employed techniques of the guided reinvention heuristic from Realistic Mathematics Education (Gravemeijer, 1998) in three second-semester calculus classrooms to engage students in reinventing the formal definition of sequence convergence, typically expressed as[1],

A sequence (a_n) converges to L provided that for every $\varepsilon > 0$, there exists an N such that $|\, a_n - L \,| < \varepsilon$ whenever $n \geqslant N$.

We posed the following research questions: 1) What cognitive challenges do students encounter while reinventing the formal definition of sequence convergence? 2) How do students resolve these cognitive challenges? 3) To what extent and in what ways do the students use approximations and error analyses in their reinvention? 4) How does a guided reinvention of the definition of sequence convergence in a

[1] The definition commonly stated in this form presumes without explicit statement that ε is quantified as a real number while N and n are quantified as natural numbers.

classroom differ from a guided reinvention with a pair of students? This research report addresses the first three questions focusing on the students' emerging understanding of the quantity typically represented by ε in the formal definition.

Literature Review

Early literature on how students understand limits focused on informal reasoning, conceptual barriers, and misconceptions (e.g., Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Tall, 1992; Williams, 1991). Cottrill et al. developed a genetic decomposition outlining a sequence of 7 mental constructions that students might make when coming to understand limit concepts (1996). This decomposition was one of the first attempts to understand how students reason with formal definitions. Unfortunately he was only able to find evidence support the first four mental constructions and was not able to collect any data on the last three.

Building on research by Williams (1991) and Tall and Vinner (1981) Oehrtman (2009) identified 5 different metaphors students use when attempting to reason with limits: "collapse in dimension", "approximation", "closeness in spatial domain", "physical limitations," and "infinity as a number." Oehrtman (2008) found that focused instruction could systematize students' use of approximation metaphors enabling more powerful subsequent reasoning about how limit structures were manifest in the various concepts developed throughout introductory calculus. When one teaches reinforcing the systematic exploration of approximation and error analyses reflecting limit structures in calculus, we will say that one is teaching using the approximation framework. Oehrtman (2008) also hypothesized that such a structural approach to calculus could establish a strong conceptual foundation for later formalization of limit concepts, such as creating and reasoning with rigorous definitions.

Very little existing research provides insight into *how students do come to understand the formal definition*. Multiple teaching experiments have been performed in the last few years to help better understand how students do come to understand the formal definition by having pairs of students reinvent the formal definition of convergence of a sequence of numbers (Hart-Weber et. al., 2011; Martin, Oehrtman 2010, Martin et. al., 2011; Martin et. al. 2012, Oehrtman et.al., 2011; Swinyard & Larsen, 2012). These teaching experiments begin with students constructing a rich set of examples of sequences that converge to a limit L and examples of sequences that do not converge to L serving an explicit external representation of their concept image of convergence of a sequence. Students then engage in an iterative process in which they are asked to create a definition, evaluate the definition against examples and non-examples, identify conflicts with their set of examples, then attempt to resolve these conflicts (Oehrtman et al., 2011). Oehrtman et al. identified *problems* and *problematic issues* that students encountered when reinventing formal limit convergence. *Problems* are cognitive challenges that students explicit identified as causing conflict between their concept image and their stated definition. *Problematic issues* are cognitive challenges that students do not produce cognitive conflict in their concept image, but are in conflict with the formal definition.

One of the more complex aspects of students developing an understanding of the formal definition appears to be their conception of the quantity ε and its relation to other elements in the definition. Swinyard and Larsen have proposed that an important aspect to coming to understanding the formal definition is that students develop a *y-first perspective*. When students are reasoning informally with they traditionally start with an *x-first perspective*, but in the formal definition we consider a bound on the range of *y*-values first and then consider values on the *x*-axis (Swinyard & Larsen, 2012). They also noted that the first four step of the genetic decomposition involve an *x*-first perspective.

This research proposes a model of when students adopt a *y*-first perspective what are possible trajectories that they could possibly take by focusing on error bounds. We provide a classification of different ways that students learn to reason with error bounds and the possible progression that students may take when reinventing the formal definition of sequence convergence. Our final model of how students how students reason with error bounds is framed in terms of Sfard's theoretical characterization of the condensation of processes into an entity. Sfard defines a process to be manipulation of familiar objects and an entity to be a condensed process. Sfard classifies any *process* as an operational conception, which is understood to not involve static objects. If an individual posses an entity view then one possess a structural conception, which involves a n individual thinking of an abstract object. Condensation of a process into an entity is said to have occurred if one is able to simultaneously consider a collection of process without having to consider an individual action (Sfard, 1992). Oehrtman, Swinyard, and Martin (in preparation) suggest that the condensation of the process of determining an N value based on an ε value may be a crucial step in constructing a universal quantification on ε in the definition.

Methodology

The research was conducted in three Calculus 2 classes with different instructors at two medium-sized research universities midway through the spring semester. The three Calculus 2 classes and the Calculus 1 classes taken by nearly all of the students the previous fall were taught using activities developed with Oehrtman's (2008) approximation framework. Since these activities were implemented in weekly in-class labs requiring students to collaborate in small groups, all of the students were familiar with this format. During the guided reinvention we engaged students in five class sessions working in groups of four or five to construct a definition of sequence convergence based on a collection of examples and non-examples the classes had previously generated. Each group had a large whiteboard used for writing out definitions and communicating ideas. One group in each class was selected to be video and audio recorded at all times, and a second video camera in each class captured the whole-class interactions. All documents created by the students were also collected.

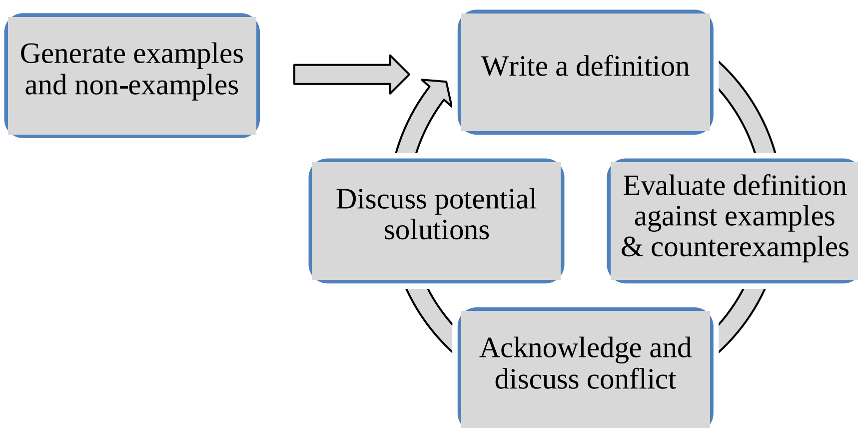

Figure 1. Iterative refinement in the process of guided reinvention of a formal definition.

Throughout the reinvention students completed reflections on the examples and nonexamples, their emerging definition, and problems they identified with their definition. For the first reflection, which occurred prior to the guided reinvention, students were asked to create graphs of as many qualitatively distinct examples of sequences that converge to five and sequences that do not converge to five as possible. The researchers compiled representative samples of these graphs on a handout for students to reference throughout the guided reinvention. On the first day of the reinvention, all groups were given the prompt "A sequence converges to 5 provided that ..." The facilitators then guided students through the iterative process of writing a definition, evaluating their definition against the examples and non-examples, identifying problems, and proposing solutions (Figure 1).

There were 6-7 groups in each class and two undergraduate facilitators in addition to the instructor and one member of the researcher team supporting the groups. The facilitators' role included keeping students engaged in the iterative refinement cycle, asking clarifying questions, and introducing cognitive dissonance when students were not aware of critical problems with their definitions.

After each class session the research team would meet to discuss interesting interactions that occurred in class and modify the protocol for subsequent days. Members of the research team would then watch the videos from each group before the next class meeting. This report focuses on our emerging hypotheses concerning how students conceived of and reasoned with the ε quantity in the formal definition and its interpretation in terms of an error bound as developed in the approximation framework. We evaluated these hypotheses against classroom events during each research team meeting and rewrote our hypotheses between each session. While reviewing the individual group video, we then developed predictions for how these ideas would develop during the next session. After the completion of the guided reinvention, we created content logs while again reviewing the video data, coding statements about the ε quantity in the students' definition. We then refined our hypotheses until we felt we could adequately understand all of the students' statements about ε and error bounds and could explain the challenges they encountered while reasoning about this quantity and the shifts that the made when reasoning with error bounds.

Results

Our initial hypothesis about students' reasoning about the ε quantity can be summarized as follows:

> Students will eventually frame their definition in terms of approximating the value of the limit L with terms of the sequence a_n and errors $| a_n – L |$. This formulation will still reflect students' initial intuitive domain-first images of terms approaching the limit and thus will not incorporate aspects of an error bound. As students encounter problems with their definition applied to specific examples (such as a damped oscillation around L) and nonexamples (such as a sequence monotonically increasing to a value slightly smaller than L), they will recognize a need to say how close a sequence needs to get to its limit. In conjunction with their language and notation about approximations, this recognition will trigger a recall and application of ideas about error bounds. Students' initial attempts will involve only a single value for ε, but as they recognize a need to rule out every possible nonexample, they will eventually construct a universal quantification for ε.

Our initial analysis agreed with our initial hypothesis that students do tend to begin with a domain-first perspective, but had difficulty explaining the difficulties that students had with developing a universal quantification of error bounds. Framing students' reasoning with Sfard's framework of condensation of processes into entities we were able to differentiate the ways that students think about error bounds. Once students begin to think of error bounds there appears to be two common trajectories that students can take. These two trajectories appear to be a reflection of the two types of questions in the approximation framework activities related to error bounds. One type of question asks students to calculate error bounds from information about the approximations. When referring to this type of conception of error bound we will use the notation E.B. When calculating error bounds in the activities a common strategy is to find an overestimate and an underestimate and use the difference between the two as an error bound. The other type of question gives students an error bound as a predetermined tolerance and then asks them to find a subset of the domain that produces approximations within that tolerance. As part of this question there is usually a follow-up question asking if it is possible to find a subset of the domain that produces approximations with error less than any tolerance one may want. When referring to such a tolerance conception of error bounds, we will use the notation ε.

The two trajectories that students usually take involve students' first thinking of error bounds as a process. One way that a student may possess a process view of error bounds is related to the type of question that ask students to calculate an error bound. A process view of error bounds related to this question would result in student manipulating objects or information about these objects to construct an error bound or a sequence of error bounds. One of the ways that students constructed error bounds is by finding an overestimate and an underestimate and subtract them, as is typically expect in the approximation framework. From these constructed error bounds students then usually will talk about the error bounds "getting smaller," "approaching zero," "decreasing," or other similar phrases. Students were not necessarily computing numerical values, but it appears that they are constructing error bounds from properties the approximations. For example in the first explicit discussion on error bounds of the participants said “the error bound between these two points like this and two points like this is going to be so close to the same thing.” Here we can see that this individual is using the points to understand the error bound and that the error bound is derived from information about the points. It appears that students may condense E.B.'s as an entity view of error bounds. To think of E.B.'s as an entity one must be able to construct a collection of error bounds simultaneously without having to consider any single action. Data seems to indicate that E.B. entity manifests themselves as "trend lines" of the error bounds that the approximations must lie within.

Another process view of error bound is related to the second question where students are asked to find approximations within a given tolerance, ε. Here students consider a single error bound and then find a relationship of the *n*-value(s) and/or the approximation(s) to this error bound. For example on the second day one of the participant’s definition included “that as n increases beyond ten thousand all of the values of a sub n gets so close to five that the error becomes less than this number here.” It should be noted that possessing this entity view of error bounds does not mean that an individual has a complete understanding of the formal definition of sequence convergence. For example one of the groups articulated that for any error bound they are able to find a single approximation with error less than any error.

Conclusion

Our data analysis identified two distinct trajectories for the development of students’ conceptions of error bounds (see Figure 2). The students in our study first developed a process view of terms of the sequence approaching the limit, which was eventually framed in terms of approximations then augmented to include associated errors. Some students then condensed their process into an entity view of the errors, enabling them to reason about an entire collection of approximations and their errors simultaneously. At this point, the students typically described errors decreasing in holistic terms such as “past some point *n*, $|a_{n+1}-5|<|a_n-5|$” or described the errors for an entire set of approximations to be small, such as “past here, all of the errors are negligible.” . Regardless of whether students developed an entity view of errors, all eventually began to think of one of the two conceptions of error bound characterized in our results. The E.B. view does not easily lead to the universal quantification of ε which is needed to understand the formal definition of sequence convergence. In fact, all of the students we observed using an E.B. view eventually backtracked and explicitly started thinking in term so of the ε quantity as a tolerance before conceiving of a universal quantification.

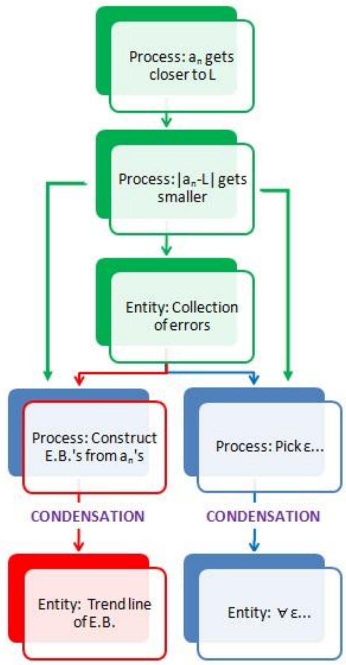

Figure 2. Trajectory of error bound conceptions

We hypothesize that develop an ability to fluidly move between E.B. and ε views of error bounds is crucial students learning to reason about formal limit definitions. In subsequent work, we are analyzing the interactions of students various process and entity conceptions of the ε quantity in formal limit definitions with the other elements of the definition.

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LESSONS LEARNED FROM A DESIGN EXPERIMENT ON STUDENTS' COMPREHENSION OF PROOFS

Previous studies in proof reading have identified several strategies to improve students' proof comprehension. In this study, we report on what we have learned from the first two iterations of a design experiment in which students were taught to apply these strategies. When coupled with modest teacher support, students made significant learning gains. Particular success was found by encouraging students to consider how they would prove the theorem, to use examples to understand assertions in the proof, and to examine definitions before reading the proof. We also discuss some of the difficulties we encountered in teaching students to implement these reading strategies, and the steps we took to attempt to resolve these difficulties.

Key words: Design experiment, proof, reading comprehension

Overview

Proof is a widely used means of conveying mathematical content in advanced mathematics courses at the university level. However, studies have suggested that undergraduates have considerable difficulty reading and understanding proofs (e.g., Selden and Selden, 2003). Even so, there are few studies examining exactly how students attempt to understand proofs, and we are aware of no studies that address how students' comprehension of proof might be improved.

Author (year) and Author (year) identified strategies that two pairs of strong mathematics majors used when reading proofs. Further, Author (year) surveyed 175 mathematics majors and 83 mathematicians and found that most mathematicians wanted students to use these strategies, but that most students reported that they did not use them. This list of strategies was compiled into a document that could be distributed to students for their use while they read proofs. We use the design research paradigm (Cobb et al, 2003) to develop an instructional intervention built around this strategy sheet to help students read and understand proofs. In this paradigm, hypotheses are formed regarding (i) how students reason upon entering the classroom, (ii) learning goals (i.e., what it means to understand the concepts in question), (iii) how students might come to achieve these goals, and (iv) how instruction can be designed to help students achieve these goals.

Regarding the first hypothesis, we assumed students tended to focus on calculations when reading proofs (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003), believed that understanding a proof was tantamount to understanding the justification for each warrant in the proof (e.g., Author, year), and did not use the strategies that we intended to teach them (e.g., Author, year). . Second, the learning goals for this study were for students to be able to come to a more complete understanding of the proofs that they read. We hypothesized that students could become better readers of proofs if they enacted the strategies that experts (mathematicians or strong students) use to read proofs. For example, in Author's (year) survey, most mathematicians said they would try to prove the theorem themselves before reading the proof. A more complete description of the strategies we included for this study is provided in the methods section. Finally, we decided that a good first approach for an instructional intervention would be to provide students with the strategy sheet and provide guidance for using these strategies through a "cognitive apprenticeship" model (Brown et al, 1989) described below. Furthermore, we decided to use a format used in Palincsar and Brown's (1984) "reciprocal teaching" studies in order to keep students actively engaged, and to be able to directly assess students' learning. The particular way in which we assist students in learning these strategies is described in the methods section. By observing how students responded to this instructional intervention, we sought confirming or disconfirming evidence for any of these four hypotheses.

Theoretical Perspective

We adopt two perspectives for this study. First, we believe that students can effectively learn through participating in a "cognitive apprenticeship" in which a community of learners work on an authentic task guided by a teacher who models the desired performance, scaffolds student learning, and then "fades"—that is, increasingly leaves more of the responsibility to the students as they gain a mastery of the skills.

We also believe that understanding a proof is more than just understanding the inferences contained in the proof (e.g., de Villiers, 1990; Rav, 1999). In particular, we define understanding a proof using Mejia-Ramos et al's (2012) model. This model asserts that understanding a proof can be assessed by evaluating a student's summary of the proof, ability to apply the techniques used in the proof to prove similar theorems, and apply the ideas of the proof to specific examples.

Methods

Participants.

All participants were undergraduate mathematics majors who attended the same large public university in northeastern United States. For the first iteration of the study, six students were recruited from a real analysis class, and had all previously passed an introduction to proof course (Eight students participated in the first study session, but two only attended the first meeting). The students met together with the first and second authors for five teaching sessions. Each session lasted approximately two hours.

For the second iteration, two students who recently graduated with a degree in mathematics agreed to participate in this study. Both students were part of a five-year program to become secondary mathematics teachers, and had previously completed an introduction to proof course and a course in real analysis. These students met with the first author for five sessions, each lasting about 2 hours.

Materials.

As discussed before, the strategy sheet that was distributed to students was largely based on the strategies discovered from exploratory studies with strong students. The strategy sheet consisted of three parts: strategies to implement before reading the proof, while reading the proof, and after reading the proof. Pre-reading strategies included recalling definitions of the terms in the theorem statement, illustrating the theorem with an example, and attempting to prove the theorem. Strategies to be implemented while reading the proof included dividing the proof into parts and identifying the proof technique used. Post-reading strategies consisted of summarizing the proof and comparing the methods used in the proof to one's own approach developed before reading.

The proofs that students read came from two sources. First, some proofs were obtained from the textbook used for the real analysis course students took. Other proofs were either designed for this study or used from previous studies we had conducted. We intended for all proofs to possess several qualities. First, they were to be accessible to students with the particular background knowledge we had assumed. As such, proofs were chosen to rely on content from basic number theory, calculus, and real analysis. Second, proofs were intended to be similar to those they would see during the course of their programs of study. This was so that inferences could be drawn about students' actual behavior, such as reading lecture notes when studying for an exam. Finally, proofs were chosen to be moderately difficult, so that we could observe students encountering real difficulties in understanding the proofs they read.

Procedure.

We adapted Palincsar and Brown's (1984) "reciprocal teaching" for teaching these comprehension strategies. In this method, a teacher first models the strategies he or she wants students to use, then allows students to assume the role of teacher by demonstrating how they would implement the strategies in given texts. The teacher provides scaffolding support for students whenever necessary. The teacher progressively reduces this support, allowing them to take increasing responsibility of the strategies

During the scaffolding phase of this study, students were encouraged to give their feedback on how their peers implemented the strategies. The instructor provided corrective feedback when students failed to identify the incorrect use of a particular strategy or showed evidence of serious misunderstandings with particular mathematical concepts. In addition, when a student showed exemplary use of a particular strategy, the instructor highlighted this for the other students, and built upon productive aspects of students' contributions whenever possible.

After students had read and discussed a proof, the written proof was taken away and the instructor asked individual students comprehension questions. These questions typically included asking a student to summarize the ideas of the proof, provide justification for particular steps in the proof, or to determine if the method the proof used could be applied in a similar setting. These assessments allowed us to constantly monitor student understanding through the course of instruction.

Throughout both iterations, students were asked for their feedback on the strategies, including which strategies they found helpful and which they found unhelpful.

Analysis.

Each meeting with the students was videotaped. These videotapes were analyzed for critical events where confirming or disconfirming evidence was found for any of the hypotheses (i) through (iv). These events were transcribed and organized into themes. A second pass through the video was made to find additional instances for each theme. Finally, student feedback on the strategies was used to supplement these themes.

Results

The result of the analysis was ten lessons that we learned by analyzing these two iterations. For brevity, we describe only three of these lessons in this proposal.

Lesson 1: Attempting to prove the theorem and comparing one's proof attempt to the method used in the proof was helpful for some students.

Some students mentioned this strategy as being particularly helpful for them, both during the teaching sessions and in the feedback forms. This is illustrated in the following excerpt, in which a student discusses a proof establishing that the equation $x^3+5x = 3x^2+\sin(x)$ has only one root. The proof for this theorem relies on proving that the function $f(x) = x^3+5x - 3x^2 - \sin(x)$ is increasing, and hence has only one root by Rolle's Theorem. Students typically found this proof difficult because they expected the proof to rely largely on manipulating the equation algebraically.

> Rahim: "*When I first looked at the [theorem], it seemed to me it would just be arithmetic*. I guess just at first glance, it seemed like a pretty basic proof where you could pretty much just do it [another student says 'plug and chug'] yeah you could just plug and chug, and you could just put steps together. *Now I know what to do next time*, I guess, to just think a little bit more outside the box. Because even though it may use more difficult concepts, it'll become a simpler proof altogether because I guess with plugging and chugging with this would be

> much longer than a nine-step proof, and *calculus made it a lot easier*, even though you wouldn't automatically think to use Rolle's Theorem." (italics were our emphasis)

In this excerpt, the student appears to gain an appreciation for the method used in the proof after comparing it with his own method. This was in agreement with the strategy included on the sheet, which asked students to make this comparison after reading the proof.

Lesson 2: Examples were helpful for students' understanding of statements in the proofs.

The strategy sheet also encouraged students to check assertions in the proof that they did not understand with examples. In one episode, a student had difficulty understanding the assertion "$n_k \geq k$ for all k since a_{n_k} is a subsequence of a_n". After spending some time with this assertion, he refers back to an example that he has written earlier, of the sequence $1/n$ and two subsequences, $1/(2n)$ and $1/(2n+1)$.

> Tony: How is that true? …We're looking at n_k, n_k by that definition, is n_1, n_2, n_3, and so on…Right. And then k is going to be starting at 1, and going on. But your subsequence…is not necessarily 1 and going on…it could be though, which is why it says 'or equal to'. So, yeah, like in our example. This is actually what I thought back to [points to sheet where he wrote the example] Because these were the…I should say like a_1, a_3, a_5, and so on [labels these on his subsequence $1/(2n+1)$] This one was a_2, a_4, a_6, right? *So the* n *term is going to be—in any sequence it's going to start at 1 or whatever number then just keep going up*. And it doesn't have to go up by ones, it could go up by whatever you define the subsequence as. *But it could never be less than k, so greater than or equal to has to be true*.

In this excerpt, we see that this strategy prompted the student to refer to his previously constructed example. This example helped him see why the statement was true.

Lesson 3: Students sometimes had difficulty implementing the strategies, or implemented them in a superficial way.

Even after having made the aforementioned adjustments to our instruction, students still had difficulty applying the strategies. After asking students how they would prove the theorem, for example, some would reply by saying "induction" without elaborating. In other words, although they appeared to be familiarizing themselves with the strategies, they did not always understand *how to implement these strategies*. We therefore modified the strategies on the sheet to make them more prescriptive. Many strategies were broken down into sub-steps. For example, the strategy asking students how they would approach proving the theorem before the proof was modified to read:

Describe how you would try to prove the theorem:

a. Choose a proof approach that you think might work.
b. Describe what you think it might look like. What would you assume and what would you try to show?
c. What difficulties would you expect if you tried your approach?

In subsequent teaching sessions, students showed evidence of interacting with the proof at a deeper level, sometimes discovering flaws in their initial arguments and subsequently modifying them.

Lesson 4: Modeling the proof reading strategies and allowing students unguided practice to implement the strategies was not sufficient to observe clear and immediate learning gains.
Lesson 5: Students did not always take responsibility for evaluating the usage of the strategies.
Lesson 6: Some students were not comfortable dividing a proof into smaller parts before they read the proof.
Lesson 7: Students had difficulty understanding proofs that contained novel or creative ideas.

Lesson 8: Attending to the definition of each concept in the theorem aided understanding.
Lesson 9: Students did not always appear to gain insight from rephrasing the theorem in their own words.
Lesson 10: The strategies sometimes helped students understand mathematical concepts used in proofs.

Discussion

Regarding our initial hypotheses on students implementing our strategies, we found on a positive note that some of the strategies that we mentioned led students to develop an increased understanding and appreciation for the proofs that they read. Having students attempt to prove theorems before reading the proof as well as comparing the proof to their own approach after reading the proof, list definitions of the terms in the statement being proven, and verify the statement to be proven with an example prior to reading the proof seemed to help the students greatly. However, we also learned that the reading strategies that we mentioned were more complicated than our one sentence descriptions and that it was difficult to partition a proof into sub-proofs without having studied the proof in detail. Finally, asking students to rephrase the theorem statement in their own words seemed to produce little benefit, despite being useful for the strong undergraduates that we analyzed previously (Author, date).

Regarding our initial hypotheses on how our instructional environment may have encouraged students to use these strategies, we realized that the reciprocal teaching approach, as we adapted it from Palincsar and Brown (1984), discouraged students from evaluating their own and others' strategies. We also highlighted the necessity of the instructor's role in helping students comprehend proofs, especially when the choices made by the proof instructor were not transparent.

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BUILDING KNOWLEDGE FOR TEACHING RATES OF CHANGE: THREE CASES OF PHYSICS GRADUATE STUDENTS

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Over the past two decades education researchers have demonstrated that various types of knowledge, including pedagogical content knowledge, influence teachers' instructional practices and their students' learning opportunities. Findings suggest that by engaging in the work of teaching, teachers acquire knowledge of how students think, but we have not yet captured this learning as it occurs. We examined whether novice instructors can develop such knowledge via the activities of attending to student work and we identified mechanisms by which such knowledge development occurs. Data come from interviews with physics graduate teaching assistants as they examined and discussed students' written work on problems involving rates of change. During those discussions, some instructors appear to develop new knowledge–either about students' thinking or about the content—and others did not. We compare and contrast three cases representing a range of outcomes and identify factors that enabled some instructors to build new knowledge.

Keywords: Mathematical knowledge for teaching, graduate student professional development, rates of change.

Introduction and Rationale

From research on teachers, it appears that particular types of knowledge used in teaching correlate with reform-oriented teaching practices and with student achievement (Ball, Hoover Thames, & Phelps, 2008; Fennema et al., 1996; Hill, Ball, & Schilling, 2008). Knowledge of student thinking, a subset of pedagogical content knowledge (PCK) (Shulman, 1986), has been found to play prominent roles in teachers' practices. Although sometimes learned in teacher preparation or professional development, findings suggest that it is often via the work of teaching that teachers have opportunities to acquire knowledge of how students think (see, e.g., Franke, Carpenter, Levi, & Fennema, 2001). To date, however, researchers have not captured this learning as it occurs.

We sought to document the use and genesis of some knowledge for teaching through task-based interviews with physics graduate students as they examined student solutions to physics problems. Although drawn from research on student thinking about physics, the task is highly mathematical in nature and is, at its core, about rates of change. Our findings indicate that, during these discussions, some graduate students appeared to develop new knowledge–either about student thinking or about the content–and others did not. We compare and contrast three cases that represent a range of outcomes. Our analysis sheds light on mechanisms that enabled two of the instructors to build new knowledge.

Theoretical Perspective

We investigate issues of knowledge development from a cognitive theoretical perspective because of the prevalence of the cognitive perspective in the research on knowledge and knowledge development as well as the primarily individual nature of the out-of-classroom teaching work that is the focus of our investigation. This perspective has been used productively to examine teachers' knowledge and its roles in teaching practices (Borko & Putnam, 1996; Calderhead, 1991, 1996; Escudero & Sanchez, 2007; Schoenfeld, 2000; Sherin, 2002). In such an approach, knowledge is seen as a key factor influencing teachers' goals and the ways they work to accomplish those goals as they plan for, reflect on, and enact instruction.

Knowledge That Shapes Teaching Practices

As noted above, research indicates that particular types of knowledge are linked with reform-oriented teaching practices and student achievement. Knowledge of student thinking, a subset of PCK plays prominent roles in this genre of research. Findings have informed the design of professional development materials and programs for elementary and secondary school teachers. This is a very positive development in the education community that may have important impacts on the teaching and learning of school mathematics. However, teachers at the college-level typically have little or no pre-service preparation (Shannon, Twale, & Moore, 1998) and opportunities for in-service professional development focused on teaching are scarce. As a result, much of the development that turns a novice college teacher into a knowledgeable, experienced teacher occurs "on the job" and the need to understand how learning occurs for this population of teachers while doing the work of teaching is especially important.

What the community lacks is insight into how and why that on-the-job learning occurs as well as the particular conditions and contexts needed for that learning to occur. Those insights could inform the design of professional development so teachers have the best opportunities possible to acquire knowledge relevant for teaching. To assist the community with these issues, we pursued these research questions: Can graduate student instructors develop knowledge while doing the work of teaching? If so, how does that knowledge development occur?

Methods

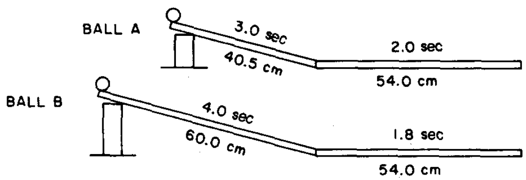
One-on-one task-based interviews were conducted with seven physics graduate student instructors. During the interviews, participants considered several kinematics problems drawn from literature on student difficulties with the concepts of velocity and acceleration (Beichner, 1994; Trowbridge & McDermott, 1981). For each problem, participants were asked to (i) solve it and give reasoning, (ii) discuss what one would need to know to solve it, (iii) generate and discuss possible student approaches, and (iv) examine and discuss samples of student work.

From the seven audio-recorded interviews, we selected three for cross-case analysis (Yin, 1989). One interview was selected for closer analysis based on its *inherent interest* to our research questions. During this interview, Jamie developed an increasingly specific and accurate diagnosis of a known common student difficulty. The other cases were selected for their *contrastive value*. Sam developed better understanding of the underlying physics by attending to student work and Alex struggled to understand the problem or student thinking.

To conduct the analysis, we drew on research findings about student thinking about these problems (Trowbridge & McDermott, 1981) and methods for fine-grained analysis of teacher thinking and practices (Schoenfeld, 2000, 2008). Particular attention was given to moments when participants utilized content knowledge to either make sense of a problem or to interpret student work. Participants' statements about their own understanding or about students' understanding were compared and constrated *within* cases and *across* the cases to generate and explore hypotheses about roles that content knowledge played in developing new knowledge.

The Ramp Problem

Here we focus on just one of the kinematics tasks used in the interviews. It was adopted from Trowbridge and McDermott's (1981) investigation of student difficulties with acceleration. In it, two balls roll down ramps starting from rest. Students are asked to compare the accelerations. Ball A gains 27 cm/s in 3 s (9 cm/s^2), while Ball B gains 30 cm/s in 4 s (7.5 cm/s^2):



The problem is particularly challenging because it invites two kinds of mistakes. One is to only consider the velocities. This can result from failing to distinguish between the concepts of acceleration and velocity. A second mistake is to calculate speed using v = x/t, and then acceleration using a = v/t. While this results in a correct comparison, the thinking behind it fails to distinguish between change in velocity and average velocity.

Although this exact kind of task is not as common in mathematics courses as it is in physics, students encounter ideas connected to velocity, acceleration, and various sorts of averages in differential calculus. In particular, with tasks such as this, many students do not clearly distinguish between the concepts of velocity and acceleration. Some will say that things that are moving fast are accelerating fast (even if they are moving fast with constant velocity). Confusing quantities and their rates of change is common across concepts and representations and is especially common in cases where the quantity itself is a rate of change, and in this case when they are tied to informal everyday language. Considerable research on student thinking about these concepts is found in the physics education research literature base but these and other derivative-related ideas are also known to be challenging for mathematics students (see, e.g., Zandieh, 2000).

Three Cases

Here we present data and analysis to illustrate that Jamie emerged from the activity of examining student work with richer knowledge of student thinking than was apparent at the start of the interview. We provide brief sketches of the other two cases where neither instructor appeared to develop PCK during the interview but one does appear to develop content knowledge.

Jamie Develops PCK

Jamie was the only participant who had little difficulty solving the ramp problem. After solving it, he was prompted to generate examples of student work. Jamie offered the following:

> So the first thing they might do is they might say that, acceleration is meters per second squared. And they might just be…well it's only being accelerated on this side [along the ramp], so I'm going to ignore what's on this side [along the flat track]… And I'm gonna say, this one is 40.5 over 3 times 3. And this one is 60 over 4 times 4. So, then I'd say like, 'A' is greater acceleration.

This quote illustrates how Jamie generated an example student solution, in part, by considering the units of acceleration as a piece of content knowledge a student might know and apply to the problem. While the student approach seems plausible, it is not actually a common student approach. The solution is, however, numerically similar to an approach actually taken by students. Students commonly use the ramp numbers to construct the same ratio that Jamie does, but students do so by substituting into x =v/t and a=v/t, not by considering the units.

When asked about what the approach would mean in terms of understanding, Jamie adds,

> [It] would mean that the student understands the units of acceleration, but that would be about it. [The student] doesn't understand how acceleration, velocity, distance, and time are inter-related, and how to go from those pieces of information.

Here we see that Jamie makes a rather broad diagnosis of the student difficulty, naming each of the four major kinematical quantities as among the things the student is having trouble relating. Based on these and other statements made by Jamie early in the interview, we describe Jamie's ideas about how students might solve the problem as neither accurate in terms of anticipating student approaches nor very specific in terms of diagnosis student difficulties.

Later in the interview, however, Jamie offers a very different decsription and diagnosis of the same student work:

> They are doing something different, because they are not finding a final velocity, which you have to use. They are finding an average velocity… So this is what this students' approach doesn't understand: the difference between change in velocity and the average velocity. What's missing from this student's understanding is that to find our acceleration, what we need is the change in velocity not the average velocity.

Without going into too much detail of the process by which Jamie changed his thinking about the student work, we highlight here several key differences between his earlier and later thinking. Earlier, Jamie characterized the student approach as taking the units, but he later characterizes this same student work in terms of finding an average velocity. This re-description is important in two ways. First, it brings Jamie's account in closer alignment with what students actually do when solving the problem. Second, Jamie's re-description of the student resulted from him bringing additional content knowledge to bear on the student work. Specifically, Jamie worked to re-express the student's calculation from units composed as "m/s^2" to units composed "(m/s)/s". He then recognized this second ratio as an expression containing an average velocity. In re-describing the student work, we see Jamie drawing on content knowledge about ratios and average velocity. Importantly, the object toward which Jamie applies this content knowledge became the student work itself, and in this phase of his work new content knowledge appeared that Jamie did not use while he solved the problem for himself.

In addition to changing his description of the student work, Jamie changes his diagnosis of the student difficulty. He originally located the difficulty quite broadly among many kinematics variables, but Jamie later locates the difficulty more specifically as being between change in velocity and average velocity. This is more specific in two senses. First, Jamie narrows the space of possible quantities the student is having trouble relating. Second, it is more specific with respect to physics content, with Jamie now specifying three different velocity-related quantities in his diagnosis. Finally, Jamie's diagnosis is also more *accurate* because it is more aligned with findings from the original research using the problem (Trowbridge and McDermott, 1981).

We claim that Jamie's statements made near end of the interview convey insights into student thinking that Jamie did not appear to know earlier in the interview. These insights emerged, in part, as Jamie sought to understand the nature of the student work and its validity. This process involved substantial considerations of the student work in terms of content knowledge.

Sam Develops Content Knowledge and Alex Struggles with the Problem

Sam's initial attempts to solve the problem included common student difficulties documented in research using this task. Sam did not explicitly consider the time it takes the ball to speed up. In addition, Sam states that the ball "reaches" an average velocity, when it is (arguably) more appropriate to speak of the ball reaching a final (instantaneous) velocity. In this sense, neither Sam's calculations nor ways of talking about the problem clearly distinguish between the concepts of average velocity and instantaneous velocity.

Later, after examining student solutions, Sam's thinking about the content is quite different and takes into account the amount of time the ball accelerates. More importantly, Sam's discussion of the problem becomes increasingly differentiated with respect to velocity concepts, distinguishing and relating instantaneous and change in velocity.

Like Sam, Alex struggled with the ramp problem. Alex did not to distinguish among different velocity concepts and, in essence, found an arithmetic difference between final velocity and average velocity, rather than a change in instantaneous velocity. Alex also viewed the sample student work through the framework that acceleration is "velocity" over "time" with it being possible to calculate different accelerations.

Discussion of Cases

The three cases illustrate the range of outcomes we observed during our investigation. While Jamie and Sam did articulate new insights during the interview, we cannot claim that these insights necessarily represent new stable forms of knowledge that would be brought to bear in other contexts. We are, however, still concerned with understanding *how* Jamie and Sam were able to learn something new from the activity of attending to student thinking We see in these cases the beginnings of processes that support stable knowledge development, and thus hope to better understand the conditions, contexts, and practices that enable it.

Based on our more extensive analyses, we hypothesize two specific processes that supported the beginning of such new knowledge development:

- First, for both Jamie and Sam, the activity of interpreting student work elicited content knowledge they had not used while solving the problem or generating student work.
- Second, Jamie and Sam developed specific and accurate interpretations of student thinking, which were then used as the basis for making more general claims.

In Jamie's case, we see a shift from describing the student work as "using units" to "taking an average velocity." This re-description required the application of knowledge about average velocity–knowledge that Jamie did not appear to use while solving the problem. Jamie was then able to leverage this interpretation to make a more general claim that the student work represents difficulty understanding the difference between average velocity and change in velocity. Jamie's case, thus, is an existence proof that even novice instructors (Jamie was a first time instructor) may be able to develop teaching-related knowledge via activities of attending to student work.

In Sam's case, we also see content knowledge used to interpret student work, Sam eventually made key content connections while examining an incorrect piece of student work. Based on this insight, Sam was able to generate a more general statement about what the error in his own understanding had been. Alex, on the other hand, utilized the same content knowledge when solving the problem and interpreting student work. Ultimately, the nature of this knowledge was not only insufficient for understanding the problem but also insuffficent for making progress in understanding the nature of the student difficulty.

Conclusions and Implications

These cases provide some insights into interactions that can occur between content knowledge and knowledge of student thinking as instructors engage in teaching-related tasks. Participants not only interpreted student work through lenses of their content knowledge, but those interpretations were (for some) generative of new knowledge. For Jamie, strong content knowledge enabled him to emerge with new knowledge of student thinking. Sam was able to develop new content knowledge but Alex's content knowledge did not seem to support new knowledge development. These kinds of fine-grained examinations of knowledge contribute to the research community's understanding of interactions and inter-dependencies of different types of knowledge (e.g., content and PCK). This builds on work such as that of Sherin (2002) where influences of types of knowledge on one another were found to shape how teachers implemented novel curricular materials. Findings shed light on how teachers learn while engaged in the work of teaching and could be used to inform the design of professional development activities to support such learning for novice teachers. In particular, findings suggest that it may be productive (and necessary) to provide teachers with opportunities to strengthen their knowledge of the content as part of professional development. It appears that without strong content knowleldge there may be some apects of knowledge of student thinking that teachers are unable to develop from their "on-the-job" teaching experiences.

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On the Plus Side: A Cognitive Model of Summation Notation (Contributed Research Report)

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This paper provides a framework for analyzing and explaining successes and failures when working with summation notation. Cognitively, the task of interpreting a given summation-notation expression differs significantly from the task of expressing a long-hand sum using summation notation. As such, we offer separate cognitive models that 1) outline the mental steps necessary to carry out each of these types of tasks and 2) provide a framework for explaining why certain types of errors are made.

Key words: *summation notation, calculus, real analysis, cognitive models, APOS*

Introduction

In the Spring of 2011, we began an Advanced Calculus teaching experiment with pairs of students. The purpose of this experiment was to lay the groundwork for a new instructional sequence for Advanced Calculus (Real Analysis). The first sequence of tasks had the students investigate notions of area, with increasing formality and rigor, in order to motivate the study of sequential limits. In the midst of this experiment, two of our students encountered interesting challenges in using summation notation to talk about area. At last year's RUME conference we detailed how those two students struggled, but the data was insufficient for us to warrant any strong claims about *why* they struggled (Strand, Zazkis, & Redmond, 2012).

The research reported here is a follow-up study designed to further understand the cognitive complexity of summation notation and contribute a framework that could help explain students' difficulties. In particular, we sought to answer the following questions:

Are student difficulties with summation notation caused *or* revealed *by working within the context of rectangular approximations to area under a curve?*

What are some plausible explanations for the difficulties students encounter?

Theoretical Perspective

Our cognitive models are similar in structure and purpose to the genetic decompositions of APOS theory. A *genetic decomposition* "is a model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept" (Brown, DeVries, Dubinsky, & Thomas, 1997). The difference between our models and a genetic decomposition is that we are not describing how the effective use of summation notation is developed by a student but rather what kinds of mental constructions are necessary in order to use the notation effectively.

In general, these "mental constructions" can be broken down into *actions, processes, objects,* and *schema.* This breakdown is a framework that is often referred to by the acronym APOS (ibid). *Actions* are procedures or transformations that are performed on objects, often in a very step-by-step manner. Through interiorization, a sequence of actions can be reflected upon and envisioned and analyzed without needing to be carried out. When an individual interiorizes a sequence of actions we say that they have constructed a *process.* When an individual is able to reflect on a process as a whole and even apply other actions to that process, we say that the individual has encapsulated that process into an *object.* Actions, processes, and objects can be coordinated into a *schema*.

These models of cognition can be useful for instructional design as well as helping to explain the causes of specific student errors and non-standard conceptions. Below we outline our cognitive models, which describe how the effective use of summation notation involves the successful coordination of a number of actions and processes into a coherent schema. In our Results section we will demonstrate how these models may be used to provide plausible explanations for some observed survey responses.

Expanding a summation-Notation Expression: summation notation can be thought of as a schema, encoding a coordinated set of mental activities or procedures :

- *Iteration* - running through the iterating values of the index
- *Function Evaluation* - This could be a conventional function in the summand of the notation, but also captures the use of indexed variables (which are themselves really just functions on the Natural Numbers).
- *Summation* – Summing the terms generated by the coordination of *iteration* and *function evaluation*. Interestingly, there may be two ways to think about this that are fundamentally different: you can either add as-you-go creating a running total, or generate the full list of terms and sum all-at-once at the end. This distinction is very important when you begin to consider infinite series and convergence.

Accurately interpreting a given summation-notation expression involves coordinating these mental activities or procedures:

For example, in order to expand

$$\sum_{k=2}^{7}(k+1)^2$$

Figure 1

one might go through the following mental actions (MA):

MA1. Recognition that the initial value of the index k is 2.
MA2. If $k <= 7$, coordination of the following actions:
(a) Action of evaluating the summand with the current value of the index (k).
(b) Action of adding the resulting value to the running total. Alternatively, the action of placing that value in a list to be added at the end.
(c) Action of incrementing the index to the next whole number.
(d) Return to Step 2 again.
MA3. Recognition that, after running through 2(a)-2(d) when $k = 7$, you are finished expanding the expression.

MA4. In the case of list generation, the final step would be to compute the sum.

After performing this sequence of actions one might interiorize that sequence and consider the whole expansion as a process. This process could then be conceived of as being constructed from the following sub-processes:

P1. Recognition that the initial value of the index k is 2.
P2. Construction of the process of k running through all of the successive integer values between 2 and 7.
P3. Coordinated with Step 2, construction of the process of the summand being evaluated over the range of k values.
P4. Recognition that these two processes together will generate a) a list of terms to be summed, b) an addition expression, or c) a number.

The process view outlined above represents a more sophisticated understanding of expanding summation notation, one that could be used to make sense of more complex ideas like Riemann sums. However, one must still go through the mental actions in order to actually generate the long-hand sum.

Expressing a Sum Using summation Notation:

Given the prompt,

Express the sum of the first five odd integers using Summation Notation,

one might go through the following mental constructions:

MC1. Construct, mentally or physically, the long-hand sum, e.g.:

$$1 + 3 + 5 + 7 + 9$$

MC2. Construct an indexing process
  a) Identify an appropriate starting value of the index.
  b) Identify successive integer values of the index with each term in the sum.
  c) Identify the appropriate terminating value of the index.

MC3. Construct a function that takes index values as its input and outputs the appropriate term of the sum. For this example, with index k, the function $(2k - 1)$ would generate the appropriate addends for integer values $1 \leq k \leq 5$.

MC4. Arrange the elements of the notation to indicate the desired sum.

We think of MC2 as outlined above as a kind of 'standard' way of constructing an indexing process. However, there are certainly non-'standard' ways of arriving at an equivalent end result. One might also first come up with a function that generates the odds and then adapt the indexing to generate the appropriate list to be summed. For instance, if you thought of odd numbers as one-greater than evens, you might want to use the formula $(2k + 1)$ to generate odd numbers. In that case, the next step would be to identify which values of k you would have to plug in to generate 1 as your first odd and 9 as your last.

Method

We designed a brief survey made up of three different tasks related to summation notation. The three tasks were designed to give students the opportunity to work with expanding a summation-notation expression and expressing a long-hand sum using summation notation. The first task was a context problem in which the students were asked to express a desired quantity using summation notation. There were two versions of this task, one whose solution involved using a composition of functions and one whose solution did not. With this first task we sought to simulate some of the challenges we noticed in our Advanced Calculus teaching experiment (Strand, Zazkis, & Redmond, 2012) *without* involving the area context. For the second task, students were asked to express the sum of the first ten odd integers using summation notation. For the third task, students were given a summation notation expression and were asked to write out the long-hand sum it represented. These pair of tasks were designed to investigate how well students could use and interpret summation notation in a simple context.

Task 1: In Physics class you and your lab partner have built a model roller coaster. In the table below, the average speed of the car (in meters per second) is recorded over each 1.5 second time interval. Using $\sum$-notation, express the total distance traveled by the car over six seconds.

Time (sec)	**Avg Speed (m/s)**
$0 - 1.5$	5.1
$1.5 - 3$	6.4
$3 - 4.5$	4.9
$4.5 - 6$	6.9

Figure 2

Task 2: Using $\sum$-notation, write an expression for the sum of the first ten odd integers.

Figure 3

Task 3: Write out the given summation longhand:

$$\sum_{k=2}^{7}(k+1)^2$$

Figure 4

The surveys were distributed to students in multiple sections of Calculus II (techniques and applications of integration), Calculus III (sequences and series), an introductory real analysis course, and a graduate-level analysis course. In this way we hoped to receive responses running the gamut of student experience with summation notation. We received 117 completed surveys, of which 98 were from the calculus sections.

The first round of analysis consisted of evaluating each survey response for correctness. During the second run-through the focus was on describing, as specifically as possible, each error that occurred. At this stage we were most interested in how errors on the simple tasks correlated with errors on the more challenging tasks. It was our initial attempt to explain the origins of the errors that led us to develop cognitive models of the mental activities involved in using summation notation.

Setting the data aside for the moment, we set about constructing the aforementioned cognitive models. These were developed a priori, drawing on our mathematical knowledge of summation notation and the constructs of APOS theory. After much thought and debate, we arrived at something we though might be useful for analyzing our data. Through the process of data analysis were able to further refine the models, until we arrived at something that was capable of explaining many of the most common and significant errors we were seeing .

Results

The cognitive models that we developed provide powerful tools for analyzing and explaining student errors when using summation notation. Due to length considerations, here we will briefly discuss some of the errors we were able to explain with the models.

A sample response to Task 3 is given in Figure 5.

$$(2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2$$

Figure 5

This student seems to have been able to evaluate the function when $k = 2$. However, there seems to be no recognition of the indexing process (MA2c), and they appear to have taken the value of 7 (Figure 4) to be the number of terms in the sum. Here is a very static view of summation notation, with almost none of the underlying processes being demonstrated with this response.

A sample response to Task 2 is given in Figure 6.

$$\sum_{i=2}^{N}(i-1)$$

Figure 6

This student also provided an unprompted example that further illuminated their thinking (Figure 7).

$$\sum_{i=2}^{4}(i-1) = (2-1)+(4-1)+(6-1)+(8-1)$$

F

igure 7

While there is evidence of an iterative evaluation of the function ($i - 1$), the increment of the index appears to be 2. It is impossible to tell whether the *"$i = 2$"* is supposed to suggest an increment of 2, the starting value of the index, or both. Even the successful coordination of MC2 and MC3 does not guarantee successful adherence to the convention; the sub-process of incrementing the index by 1 when a different increment is desired presents a non-trivial challenge for many students.

A sample response to Task 1 is given in Figure 8.

$$\sum_{j=0}^{6}\frac{3}{2}f(x) = \frac{3}{2}(5.1)+\frac{3}{2}(6.4)+\frac{3}{2}(4.9)+\frac{3}{2}(6.9)$$

Figure 8

It seems reasonable that the student used "*f(x)*" to stand for the Avg Speed values from the table provided and that the 3/2 represents the time interval for each measurement. Notice that there is no indexed variable in the summand. We intentionally designed this problem so that there is no obvious rule that takes an index value to the list of average speeds (Figure 2). Thus it is not surprising that the student was unable to use an indexing process to generate the desired long-hand sum (which itself is a correct representation of the stated problem). However, the values for j going from 0 to 6 suggest that there may be further difficulties with MC2. While it is unclear exactly how they envisioned *f(x)* being evaluated, they did not demonstrate an ability to coordinate an indexing process with a function-evaluation process (MC2 and MC3).

Conclusions

The cognitive models outlined above provide a framework for analyzing student thinking about summation notation. Additionally, they could be used to design instructional tasks that would help students to develop an understanding of the underlying processes *and* ways in which they can successfully be coordinated (Weber, and Larsen, 2008; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997). It is worth noting, especially for educators, that summation notation encodes multiple potentially challenging processes for students. More than that, successful use of summation notation involves *coordinating* these challenging processes. The struggles we found with Task 1 suggest that it is likely that our original work in the context of area under a curve *revealed* difficulties with summation notation rather than *causing* them.

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UNDER THE RADAR: FOUNDATIONAL MEANINGS THAT SECONDARY MATHEMATICS TEACHERS NEED, DO NOT HAVE, BUT COLLEGES ASSUME

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High school mathematics teachers must have coherent systems of mathematical meanings to teach mathematical ideas well. One hundred five teachers were given a battery of items to discern meanings they held in with respect to quantities, variables, functions, and structure. This paper reports findings on a sample of items that, by themselves, should alert college mathematics professors that foundational understandings they assume students have in advanced mathematics courses likely are commonly missing.

Key words: Mathematical meanings, mathematical knowledge for teaching, undergraduate mathematics, mathematics teacher education

In this paper we report partial results from a project that attempts to discern teachers' *mathematical meanings for teaching secondary mathematics* (MMTsm). The project's focus is on assessing mathematical meanings teachers have, meanings they attempt to convey in instruction, and relationships between them (Thompson, in press). For the present purpose, however, we focus on results of an assessment given in the summer of 2012 to 105 secondary mathematics teachers in two states in terms of their implications for teachers' undergraduate mathematical preparation.

The assessment consisted of 36 paper items and 3 animation items. We will not discuss the animation items here. The 36 paper items were distributed among three forms, each form consisting of 9 items that were common to all forms and 9 items unique to each form.

We administered the assessment to 141 teachers in June and July of 2012. The teachers were in four groups: Three groups (126 teachers) from an MSP project conducted at a major Midwestern university and one group (15 teachers) from an MSP project conducted at a major Southwestern university. One hundred five (105) of these teachers were currently teaching high school mathematics or were starting to teach in Fall 2012. Table 1 gives a breakdown of teachers' teaching experience (number of courses taught, "Crs Tgt") in relation to formal mathematical preparation.

Table 1. Teachers' experience

	Major			
Crs Tgt	***Math***	***MathEd***	***Other***	***total***
0	1	5	0	6
1-5	7	2	10	19
6-10	6	3	11	20
11-20	5	15	8	28
>20	10	18	4	32
total	29	43	33	105

The items were drawn from the areas of variables and variation (4), covariation (4), functions (7), proportionality (5), rate of change (7), and structure (9). No item that has face validity to teachers can focus on just one these areas. Items therefore have aspects of two or more areas but a greater reliance on one. The item's category is our estimation of the most prominent meaning involved.

In this paper we will discuss results from three specific items: one on function, one on structure, and one on rate of change.

Function Item

The function item is given in Figure 1. It was given to 34 teachers. The item draws upon a scheme of meanings entailed in the use of function notation, namely that, for example, "*w*" is the function's name, "*u*" represents an input value, and that "$w(u)$" represents the output value that is determined by the rule "$\sin(u - 1)$ if $u \geq 1$" when a value of *u* is given as input. The item also entails an additional aspect of a process conception of a function definition (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992): teachers need to envision a process whereby values of variables are "passed" from one function definition to another.

Here are two function definitions.

$$w(u) = \sin(u-1) \text{ if } u \geq 1$$
$$q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1$$

Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined.

$$c(v) = \begin{cases} q(\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ w(\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

Figure 1. Item F07v1: Understanding function notation.

The letter "v" should be placed in each blank because *v* represents the value of the function's input, and the right hand side gives a rule for how to produce *c*'s output when given a value of *v*.

Table 2 shows that only 9 of 34 teachers (26%) wrote *v* in the blanks, and that 17 of 34 teachers (50%) wrote *r* in the blanks associated with *q* and wrote *u* in the blanks association with *w*. Table 2 also shows that only 1/3 of teachers with a BS in Math or Math Ed wrote *v* in the blanks of *c*'s definition. Others either wrote *r* and *u* or a combination of *r*, *u*, and *v*. Table 3 shows that teachers who had taught precalculus, calculus (AB or BC), or differential equations were not likely to have written *v* in the slots in *c*'s definition.

Table 2. Responses for Item F07v1

	Math	*MathEd*	*Other*	*total*
R U	5	5	7	17
V	2	5	2	9
Mix	0	2	2	4
I don't know	0	1	1	2
No Answer	0	2	0	2
total	7	15	12	34

Table 3. Responses to F07v1 by Precalculus+ teachers

	Math	*MathEd*	*Other*	*total*
R U	2	2	0	4
V	0	1	2	3
I don't know	0	1	1	2
Mix	0	1	0	1
total	2	5	3	10

Teachers' responses on other items in the function group shed light on their scheme for function notation. It is that many teachers think of function notation idiomatically. As an idiom, the constituent elements of "$w(u)$" have no direct relationship to the meaning of "$w(u)$". Rather, the idiom's meaning is figurative, and is made through the use of the expression in its entirety. It is as if "$w(u)$" is the function's name, whereby a teacher writing something like “$w(u) = 3x + 5$” is expressing something like “the function named ‘w(u)’ is $3x + 5$”.

Structure Item

An important aspect of seeing mathematical structure is to see something complex as also being something simple, and to see something simple as entailing an internal complexity. The structure item in Figure 2 requests teachers either to see four terms as three terms or two terms as three terms.

> **Δ is a binary operation over the real numbers. It is associative. This means:**
>
> **For all real numbers a, b, and c, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.**
>
> **a) Let u, v, w, and z be real numbers. Can the associative property of Δ be applied to the expression below? Explain.**
>
> $$(u \Delta v) \Delta (w \Delta z)$$
>
> **b) Why might a teacher ask this question?**

Figure 2. Structure item S01v3.

Viewed structurally, the expression $(u \Delta v) \Delta (w \Delta z)$ can be seen as

$$\overbrace{(u \Delta v)}^{a} \Delta (w \Delta z)$$

or as

$$(u \Delta v) \Delta \overbrace{(w \Delta z)}^{c}.$$

In either case, the associative property of Δ can be applied to the resulting 3 terms.

Table 4 shows that 16 of 111 high school math teachers (5 Math, 5 MathEd, and 6 Other) said that the associative property of Δ cannot be applied to $(u \Delta v) \Delta (w \Delta z)$. The most common reason was that associativity requires 3 terms and $(u \Delta v) \Delta (w \Delta z)$ has either two or four terms. Table 5 shows that 74 of 111 high school math teachers said “yes”, that the associative property of Δ could be applied to the expression $(u \Delta v) \Delta (w \Delta z)$. Of those 74 teachers, 16 gave a valid explanation that was either a correct application of associativity or a statement that two terms could be considered one. Fifty-eight (58) teachers said “yes”, that the associative property of Δ can be applied to the expression $(u \Delta v) \Delta (w \Delta z)$, and then gave an invalid demonstration, a non sequitur justification, or no justification.

Table 4. Explanations by High School Math Teachers' Who Said "No"

Explanation	***Math***	***MathEd***	***Other***	***total***
Associative property requires 3 elements	2	2	3	7
Interpreted problem as about something other than associativity	2	0	2	4
Imported the properties of an arithmetic operation	1	1	0	2
No justification	0	1	1	2
Used commutativity	0	1	0	1
total	5	5	6	16

Table 5. Explanations by High School Teachers' Who Said "Yes"

Explanations by teachers who answered "Yes"	*Math*	*MathEd*	*Other*	*total*
Imported the properties of an arithmetic operation	3	6	8	17
Explained that (uv)(wz) can be thought of as a(wz), where a=(uv), and associativity of Δ applied to it	5	11	0	16
You said in the item's stem that Δ is associative	4	5	5	14
Parentheses don't matter for associativity	5	4	4	13
No justification	5	4	0	9
Used commutativity	1	1	2	4
Interpreted problem as about something other than associativity	0	0	1	1
total	23	31	20	74

Other items in the Structure group clarify teachers' difficulty with S01v3. Together they suggest that many teachers' thinking is constrained to one level of organization—that they see complex mathematical statements as unstructured strings. If this is indeed the case, then it is understandable that teachers with this way of thinking are challenged by sophisticated concepts—concepts having nested levels of entailed meanings—and by complex mathematical statements.

Rate of Change

Rate of change, which entails both ideas of relative change and ideas of accumulation, is *the* foundational concept in the calculus (Thompson, 1994a; Thompson, 1994b). Developing students' understandings of rate of change is a primary task of secondary school mathematics instruction. Teachers without a rich scheme of meanings for rate of change will be limited in helping students support a rich scheme of meanings. The rate item R08v1 appears in Figure 3. It is a standard algebra question, usually included under the heading "weighted averages". The key to reasoning to a solution is to understand that a round trip will take 3 hours (180 miles at 60 mi/hr), and that the first part takes 2.25 hours (90 mi at 40 mi/hr), leaving 0.75 hours to travel the returning 90 miles. Thus, the car would need to have an average speed of 120 mi/hr to have a round-trip average speed of 60 mi/hr.

> A car went from San Diego to El Centro, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to San Diego if it were to have an average speed of 60 miles per hour over the round trip?

Figure 3. Rate of change item R08v1.

Thirty-four teachers received item R08v1. Table 5 gives teachers responses. Only 7 of 34 teachers (21%) answered correctly. The most common response was 80 mi/hr. Teachers' work made clear that they arrived at 80 by solving the equation $(x + 40)/2 = 60$. We hoped that teachers who had taught precalculus, calculus (AB or BC), or differential equations would have a higher success rate. Table 6 shows a success rate of 40% among these teachers. While it is higher than other teachers, it is still surprisingly low.

Table 6. Responses to R08v1.

	Math	*MathEd*	*Other*	*total*
80 mi/hr	4	9	3	16
120 mi/hr	2	2	3	7
Analyst could not interpret	0	1	4	5
No answer—no written work	1	1	2	4
Computed time for first trip	0	2	0	2
total	7	15	12	34

Table 7. Responses to R08v1 by Precalculs+ teachers

	Math	*MathEd*	*Other*	*total*
80 mi/hr	1	2	1	4
120 mi/hr	1	1	2	4
Computed time for first trip	0	1	0	1
No answer—no written work	0	1	0	1
total	2	5	3	10

Other items in the Rate group shed light on teachers' meanings for rate of change. In essence, their scheme involves only one quantity – rate, and that quantity itself is more a number than a quantity. A mature scheme of meanings for rate of change involves three quantities – two quantities changing simultaneously and a third quantity (rate) that entails a multiplicative relationship between them. If this is correct (it is consistent with other research on students' understanding of the calculus), then these teachers had these meanings as students of calculus and their students will have a high probability of having a one-quantity meaning of rate when they enter calculus.

Discussion

Our research suggests that a significant percentage of teachers have meanings and schemes of meanings that are poorly developed. We see two possibilities, neither of which speak well of university mathematics education. Either teachers developed these weak meanings as undergraduate students, or they developed many of these meanings when they were high school students and they carried these meanings throughout their undergraduate mathematics coursework. In either case, they assimilated their undergraduate mathematics instruction into these schemes and their understanding of undergraduate mathematics instruction was built upon these schemes. We hope that by making college mathematics instructors aware of how fragile the base of understanding is among students they are teaching, that mathematics departments will become proactive in adjusting their introductory mathematics curriculum and instruction.

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UNDERSTANDING ABTRACT ALGEBRA CONCEPTS

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ABSTRACT: *This study discusses various theoretical perspectives on abstract concept formation. Students' reasoning about abstract objects is described based on proposition that abstraction is a shift from abstract to concrete. Existing literature suggested a theoretical framework for the study. The framework describes process of abstraction through its elements: assembling, theoretical generalization into abstract entity, and articulation. The elements of the theoretical framework are identified from students' interpretations of and manipulations with elementary abstract algebra concepts including the concepts of binary operation, identity and inverse element, group, subgroup.*
To accomplish this, students participating in the abstract algebra class were observed during one semester. Analysis of interviews and written artifacts revealed different aspects of students' reasoning about abstract objects. Discussion of the analysis allowed formulating characteristics of processes of abstraction and generalization. The study offers theoretical assumptions on students reasoning about abstract objects. The assumptions, therefore, provide implications for instructions and future research.

KEYWORDS: Abstraction, Generalization, Abstract Algebra, Group Theory

<u>Introduction.</u> Abstract thought is considered to be the highest accomplishment of the human intellect as well as its most powerful tool (Ohlsson, Lehitinen, 1997). Even though some mathematical problems can be solved by guessing, trial and error, or experimenting (Halmos, 1982), there is still a need for abstract thought. There is support (Ferguson, 1986) for the hypothesis that abstraction anxiety is an important factor of mathematics anxiety, especially concerning topics which are introduced in the middle grades. By understanding an abstract concept formation we will be able to help students to overcome this anxiety.

This paper presents results of the exploration of the process of abstraction and gives a description of its components and outcomes. The goal is to understand the nature and acquisition of abstraction, so we can help students to bridge the gap from the abstract to concrete. Qualitative approach has been used to reach the goal. Analysis of students' concept formation (knowledge of abstract/mathematical object) is consistent with the tradition of a grounded theory (Charmaz, 2003; Glaser & Strauss, 1967). The study was conducted in the content of group theory.

<u>Theoretical Framework.</u> Piaget (1970a, 1970b) considers two types of cognition: association and assimilation, stating that assimilation implies integration of structures. Piaget distinguishes three aspects of the process of assimilation: repetition, recognition and generalization, which can closely follow each other. In his papers about advanced mathematical thinking, Dubinsky (1991a, 1991) proposes that the concept of reflective abstraction, introduced by Piaget, can be a powerful tool in the process of investigating mathematical thinking and advanced thinking in particular.

In the late 1980s Ed Dubinsky and his colleagues (Clark at al., 1997) started to develop a theory that describes what can possible be going on in the mind of an individual when he or she is attempting to learn a mathematical concept. In recent years, the mathematics education community at large started to work on developing a theoretical framework and a curriculum for undergraduate mathematics education. Asiala (Asiala et al, 1996) reported the results on their work: based on the theories of cognitive construction developed by Piaget for younger children, Dubinsky and his colleagues proposed the APOS (action – process – object – schema) theory. A number of studies on topics from calculus and abstract algebra (Zazkis & Dubinsky, 1996; Dubinsky at al, 1994; Brown at al, 1997; etc) were reported using this framework.

The theoretical approach, described by Davydov (1972/1990), is highly relevant to educational research and practice. His theory seems incompatible with the classical Aristotelian theory, in which abstraction is considered to be a mental shift from concrete objects to its mental representation – abstract objects. By contrast, for Davydov, as well as for Ohlsson, Lehitinen (1997), Mitchelmore and White (1994, 1999), Harel and Tall (1991, 1995), abstraction is a shift from abstract to concrete. Ohlsson and Lehitinen provide us with historical examples of scientific theories development; Davydov also gives historical examples and, at the same time criticizes the empirical view on instruction by claiming that empirical character of generalization may cause difficulties in students' mathematical understanding.

Following Piaget (1970 a), the framework for this study considers the process of abstraction as a derivation of higher-order structures form the previously acquired lower-order structures. Moreover, the two types of abstraction are distinguished. One of these types is simple or empirical abstraction – from concrete instances to abstract idea. The second type then is more isolated from the concrete. Davydov (1972/1990) calls this type of abstraction "theoretical abstraction". Theoretical abstraction, based on Davydov's theory, is the theoretical analysis of objects (concrete or previously abstracted) and the construction of a system that summarizes the previous knowledge into the new concept (mathematical object), so it is ready to be applied to particular objects. This abstraction appears from abstract toward concrete and its function is the object's recognition. According to present research, the second type of abstraction is commonly accepted as essential in the process of learning deep mathematical ideas. Similarly, there are two types of generalization – generalization in a sense of Ohlsson and Lehitinen perspectives (which coincides with empirical perspective, described by Davydov); and theoretical generalization. Theoretical generalization is the process of identifying deep, structural similarities, which in turn, identify the inner connections with previously learned ideas. The process of theoretical abstraction leads us to the creation of a new mental object, while the process of theoretical generalization extends the meaning of this new object, searching for inner connections and connections with other structures.

In summary, the genesis of new abstract idea looks like following: (0) initial abstractions; (1) grouping previously acquired abstractions (initial abstractions in a very elementary level); (2) generalization to identify inner connections with previously learned ideas; (3) the shift from abstract idea to a particular example to articulate a new concept. Note that at some level of cognitive development initial abstractions become obsolete since sufficient more complex and concrete-independent ideas are already acquired. The result of this genesis is a new structure which is more complex and more abstract compared to the assembled ideas. Hence, we have hierarchical construction of knowledge, where the next idea is more advanced than the previous one. Moreover, cognitive function of abstraction (from now on, abstraction and generalization are theoretical abstraction and generalization, as defined above) is to enable the assembly of previously existed ideas into a more complex structure. The main function of abstraction is recognition of the object as belonging to a certain class; while construction of a certain class is the main function of generalization, which is making connections between objects (see Fig 1). The framework suggests the design of the study and helps to ground the methodology and data collection.

Methodology. To answer the questions above, 22 students, participating in undergraduate Abstract Algebra course were observed during class periods during one semester. Written assignments (quizzes, homework, exams) were collected from all participants. A group of participants (7 students) was interviewed three times during the semester.

Research Questions. The following questions were formulated based on the theoretical framework:

- What notions and ideas do students use when they recognize a mathematical object, and why? (what are students using: definitions, properties, visualization, previously learned constructs, or something else?)
- What are the characteristics of students' mathematical knowledge acquisition in the transition from more concrete to more theoretical problem solving activity?

Discussion. The data analysis revealed a different aspect of students reasoning about abstract algebra concepts.

Understanding the concept of a binary structure

The term "binary structure" and the notation $(S, *)$ normally used to represent a binary structure is usually understood by students as a mathematical object with two entrees: a set and an operation. The term and notation do not imply any necessary correspondence or relations between them. Dubinsky and colleagues (1994) discussed this problem analyzing students' understanding of groups and their subgroups. The study proposed that there are two different visions of a group: 1) a group as a set; and 2) a group as a set with an operation. Similarly for a subgroup: 1) a subgroup as a subset; and 2) a subgroup as a subset with an operation. Analysis of the data collected for this study showed related trends:

Binary Operation. Closure

The number of solutions in the data (Figure 2 and Figure 3) suggested that the students still try to assimilate the concept of a binary operation through familiar operations. Davydov (1972/1990) has proposed that the students who experience this problem try to make sense of a binary structure using empirical thoughts (empirical generalization and abstraction). Students assemble ideas of a set, its elements, an operation on any two elements, and the result of the operation on any two elements. By a simple generalization process they develop a simple abstract idea or, in other words, there is a shift from concrete operations (such that addition or multiplication, for instance) to abstract (such as operation "star" defined on set $\{a, b, c\}$).

Thus, often the process of understanding a binary operation is empirical rather than theoretical. The data provided evidence for the failure of empirical thought about binary operation during the object recognition stage. For example, when answering the following question: "Give an example of an operation on $\boldsymbol{Z}$ which has a right identity but no left identity", students often responded that division is this type of operation on $\boldsymbol{Z}$ (Figure 2). Indeed, division is not defined on $\boldsymbol{Z}$ since $\boldsymbol{Z}$ is not closed under division, and division by 0 is undefined. However, many students recognize division as a binary operation on $\boldsymbol{Z}$.

Binary Structures. Group as a set of discrete elements

Understanding a group as a structure consisting of two objects that interact with each other is complicated and novel for students. The data collected during this study suggest that some students understand a group as a set of elements. The operation in this case does not play an important role in the structure. Figure 6, for example, illustrates how students switched from one operation to another. It suggests that for students operation is not an attribute of a binary structure but rather a separate object which may be used if needed.

At the early stage of understanding the binary structure concept, students construct their knowledge based on previously learned objects. To understand a complex idea such as binary structure, students must have other ideas as parts. The process of generalization initializes connections between the elements and groups these elements in a set. Thus, the new created abstract entity simply repeats the one that already exists. In this case the operation defined on binary structure is not a part of the assembling process and exists disjointedly from the set. It follows that the abstract idea is not complete; further the main function of abstraction (object recognition) fails.

Groups and their subgroups as Binary Structures

The data showed that students often have difficulty understanding connections between a group and its subgroups, both operational and via elements. Student's responses revealed three major misconceptions about subgroups. First, for some students understanding of a subgroup is similar to the understanding of groups as sets. Interestingly, those students who at first understood a group as a set would not necessarily transfer this understanding onto subgroups and vice versa. For some students a group is a set with the operation, whereas a subgroup is just a subset, a part of a bigger structure. A subgroup exists if a subset exists. Several students claimed that the set of odd integers is a subgroup of ($\boldsymbol{Z}$, +). Second, students have problems seeing structural connections between groups and its subgroups. Sometimes they comprehend only elements connection. Students realize that a subgroup is a group itself under an assigned operation. It is not merely a subset of a bigger set; it is a structure. Nevertheless, the assigned operation is not necessarily the group operation. For example, some of the responses defended that $(\boldsymbol{Z}_n, +_n)$ is a subgroup of ($\boldsymbol{Z}$, +), since it is a group and $\boldsymbol{Z}_n$ is a subset of $\boldsymbol{Z}$. A change of subgroups operation from the group operation to a completely different operation was also observed during problem solving activity (Figure 6). Third, some responses did not only demonstrate students' understanding of a subgroup as a subset of a given structure but, in addition, this subset is assumed to be a group itself under the given group operation. However, the concept of binary operation caused difficulty. It is illustrated by the following student's response: "the set of odd integers together with 0 is a subgroup of ($\boldsymbol{Z}$, +)".

The data also showed that students find it easy to work with concrete examples of cyclic groups. Moreover, they are very comfortable listing their subgroups and describing them. Not all the students, however, appreciate theorems which help to minimize steps in the problem solving process. The fact that students often used cyclic groups as concrete examples during problem solving suggests that cyclic groups proved themselves very useful objects for the articulation process in the group concept formation. Nevertheless, sometimes this articulation is based on empirical generalization (students observe several examples of subgroups of a cyclic group and conclude that they all must be cyclic), rather than on analysis of the inner connection within the structure. As a result, students accept the idea that if G is a group, then it is closed under the assigned operation. It follows that every nonidentity element generates a nontrivial cyclic subgroup. However, students' view of the inner connections is still not comprehensive and a group is perceived as a union of such cyclic subgroups (Figure 4).

Data analysis and theoretical perspectives suggest that when learning concepts of cyclic groups, their subgroups and cyclic subgroups, students often rely on empirical generalization since the concepts are well illustrated by a variety of concrete examples. Instead of recognizing concepts in the examples, students are looking for commonalities via empirical thought rather than theoretical.

Definitions of objects. How students use them

The theoretical framework suggests that a definition is the initial stage of concept formation. A definition suggests ideas for assembling. For example, a group is a set, closed under an assigned operation, the operation must be associative; an identity element must be in the set, and every element of the set must have an inverse. The definition puts forward some previously abstracted ideas for assembling. Analysis of the connections between the ideas, and articulation follow the assembling. Later, when concepts are being recognized in concrete problems student also must refer to definitions to collect objects from the assembling process, which must be recognized first. The data shows that students had no troubles using definitions to recognize objects but could not use definitions to construct them

The data suggests that there is a gap between the abstract entity students have constructed from the definition and the articulation process, the recognition per se. Another important issue that came from the analysis of students' responses is the use of quantifiers and understanding of quantification in general.

Quantifiers

The study did not intend to explore students' discourse or use of quantifiers. However, this problem could not be disregarded. Some students who participated in the study did not use quantifiers at all when defining objects. Sometimes, missing quantifiers did not mean that the concept was not recognized or used properly during problem solving process. The preliminary analysis of the interviews suggested looking more carefully at the written work in terms of the presence of quantifiers. Students used quantifiers more often when writing statements but sometimes students changed the order of quantifiers they used. For example, instead of writing $\exists\forall$ statement they had $\forall\exists$ statement (Figure 5). Quantification question is very important for concept formation and requires more exploration.

Conclusions and implications: The study showed that one needs to have previously abstracted ideas to understand a new abstract structure. Moreover, data analysis and further discussion revealed that an abstract concept cannot be learned without concrete examples and problems that involve the concept. In other words, the articulation of an abstract concept is required for coherent structure formation.

At the first stage of the learning process, students are often given a definition of a concept being studied. Sometimes several simple examples precede the definition. These activities give students a chance to generate a preliminary set of objects for assembling. All these objects are previously learned abstract ideas. The process of assembling is followed up by the process of theoretical generalization. Since a definition usually gives only a preliminary set of ideas for assembling, it is most likely impossible to coherently understand inner connections between the ideas and form a plausible abstract entity, which means that we deal with a preliminary generalization. The next standard instructional step is illustration of the concept via various examples. During this stage students are getting the first articulation experience and make first attempts to concept recognition. At this stage a student should be able to exemplify and counter exemplify the concept. It means that when the concept is learned, the process of abstraction of these objects gets into the following static form: 1) connected assembled ideas; 2) complete understanding of meaningful inner connections; 3) and open-minded recognition of the object. At this stage, a student also should be able to interchange from object recognition to assembled ideas, if needed. It follows that all stages of abstract concept formation are interconnected. There is a constant interaction between processes (assembling and articulation) within the process of abstraction. This observation implies that if there is a problem with one process the abstract concept cannot be appropriately formed. This discussion leads to the following summary of possible predicaments for concept formation: 1) Empirical generalization and abstraction instead of theoretical. Students are trying to learn concepts by extracting commonalities from given concrete objects and examples. 2) Assembling of unsuitable ideas. Students mistakenly assemble some ideas which are not supposed to be assembled to learn a certain concept. As a result, theoretical generalization results in a misleading abstract entity and further in false conclusions which look true under students' arguments. 3) Insufficient number of assembled ideas. 4) Making the object of recognition (during problem solving) one of the ideas for assembling. 5) Insufficient articulation. Students find it difficult to provide examples, especially counterexamples. 6) Isolation of concrete examples from objects of assembling. Sometimes students do not see the interaction between the concrete examples and the abstract structure. A concrete example is considered to be a static object with fixed properties.

Awareness of these predicaments can help to create meaningful instructional activities and classroom settings, giving enough examples and time so that students can articulate the concept they study. The theoretical conclusions can be applied to different mathematical courses at various levels. They are not limited by mathematics only and can be applied in other areas of study. To elaborate on these predicaments, more exploration, possibly within a different mathematical content, is needed.

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List of figures:

Figure 1: Process of Abstraction

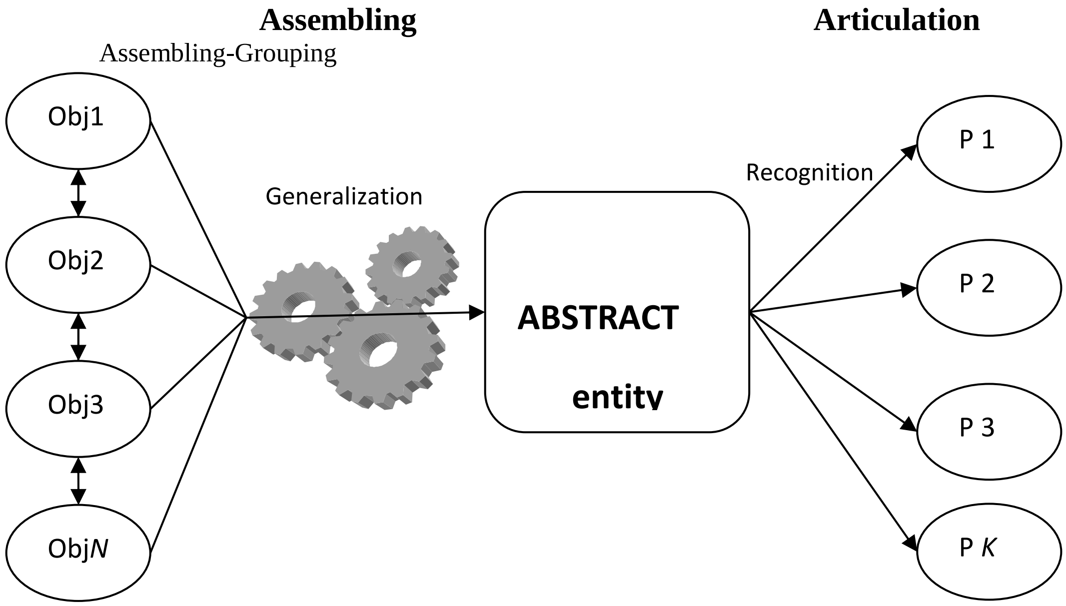
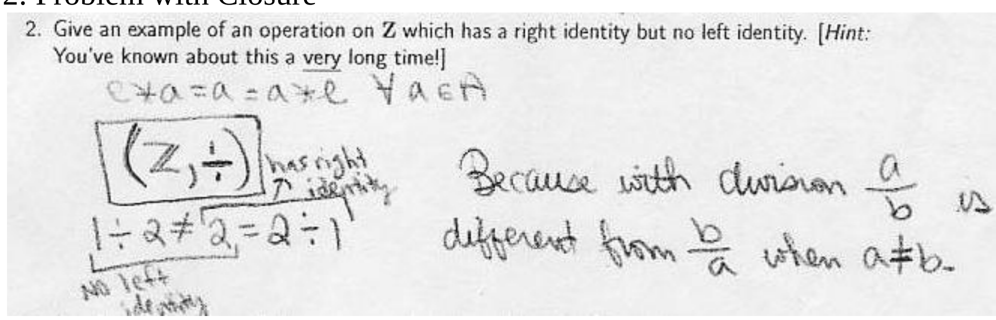


Figure 2: Problem with Closure


2. Give an example of an operation on **Z** which has a right identity but no left identity. [*Hint:* You've known about this a very long time!]

$e * a = a = a * e \ \forall a \in A$

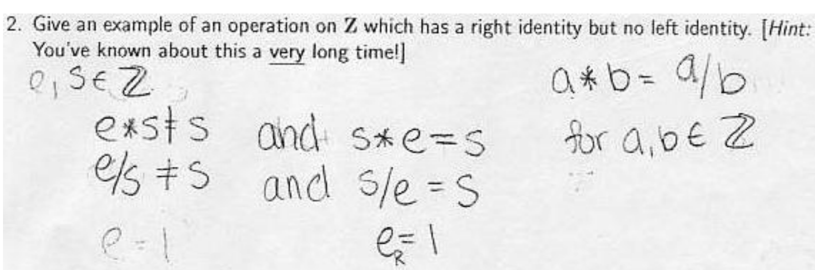
$\boxed{(\mathbb{Z}, \div)}$ has right identity

$1 \div 2 \neq 2 = 2 \div 1$

No left identity

Because with division $\frac{a}{b}$ is different from $\frac{b}{a}$ when $a \neq b$.

Figure 3:


2. Give an example of an operation on **Z** which has a right identity but no left identity. [*Hint:* You've known about this a very long time!]

$e, s \in \mathbb{Z}$

$e * s \neq s$ and $s * e = s$

$e/s \neq s$ and $s/e = s$

$e = 1$ $\quad e_R = 1$

$a * b = a/b$ for $a, b \in \mathbb{Z}$

Figure 4: Student's response to the question: *Is it possible to find two nontrivial subgroups H and K of (**Z**, +) such that H∩K={0}? If so, give an example. If not, why not?*

(3) Yes it is possible.
Consider $2\mathbb{Z}$ (the even integers)
and the set of all odd integers unioned with $\{0\}$.
Certainly these are not trivial subgroups and
the intersection is equal to $\{0\}$.

Figure 5: Student defines an identity

b) given $(S,*)$ $\forall a \in S\ \exists e_s \in S$ s.t. could give alternate

$e_s * a = a = a * e_s$

If $e_s * a = a$ Then e_s is a left idenity

If $a = a * e_s$ Then e_s is a right idenity.

Figure 6: Switching from group operation (addition) to another operation (multiplication):

The subgroup criterion is:
H must be a non-empty subset with
$a, b \in H$ and $ab^{-1} \in H$

$2\mathbb{Z} = 2, 4, 6, 8, 10$
$3\mathbb{Z} = 3, 6, 9, 12$
$2\mathbb{Z} \cup 3\mathbb{Z} = \{2, 3, 4, 6, 8, 9, 10, 12\}$

No, $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subgroup of $(\mathbb{Z}, +)$ because $2\mathbb{Z} \cup 3\mathbb{Z}$ is not closed.
for example $2+3=5$ but 5 is not in the union of $2\mathbb{Z}$ and $3\mathbb{Z}$.

$H = \{a^n \mid n \in \mathbb{Z}$
$x = a^n$ $xy = xy^{-1}$
$y = a^n$
disregard

1

using the prop.
$x = 2\mathbb{Z}$
$y = 3\mathbb{Z}$
$x, y \in \mathbb{Z}$ and $xy^{-1} \in \mathbb{Z}$
? $2\mathbb{Z} \cdot 3\mathbb{Z}^{-1} \in \mathbb{Z}$, $3\mathbb{Z}^{-1}$ is not in $\mathbb{Z}$. Therefore $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subgroup.

UTILIZING TYPES OF MATHEMATICAL ACTIVITIES TO FACILITATE CHARACTERIZING STUDENT UNDERSTANDING OF SPAN AND LINEAR INDEPENDENCE

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The purpose of this study is to investigate students' concept images of span and linear (in)dependence and to utilize the mathematical activities of defining, example generating, problem solving, proving, and relating to provide insight into these concept images. The data under consideration are portions of individual interviews with linear algebra students. Grounded analysis revealed a wide range of student conceptions of span and/or linear (in)dependence. The authors organized these conceptions into four categories: travel, geometric, vector algebraic, and matrix algebraic. To further illuminate participants' conceptions of span and linear (in)dependence, the authors developed a framework to classify the participants' engagement into five types of mathematical activity: defining, proving, relating, example generating, and problem solving. This framework could prove useful in providing finer-grained analyses of students' conceptions and the potential value and/or limitations of such conceptions in certain contexts.

Keywords*:* Span, Linear Independence, Linear Algebra, Mathematical Activity, Concept Image

The purpose of the study is to investigate student thinking about the important ideas of span and linear independence in linear algebra and to contribute to the body of knowledge regarding how individuals understand undergraduate mathematics. In particular, our research goals are:

1. To classify students' conceptions of span and linear (in)dependence.
2. To investigate how students use these conceptions to reason about relationships between span and linear (in)dependence.

The present study focused on interview data that elicited student reasoning about span and linear (in)dependence. We began our analysis through a grounded theory approach in order to identify student conceptions of span and linear (in)dependence. We noticed that, in coding students' concept images, our analysis was facilitated by noting the type of mathematical activity in which the students were engaged as they were sharing their ways of reasoning. In other words, the interview question to which they were responding had the potential of eliciting different aspects of the students' concept images. This is consistent with Vinner's (1991) notion of *evoked concept image.* For example, students' reasons why a claim was true or false revealed ways of thinking about the concepts involved different than did their response to "how do you personally think about this concept?" As such, we identified within the data set five mathematical activities in which students engaged during the interviews: describing, proving, relating, example generating, and problem solving. Within this study we show how these mathematical activities can be used as a lens to further refine characterizations of students' understanding of span and linear (in)dependence.

Given this framework, our refined research objectives are (a) to investigate students' concept images of span, linear (in)dependence, and relationships between the two concepts and (b) to utilize the mathematical activities of defining, example generating, problem solving, proving, and relating to provide insight into these concept images. Our results section briefly details the four concept image categories that grew out of our data: travel, geometric, vector algebraic, and matrix algebraic. We also define the five mathematical activities and provide an example of how coordination of the frameworks informed analysis of student thinking. Additional results will be discussed during the presentation.

Literature Review

There exists a growing body of research into student understanding of span and linear (in)dependence. For instance, Bogomolny (2007) makes use of APOS Theory (Dubinsky & McDonald, 2001) to examine how example generation tasks can influence student understanding of linear (in)dependence. Stewart and Thomas (2010) combine APOS with Tall's (2004) Three Worlds of Mathematics to generalize student understanding of linear independence, span, and basis according to the authors' genetic decomposition of the concepts. Hillel (2000) offers three modes of reasoning (geometric, algebraic, and abstract) within linear algebra, and Sierpinska (2000) suggests three modes of thinking in linear algebra: synthetic-geometric, analytic-algebraic, and analytic-structural. These modes of thinking are attributed to the historical development of linear algebra and align somewhat with Tall's three worlds framework in that the first focuses on spatial reasoning, the second on algebraic manipulation and representation, and the third on formal, theorem-based and axiomatic thinking.

While these studies expand our knowledge of student conceptions of span and linear (in)dependence, the current study differs in that our analysis of student conceptions are grounded with no a priori categorizations. Furthermore, our framing of student conceptions via the construct of mathematical activity adds a level of nuance into both powerful and problematic ways in which students reason about span, linear (in)dependence, and how they are related.

Setting and Participants

The data for this study comes from a semester-long classroom teaching experiment (Cobb, 2000) conducted in an introductory linear algebra course at a large public university. Classroom instruction was guided by the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991), with the goal of creating a linear algebra course that builds on student concepts and reasoning as the starting point from which more complex and formal reasoning develops. The class engaged in various RME-inspired instructional sequences focused on developing a deep understanding of key concepts such as span and linear independence (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, in press), linear transformations (Wawro, Larson, Zandieh, & Rasmussen, 2012), Eigen theory, and change of basis.

The five students analyzed in this research - Abraham (a junior statistics major), Giovanni (a senior business major), Justin (a sophomore mathematics major), Aziz (a junior chemical physics major), and Kaemon (a senior computer engineering major) - participated in semi-structured individual interviews (Bernard, 1988) the week after final exams. Each interview lasted approximately ninety minutes. The purpose of the interview was to investigate how students reasoned about the concept statements that comprise the Invertible Matrix Theorem; the entirety of the interview protocol can be found in Wawro (2011). The current study considered only a portion of this data: students' conceptions of span, linear (in)dependence, and how they relate to each other. The interview questions analyzed in this study are given in Figure 1. Researchers analyzed video recordings and transcripts of the interviews, as well as all written work.

Methods

Videos and transcriptions of the participants' responses to Question 1a and 1b were iteratively analyzed. In the first analysis, the researchers focused on the logical progression of the participants' argumentation and what mathematical objects the participants attributed as 'acting' in different parts of their discussion (i.e. "the matrix spans $\mathbb{R}^3$" or "the vector moves in this direction"). A summarizing process that described the participants' general progression followed this analysis. A second analysis parsed out students' conceptions of linear (in)dependence and span, separating general discussion from instances when the interviewer directly asked the student to define the term. Quotes were drawn from the transcript and grouped by which concept the student was arguing with or describing. It was in this iteration of analysis that distinctions between types of activity became clear and led to the formation of the five categories of activity. In the next iteration of analysis the researchers categorized student quotes according to these five activities and separated the quotes according to span, linear independence, or linear dependence. These quote collections were then compared for categorical similarities and differences. At every stage of this process, the two researchers continually questioned and challenged each other's decisions, such as motivation for choice of categorization or interpretation of a student's quote.

"Suppose you have a 3 x 3 matrix *A*, and you know that the columns of *A* span R^3. Decide if the following statements are true or false, and explain your answer:"

Question 1a

The column vectors of *A* are linearly <u>de</u>pendent.

Follow-ups. Skip if redundant:

- "How do you think about span?"
- "How do you think about what it means for vectors to be linearly <u>de</u>pendent?
- "How does linear <u>de</u>pendence relate to span of a set of vectors?

Question 1b

The row-reduced echelon form of A has three pivots.

Follow-ups. Skip if redundant:

- "How do you think about what a pivot is?
- "How do pivots of a matrix relate to span of a set of vectors?"

Figure 1. Interview questions analyzed for this study.

Results

The participants in this study used a variety of language to describe their understanding of span, linear (in)dependence, and how the two concepts relate to each other. We organized this variety into four concept image categories: travel, geometric, vector algebraic, and matrix algebraic. We also identified five mathematical activities in which students engaged during the interview: defining, proving, relating, example generating, and problem solving. We present a table that coordinates these analyses for linear (in)dependence and briefly explain.

Categories of student conceptions

The *travel* category captures students' description of span and linear (in)dependence in terms indicative of purposeful movement. While this category is consistent with spatial and geometric reasoning, it is more specific in that it captures notions of "getting" or "moving" to locations in the vector space under consideration. The participants' travel conceptions of span were indicated by phrases such as "everywhere you can get" (stated by Justin when describing the span of a set of vectors) or "the vectors can take you anywhere [in $\mathbb{R}^3$]" (stated by Giovanni when describing what it means for vectors to span $\mathbb{R}^3$). With respect to linear independence, participants' travel conceptions included phrases like "[the vectors] only move farther away" (Justin) and "the vectors go in different directions" (Giovanni). A travel conception of linear dependence was generally indicated by phrases such as "then that would make that linearly dependent because I can, I can kind of get there and take that vector back" (Abraham), and "you can move 1 way on 1 vector, 2nd way, and then take the 3rd one back to the origin." (Aziz). These were often given as the inverse of phrases used to describe sets of linearly independent vectors. The *geometric* category includes student discussions in which a matrix or set of vectors are described as "covering an area," and when vectors are represented graphically on 2- or 3-dimensional axes. Most geometric examples of linear dependence showed either two collinear vectors or three vectors placed head to tail to form a triangle with one vertex at the origin.

The *vector algebraic* category captures participants' use of operations on algebraic representations of vectors in order to describe span and linear (in)dependence. This includes scalar multiplication, vector addition, and linear combination of vectors written as n x 1 matrices, or designated by variables (i.e. $2v + 3w$) as well as the use of the equation $A\mathbf{x} = \mathbf{b}$. Vector algebraic conceptions of span included "every vector you can make with linear combinations of the columns" (Justin) and "in order to span $\mathbb{R}^3$, vectors have to be different" (Giovanni). Vector algebraic conceptions of linear independence consisted of some form of the notion that only the trivial linear combination of linearly independent vectors would equal the zero vector. One participant, Abraham, described linear independence as when the equation $A\mathbf{x} = \mathbf{b}$ has one unique solution. This notion is included in this category since Abraham tended to focus on the product as a linear combination of column vectors of A rather than on the matrix as an entity. Vector algebraic conceptions of linear dependence include when a nontrivial linear combination yields the zero vector and the process of scaling one vector or taking a linear combination of two vectors in order to produce a vector that is linearly dependent.

The *matrix algebraic* category is based on instances in which participants used matrix-oriented algorithms such as Gaussian elimination through elementary row operations. Participants also demonstrated attention to whether a matrix was square and noted that elementary row operations maintain solution sets. Although these latter data describe matrices more generally and so interact with or inform conceptions of span, but are not conceptions of span, LI, or LD proper, they were also classified as matrix algebraic. The most prevalent notion of a matrix algebra conception was participants' reliance on the row-reduced echelon form of a matrix equaling the identity matrix (see Figure 2 for an example), with most of these participants discussing the pivots of the matrix. The participants frequently used this idea when discussing both span and linear independence – [square] matrices with column vectors that span the vector space or that are linearly independent row-reduce to the identity matrix. For a matrix algebraic conception of linear dependence, such a matrix would not row-reduce to the identity matrix.

Interviewer:	Um, how do you think about linear dependence in general?
Kaemon:	Um, dependence for me is just, first I just try to look with, if there's a matrix to see if it's, like, if it's already reduced, then I see if there's a variable, or I see that they're multiples, that they're 0 vector, just something to show it's dependent. And then if I can't find that, like, right away, then maybe I'll try to then, I don't know, try to reduce it, whatever, just until I could figure it out.
Interviewer:	So when you say, 'if I can look at it and see if it's dependent,' … do you have an idea behind what it means to be dependent?
Kaemon:	If I could find a linear combination between the vectors, then that doesn't, like, well zero's a scalar I guess, then, well no, that's dependent. It's, it's—every question always has like a different, like, indicator that it's dependent. So, like, it really depends what the question is. But usually, yeah, I just look, just look at the matrix first and then try to manipulate it, if it's not obvious.

Figure 2. Kaemon's matrix algebraic conception of linear dependence.

Types of Mathematical Activity

The construct of types of mathematical activities emerged from our grounded analysis of the data. As we analyzed students' understanding of span and linear (in)dependence in light of the concept image construct, we found ourselves continually drawn to notice the *type of activity* in which students were engaged as they responded to the interview questions. For instance, if a student spoke of span using a phrase such as "get everywhere," was that student engaged in explaining how span related to linear independence, explaining how s/he thought about the concept of span itself, or some other activity? As such, we identified five mathematical activities within the interview data: defining, proving, example generating, problem solving, and relating. We contend that considering the five mathematical activities provides insight into a student's understanding of a concept. We can think of these as facets of a student's interaction with the world based on what s/he understands a concept to be. These activities do not always occur in isolation. Furthermore, an activity may arise naturally based on the interview prompt or may occur spontaneously. Here we provide short descriptions of each activity; further detail and examples from the data set will be discussed during the presentation.

We use the term *defining* to mean the act of describing a concept's essential qualities. During the interviews, students were not asked to create definitions for concepts that were new to them, but rather to explain their notion of a concept's definition. As such, this use of defining may be of a slightly different connotation than the discipline-specific practice of defining (e.g., Zandieh & Rasmussen, 2010). Also, if students spontaneously (i.e., without prompting) described a concept, we put that within the "defining" activity. We use the term *proving* to mean the act of providing a justification to a claim. This reasoning process may be of various levels of mathematical rigor, and it may be carried out for the participant's personal conviction or to convince the interviewer. As such, we use "proving" similarly to Harel and Sowder's (1998) use of "the process of proving," which included the subprocesses of ascertaining and persuading. We use the term *relating* to denote any participant activity that compares, contrasts, or explains relationships between two concepts. The activities of proving and relating are similar, but distinct. We distinguish between these activities based on the participants' intentions. For instance, participants carried out relating activities while engaged in proof activity. Our analyses focused on the participants' purposes for expressing the relationship, and we categorized the activities accordingly. For instance, in Figure 2 Kaemon relates his notion of linear dependence to a matrix's appearance or operations that he might carry out on a matrix. The activity of *example generating* denotes when participants create cases of certain concepts or properties (e.g., a set of three linearly dependent vectors in $\mathbb{R}^3$). As with the other activities, this may be prompted by the interview explicitly or spontaneously done by the interviewee. Finally, the activity of *problem solving* is engaging in some calculation or reasoning with a specific goal to determine a previously unknown result.

Coordinated Analysis: Linear (In)dependence

A summary of the coordinated analysis between students' concept image categories and types of mathematical activity, within linear independence and dependence, is given in Table 1. Students' names are italicized to differentiate linear dependence from linear independence. It is worth noting that a student may understand linear (in)dependence in a way that is not indicated in this table; it may merely be the case that this particular interview did not evoke that understanding from the student at that time.

To lend insight into how a coordinated analysis informed the researchers, consider one student's struggle to coordinate his understanding of linear independence and span. Aziz's name appears in Table 1 in the travel row under the relating and example generating columns as well as in the geometric row under these same columns (among other places). These specific categorizations emerged as Aziz related linear dependence and span by generating geometric examples of linearly dependent vectors via a travel conception. Aziz generated these vectors, stating “they're linearly dependent, because you can use a combination of all 3 to get back to the origin” (geometric and travel). When trying to relate linear independence to span, however, Aziz stated that, “they're linearly dependent. Um...that's a problem I always thought, because if it’s…they move in 3 different directions, they should technically span $\mathbb{R}^3$.” Here, Aziz is referring to his previous statement that vectors spanning $\mathbb{R}^3$ need to move in three different directions (travel conception of span). Aziz makes sense of this seeming contradiction by noticing that the three vectors, “move on the same plane in 3 different directions, but not out of that plane.” This geometric conception of linear dependence allows Aziz to distinguish between the “3 different directions” of vectors that span $\mathbb{R}^3$ (3 dimensions) and vectors that are linearly dependent (3 ordinal directions in the same plane) and hence, make a meaningful comparison between span and linear independence when he states, “but it technically spans, no, makes it a plane in $\mathbb{R}^2$, $\mathbb{R}^3$. I got it, I figured it out.” Attending to the different activities that Aziz engaged in allows a more nuanced analysis of his different conceptions of span and linear dependence.

Table 1. Coordinated analysis of conception categories and mathematical activities.

Linear Independence/*Linear Dependence*					
	Defining	Relating	Proving	Ex. Generating	Pr. Solving
Travel	Aziz	Aziz/*Aziz*	Aziz	Aziz/*Aziz*	
	Abraham		*Abraham*		*Abraham*
	Giovanni/*Giovanni*	Giovanni/*Giovanni*			
	Justin/*Justin*	Justin/*Justin*			
Matrix Alg.	Aziz				
	Abraham	*Abraham*			*Abraham*
				Giovanni	
		Justin	*Justin*		
	Kaemon/*Kaemon*	*Kaemon*			*Kaemon*
Vector Alg.		*Aziz*	Aziz		
	Abraham	Abraham		*Abraham*	*Abraham*
				Giovanni	
				Justin	
	Kaemon			*Kaemon*	
Geometry		Aziz		*Aziz*	
				Abraham	
				Justin/*Justin*	

Conclusion

This proposal summarizes our work in categorizing students’ concept images of span and linear (in)dependence and our use of the construct of mathematical activity to provide insight into these conceptions. We note that the concept image categories that arose may be an artifact of the type of instruction and curriculum that these students experienced. We also note that the types of mathematical activity are not meant to be exhaustive; rather, these five activities were determined from analysis of this small data set. Analysis of classroom data or problem-solving interviews, for instance, would likely give rise to additional types of mathematical activity. As such, our future work involves a further examination and refinement of the framework of mathematical activity as a way to gain insight into students' conceptions of mathematical ideas. In addition, we also plan to examine additional data (classroom, a mid-semester interview) of these same five students in order to gain a more complete analysis of their understanding.

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Prospective Elementary Teachers' Prior Knowledge as a Resource in their Number Sense Development
Contributed Research Report

Abstract
Researchers have documented deficiencies in prospective elementary teachers' mathematics content knowledge. The construct of number sense captures the qualities of mathematical reasoning that mathematics teacher educators wish for elementary teachers. However, few researchers have documented interventions that were successful in improving the number sense of prospective elementary teachers. The field is in need of theory concerning prospective elementary teachers' number sense development. I report findings from a study of number sense development in the setting of a mathematics content course for prospective elementary teachers. Drawing on analyses of collective activity and individuals' number sense development, I highlight findings that illuminate our understanding of the phenomenon of prospective elementary teachers' number sense development.

Keywords: prospective teachers, number sense, knowledge as resources, elementary mathematics

Researchers have documented deficiencies in prospective elementary teachers' mathematics content knowledge (e.g., Ball, 1990; Simon, 1993; Ma, 1999; Yang, 2007). The qualities of mathematical reasoning that mathematics teacher educators wish for elementary teachers are well captured by the construct of number sense (Yang, Reys, & Reys, 2009). However, few researchers have documented interventions that were successful in improving the number sense of prospective elementary teachers (Authors, 2007). The field is in need of illuminating findings concerning prospective elementary teachers' number sense development. I report findings from a study of number sense development in the setting of a mathematics content course for prospective elementary teachers. Drawing on analyses of collective activity and individuals' number sense development, I highlight results that illuminate our understanding of the phenomenon of prospective elementary teachers' number sense development. In particular, I identify ways in which prospective elementary teachers' prior knowledge can play a productive role in their learning. I focus especially on the roles of procedural knowledge and commonplace ideas in prospective elementary teachers' number sense development.

Theoretical Perspective

The emergent perspective (Cobb & Yackel, 1996) informs this study and the larger research program to which it belongs. Learning is viewed through both psychological and social lenses, and these are seen as reflexively related. The study reported here is most directly influenced by a knowledge-as-resources view (Hammer, 1996; Smith, diSessa, & Roschelle, 1993). I value students' prior conceptions and attempt to account for the role of these in the learning process.

Background

Prospective elementary teachers come to teacher education with an understanding of mathematics that is largely procedural (Ball, 1990) and has been characterized as "disconnected" (Simon, 1993). A popular goal in mathematics teacher education is to enhance prospective elementary teachers' conceptual understanding of elementary mathematics (e.g., Hill & Ball, 2004). For example, prospective elementary teachers typically know the standard written algorithms that they were taught in school, but they do not understand why these algorithms work (Simon, 1993; Thanheiser, 2009, 2010; Zazkis & Campbell, 1996).

An influential perspective in mathematics and science education is the view of students' prior knowledge as a resource in their learning (Hammer, 1996; Smith, diSessa, & Roschelle, 1993). In the literature concerning prospective elementary teachers mathematics content knowledge, however, there is little evidence of the influence of such a perspective. Perhaps it is because of the nature of the student population that this so. Prospective elementary teachers, being college students, have a great deal of experience with elementary mathematics. In mathematics content courses, they are expected to revisit elementary mathematics and to come to understand it in new ways (e.g., Sowder, Sowder, & Nickerson, 2010). We know that prospective elementary teachers come to these courses familiar with the standard algorithms of elementary mathematics, as well as with a variety of commonplace mathematical ideas (e.g., Simon, 1993; Tirosh & Graeber, 1989). However, this knowledge has not typically been written about as a resource for their learning. On the contrary, prospective elementary teachers' prior knowledge seems not be regarded as valuable, much less as a productive resource (e.g., Graeber, Tirosh, & Glover, 1989).

This paper presents findings from a study of number sense development in the setting of a mathematics content course for prospective elementary teachers. Several analyses were conducted, and a comprehensive presentation of results is beyond the scope of this paper. Instead, I focus on particular findings that have led me to think differently about the role of prospective elementary teachers' prior knowledge in their number sense development. Results from analyses of collective activity and from individual case studies are presented in order to offer examples of ways in which prospective elementary teachers' prior knowledge served as a productive resource in their development of number sense.

Methods

The setting for this study was an elementary mathematics content course taught at a large, urban university in the southwestern United States. I conducted research in this course in an earlier semester, and previous analyses indicated improvement in the prospective elementary teachers' number sense (Author, 2007). In the more recent study, I investigated the processes by which prospective elementary teachers' number sense development occurs. Data collection took place during Fall Semester 2010. There were 39 students enrolled in the course, and 38 of them were female. The majority of the students were freshmen Liberal Studies majors. The instructor of the course was a mathematics educator and experienced teacher of mathematics courses for prospective teachers.

I report here on selected findings from two different sets of analyses. Prospective elementary teachers' number sense development was investigated on both the collective and individual levels. Collective activity was analyzed in terms of as-if shared ideas and classroom mathematical practices, using the methodology of Rasmussen and Stephan (2008). Individual learning was analyzed in the form of qualitative case studies (Merriam, 1998; Yin, 1994). The present report draws on the results of both analyses. I identified ways in which prospective elementary teachers' prior knowledge served as a resource in their learning. With respect to both data sets, the notion of knowledge serving as a resource was operationalized in terms of argumentation. Ideas were considered to serve as resources if they were used in students' arguments to justify new, valid mathematical conclusions.

Results

The analyses focused on two distinct content strands: (1) whole-number place value, addition, and subtraction; and (2) whole-number multiplication. The major finding was that prospective elementary teachers' procedural knowledge of the standard algorithms and their knowledge of commonplace mathematical ideas served as resources in their number sense development. I give examples of these from both content strands.

Knowledge as Resources in Place Value, Addition, and Subtraction

Early in the semester, the standard algorithms functioned authoritatively in class discussions. Applications of standard algorithms went unquestioned, whereas the use of nonstandard addition and subtraction strategies required justification. The authoritative function of the standard algorithms was leveraged productively in that it motivated the need to justify nonstandard strategies and, thus, led to argumentation in which mathematical ideas were established that were important to the learning route. Students' knowledge of the standard addition and subtraction algorithms served as a resource in that it afforded the transition to the Separation strategies, Right to Left and Left to Right (Heirdsfield & Cooper, 2004), e.g., computing 88 + 47 as 8 + 7 = 15, 80 + 40 = 120, 15 + 120 = 135.

A commonplace mathematical idea, reasoning about subtraction as a take-away process, came to function as if shared. This idea was used rather routinely in discussions of the standard subtraction algorithm. However, it was later used productively to justify nonstandard subtraction strategies. Reasoning about subtraction as a take-away process served as backing for the idea that subtraction functioned cumulatively, and the latter idea was used to justify both Aggregation and Compensation strategies (Heirdsfield & Cooper, 2004).

Knowledge as Resources in Multiplication

The class progressed from assuming the authority of the standard algorithms (including the multiplication algorithm) to reasoning flexibly about computing products. To do this, students had to overcome difficulties in accounting for partial products when it came to multiplying double-digit numbers. The commonplace idea of reasoning about multiplication in terms of repeated addition served as a resource in this process as students used reasoning about "groups of" to determine whether different ways of decomposing products into partial products were valid or invalid.

Valerie, who was the subject of a case study, initially relied on an invalid multiplication strategy. For example, she (along with many others in the class) computed the product of 23 and 23 mentally as 20 x 20 + 3 x 3. Valerie was eventually able to correctly account for partial products in double-digit multiplication. She did this by making use of tools that helped her to organize her thinking. On one occasion, she used her procedural knowledge of the standard multiplication algorithm to correctly form and account for all four partial products. On another occasion, a rectangular area drawing served a similar purpose. The latter tool was introduced in class and was intended to serve as a model for reasoning about products. The standard algorithm, on the other hand, had not been anticipated as a tool that would serve such a purpose.

Discussion & Implications

Researchers have documented deficiencies in prospective elementary teachers' mathematics content knowledge. In particular, this population has been characterized as having poor number sense. Less is known about how to support prospective elementary teachers' number sense development. This report derives from an effort to identify prospective elementary teachers' mathematical knowledge that serves as a resource in their learning. The noteworthy finding reported here is that prospective elementary teachers' procedural knowledge of elementary mathematics and their knowledge of commonplace mathematical ideas served as an important resource in their number sense development.

Many in the mathematics education community agree that it is important for teachers to take into account students' prior knowledge. The view of students' prior knowledge as a resource in their learning is consistent with a constructivist perspective. Such a view appears explicitly or implicitly to influence a great deal of current mathematics education research. However, when it comes to prospective elementary teachers, little attention has been given to their prior knowledge from a knowledge-as-resources perspective. We know that prospective elementary teachers come to teacher education familiar with standard algorithms and commonplace ideas. It is typically a goal of mathematics teacher educators to facilitate prospective elementary teachers' development of conceptual understanding of mathematics. This being the case, it is easy not to regard prospective elementary teachers' prior knowledge as valuable.

There is a dichotomy between procedural and conceptual. To promote the development of conceptual knowledge, we may try to steer prospective elementary teachers away from using their knowledge of procedures. However, in so doing, we may neglect to take seriously the role of their prior knowledge in their learning. Similarly, it is easy to undervalue commonplace ideas, such as reasoning about subtraction as a take-away process. The take-away meaning is familiar to prospective elementary teachers, and mathematics teacher educators may be more interested in encouraging them to reason about subtraction in other ways, such as in terms of additive comparisons. However, commonplace ideas can be used productively, and supporting prospective elementary teachers' learning requires us to recognize opportunities for their knowledge of commonplace ideas to serve as a resource in their learning.

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STUDENTS' AXIOMATIZING IN A CLASSROOM SETTING

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The purpose of this paper is to examine descriptive axiomatizing as a classroom mathematical activity. More specifically, if given the opportunity, how do students select axioms and how might their intellectual needs influence these decisions? These two case studies of axiomatizing address these questions and elaborate on how students engage in this practice within a classroom setting. The results of this research suggest that while students may at first be resistant to axiomatizing, this mathematical activity also affords them opportunities to create meaning for new mathematical content and for the axiomatic method itself.

Key words: Axiomatizing**,** Intellectual Need, Defining, Realistic Mathematics Education

Introduction

In discussing the genesis of the word, mathematizing, Freudenthal (1990) states that "Mathematizing as a term was very likely preceded and suggested by terms such as axiomatizing, formalizing, schematizing, among which axiomatizing may have been the very first to occur in mathematical contexts" (p. 30). Indeed, axiomatizing is essential to the practicing mathematician, as it serves to create and reorganize knowledge into a fundamental starting point for deductive work in both education and research, but what does it mean for mathematics students to axiomatize?

Theoretical Perspective

While Freudenthal (1973) acknowledges that axiomatizing can be traced back to the classical age of Greece, he cautions that the current usage of axioms is quite different from the function they served in Antiquity: "Axiomatics, as we now use this term, is a modern idea, and ascribing it to the ancient Greeks is, in spite of precursors, an anachronism" (p. 30). Accordingly, axiomatizing is often characterized along ontological lines with "Greek" axiomatizing relying upon selecting self-evident or observable truths, while "modern" axiomatizing is born out of logical convenience and therefore, the axioms one chooses may or may not be self-evident.

In his paper synthesizing much of the previous literature on axiomatizing, De Villiers (1986), identified two fundamentally different types—constructive and descriptive axiomatizing. *Constructive* (a priori) *axiomatizing* occurs when an existing set of axioms is altered through the omission, generalization, substitution or the addition of axioms resulting in content that can then be organized into a new logical structure. Constructive axiomatizing can be illustrated historically by the systematic discoveries and subsequent inventions of non-Euclidean geometries, and its primary function is the creation of new knowledge. On the other hand, "*Descriptive* (a posterior) *axiomatizing* is meant the **selection** of an axiom set from an already existing set of statements" (De Villiers, p. 6)". As a component of systematization, descriptive axiomatizing generally functions to reorganize existing knowledge into a starting point for deductive activity. The selection of axioms may originate with self-evident truths, as in the case of Euclid's geometry, but this is not a requirement of descriptive axiomatizing.

As noted by Freudenthal (1973), many mathematical systems may be too large or too complex for undergraduate students to axiomatize, so a structure such as a group might be more manageable for them. Hence, axiomatizing as a student activity might be readily observable if students are given the opportunity to define the group concept. More specifically, if students are encouraged to select and refine the group axioms, as opposed to being presented with them in some pre-structured form, axiomatizing can then be viewed as a mathematical process that supports their definition of group. This view is consistent with Zandieh and Rasmussen (2010), who stress the importance of including other mathematical activities as part of the act of defining as students engage in "formulating, negotiating, and revising a mathematical definition" (p. 59).

Harel's (1998) Necessity Principle states, "Students are most likely to learn when they see a need for what we intend to teach them, where by 'need', is meant intellectual need, as opposed to social or economic need" (p. 501). The Necessity Principle puts forth a conjecture about how students learn (Speer, Smith & Horvath, 2010) and has been used extensively by Harel as a component of a larger conceptual framework called DNR (Duality, Necessity, and Repeated-Reasoning) (Harel, 2001). As students move "from more computational mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus" (Selden & Selden, 1995, p. 135), one should not be surprised that there is an increased emphasis placed upon structure and axiomatic reasoning. Therefore, Harel's (2011) *intellectual need for computation* (to quantify and calculate) and *the need for structure* (to re-organize knowledge into a logical system) serve as useful constructs for investigating student axiomatizing.

Background

The setting for examining student axiomatizing consisted of two different mathematics classes that used the same inquiry-oriented curriculum for reinventing the concept of group (Larsen, 2009). An overview of the group theory unit in the TAAFU (Teaching Abstract Algebra for Understanding) curriculum is provided here:

> The reinvention of the group concept begins with an investigation of the symmetries of an equilateral triangle (Larsen, 2009; Larsen & Zandieh, 2007). Students develop symbols for the six symmetries and develop a calculus for computing combinations of symmetries. The rules the students use to compute combinations include the group axioms along with relations specific to this group. Based primarily on their work within the symmetry context, the students construct a formal definition of a group (Johnson & Larsen, 2012, p. 3).

Two community college mathematics instructors (Mr. A and Mr. B) used these group theory materials in mathematics bridge courses that were primarily designed to expose students to university mathematics in an active and supportive environment. The elective nature of these courses not only gave the instructors ample time and flexibility to engage in the TAAFU tasks, but to also pursue discussions and activities that originated from the students.

The video data for my research consisted of both whole-class and small-group episodes, taken from the two bridge classes. The primary domain of inquiry occurred during the implementation of a subset of activities in the TAAFU group theory curriculum that are described here. Following their symbolizing activity that consisted of negotiating a class-wide convention for representing the symmetries of an equilateral triangle, the students were then asked to compute all of the combinations of any two symmetries. During this process, they were encouraged to keep track of any "rules" or "shortcuts" they may have used in these computations. After sharing some of these rules in small groups, a class list was then recorded by the teachers, with little regard for economy. For example, the property that three consecutive 120-degree rotations or two consecutive flips essentially did nothing might be represented by the rules $R^3 = I$ or $F^2 = I$ respectively. After a comprehensive list of rules was created, the teachers then asked if all of the rules were necessary or could some of them be deduced from other rules, thus making them redundant. It was during this stage that students put forth different arguments about which rules to keep out of necessity and in the case of equivalent rules, which version to keep, as they tried to create a minimal list. Once the minimal list was agreed upon as a class, the students were then asked to recompute their combinations using only the minimal list of rules and then each student was assigned a particular calculation to prove, again using only the minimized list. Following this exercise, axiomatizing continued to occur sporadically as students progressed toward the formal definition of group, through the refinement of this set of class rules.

Methods

Noting the similarities to De Villier's (1986) notion of descriptive axiomatizing, I conducted a retrospective analysis (Cobb & Whitenack, 1996; Stylianides, 2005) of the students' activity in Mr. A's community college bridge course using "axiomatizing" as the lens of examination. At the conclusion of the first pass through the data, I found the students' activity to be consistent with de Villiers's (1986) notion of descriptive (a posterior) axiomatizing in the Greek sense. In this case, the act of selecting rules was at first done individually, then discussed in small groups, and later collectively negotiated to comprise a more economical list that was ultimately used as a launching point for deductive work. To further analyze the students' progression from informal to formal activity, I made a second pass through this data using Zandieh and Rasmussen's (2010) DMA (Defining as a Mathematical Activity) Framework. This construct provides a lens for examining different levels of activity in the process of defining, namely: situational, referential, general, and formal (Gravemeijer, 1999). During this stage of concept development, I concluded that the students were primarily engaged in referential activity that transitioned to general activity, as they organized a system of relations for the symmetries of an equilateral triangle that later extended to symmetries of other *n*-gons and then eventually to relations for arbitrary sets.

Following my two-stage preliminary analysis, I concluded that de Villier's construct of descriptive defining and Zandieh and Rasmussen's DMA framework globally *described* what the students were doing in the TAAFU classroom episodes, but there were still elements of the axiomatizing sequence that warranted further analysis. For instance, I was curious why some of the students in the bridge course seemed resistant to axiomatizing and why they chose certain rules instead of others. Therefore, I extended my data set to include Mr. B's class and conducted a cross-case analysis of axiomatizing that consisted of two phases: an explanatory pattern-matching analysis followed by a cross-case synthesis (Yin, 2009).

The explanatory pattern-matching analysis served as a starting point for identifying the various phases of the axiomatizing sequence and culminated with explanatory descriptions of the students' activity in both cases. Although other mathematical activities were also occurring during this timeframe, I identified particular milestones (both in terms of class time and in real time) for comparative axiomatic development. As both classes were taught using the same curriculum, patterns such as negotiating notational conventions for the rules and selecting which rules to keep in the minimal set were similar and occurred in the same chronological order. However, given that each list of rules was student generated, the order of the rules and whether certain rules were implicit or explicit constituted notable differences. The cross-case synthesis focused on examining students' intellectual needs as they pertained to axiomatizing. Therefore, I made another pass through the data looking for direct evidence that might explain why students selected certain rules to keep as part of the minimal set and why others were let go.

Results

Following the analysis of the data, several themes emerged, but only two are reported here. First, there was evidence in both case studies of students' initial resistance to axiomatizing. Consistent with Larsen's (2004; 2009), teaching experiments, some of TAAFU students did not see a need for axiomatizing the associative property. Larsen (2009) explained this resistance by noting that in the initial stages of the curriculum, it was not uncommon that students would view algebraic expressions as a sequence of actions, rather than as a binary operation acting on two elements. Although the closure property was identified very early in the curriculum, both groups of students also struggled with axiomatizing a rule for it as well. For instance, when the students used the technique of "multiplying both sides by symmetry" in order to solve a linear equation, both teachers asked which axiom justified this technique. One of Mr. B's students responded, "Isn't that just the multiplicative property of equality?" Even when Mr. B reminded them that the operation was not multiplication, some students suggested that it was obvious from their operation table or that this rule could be deduced using their existing set of axioms.

I also found that some students were resistant to axiomatizing when presented with the minimization task. For the majority of these students, economy was inhibited by what I inferred as an intellectual need for computation. For example, even after it was shown that one rule could be derived from the other, some of the students insisted on keeping two versions of the dihedral relation (i.e. $RF = FR^2$ and $RFR = F$) in their minimal list because it made certain calculations "faster". For a few students in the transition courses, there was even a global resistance to economy as illustrated in the following excerpt from a student in Mr. A's class:

Chris: Ok, so um… I just ran into this whole logical error for this whole situation…imagine you are out shopping for a calculator and you get to the store and there's one calculator that costs like five bucks. It can add, it can subtract, it can multiply, it can divide and that's all. Then there's a graphing calculator for like sixty dollars that can do all this stuff really easily. You're gonna go, I mean if a you are in an advanced math class, you are going to go for the graphing calculator, even though it costs more—even though it is a greater initial expense, uh, because it has all of these pre-programmed into it. Basically what we are doing at this point, um and it just seems like a logical fallacy, is we're reducing the capabilities of our calculator to as few as possible, which just doesn't seem all that efficient.

While there was some initial resistance to axiomatizing, this reluctance seemed to diminish as the students progressed from referential to more general activity. For instance, once Chris had established a need for minimizing the list, he suggested keeping $RFR = F$ as the dihedral rule as opposed to $RF{=}FR^2$, because it applied not only to the triangle, but also to any regular polygon. Students also suggested replacing *F*'s and *R*'s with other letters such as *A*, *B*, and *C*, so they could refer to symmetries of other figures in addition to the symmetries of an equilateral triangle.

A second theme that emerged from the data suggested that descriptive axiomatizing not only provided the students opportunities to structure existing content, but it also fostered a rich context for discussing new student-initiated mathematical ideas. For example, in Mr. A's class, a student put forth a rule that was equivalent to $R^{3n} = F^{2n} = I$, which naturally led into a conversation about modular arithmetic. While this rule was not accepted as a class rule because it was not immediately useful in the task of combining any two symmetries, the student who suggested it noted its future computational value and offered ways to modify the rule so that it would apply to other regular *n*-gons. In Mr. B's class, students chose to axiomatize the law of exponents, which precipitated a discussion about rules that were essential for describing their symmetry relations and those that were either consequences of their notational choice (as in the case of the exponent rule) or more general relations, such as the transitive property of equality.

Structural recommendations also emerged as the students revised their list of rules. For example, one student suggested reordering the list of rules, so that generic symmetries and the identity symmetry were defined into existence before they were used in a subsequent rule. The act of axiomatizing itself was even considered in both of the bridge courses when some students commented on their desire to keep a rule, but somehow relegate it as less important than the others. When a student in Mr. B.'s class asked if this distinction was the difference between a theorem and an axiom, it resulted into a lively whole-class discussion that introduced these terms as well as lemma and corollary to many students who were unfamiliar with them.

Discussion

In proof-based courses, much attention is paid to the deductive stage of the axiomatic method and yet, a common theme that is recurrent in the literature is the disconnection between how students view proof when compared to the mathematical community (Tall, 1989; De Villers, 1990; Almeida, 2000). In light of this study, an analogous conjecture might be made regarding students' views of axioms, which are utilized both implicitly and explicitly in proofs. For instance, the students in this study not only initially struggled with creating rules, but also in knowing when they were using (or not using) a rule in a calculation or a proof. These students also expressed difficulty with discerning objects that are typically viewed by the mathematics community as axioms from properties of an equivalence relation or a relation that was consequence of their notation choice, such as the law of exponents.

On the other hand, this study also suggests that if students are asked to create and refine rules, which describe relations that originate from their own mathematical activity, they may have the opportunity not only to formalize those relations, but also to consider axiomatic qualities as they reorganize their existing knowledge into a logical structure. While such activities may not be considered axiomatizing in the "modern" mathematical sense, the students in this study engaged in rule making that was consistent with "Greek" descriptive axiomatizing. In contrast to being given a pre-structured form of Euclid's *Elements*, which traditionally serves as a starting point for deductive work in Euclidean geometry, these students had opportunities to realize an intellectual need for axiomatizing a mathematical system prior to utilizing those axioms in deductive work.

Conclusion

These case studies suggest that although it may be met with initial resistance, descriptive axiomatizing can be a fruitful activity for students. In a traditional mathematics classroom, students are rarely given opportunities to axiomatize. However, as mathematics students progress through an undergraduate program, the increasing importance of axioms and reasoning from them cannot be ignored. Therefore, if we expect students to understand and reason from axioms, it might be worthwhile to foster an intellectual need for them by giving them opportunities to create and refine axioms.

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Students Reconciling Notions of One-to-One Across Two Contexts

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As part of a larger study of student understanding of concepts in linear algebra, we interviewed 10 university linear algebra students as to their conceptions of functions from high school algebra and linear transformation from their study of linear algebra. Analysis of these data led to a classification of student responses into properties, computations and a series of five interrelated clusters of metaphorical expressions. In this paper, we use this classification to analyze students' written and verbal responses to questions regarding one-to-one in the context of function from high school algebra and in the context of linear transformation from their study of linear algebra. We found that students' ability to construe sameness across the two contexts is related to their reliance on properties versus metaphors. We conjecture that this phenomenon is likely to occur for other mathematical constructs as well.

Keywords: Concept image, function, linear algebra, linear transformation, metaphor

The research reported in this paper began as part of a larger study into the teaching and learning of linear algebra. As we examined student understanding of linear transformations we wondered how student understanding of functions from their study of precalculus and calculus might influence their understanding of linear transformations and vice versa. In previous work we created a framework for analyzing student understanding that incorporates five clusters of metaphorical expressions as well as properties and computations that students spoke about when discussing function or linear transformation. In this paper we apply this framework to the setting of students reconciling their understandings of one-to-one in the context of function with their understandings of one-to-one in the context of linear algebra. Ideally we would like students to be able to recognize a similar structure for one-to-one in each context, and thereby to strengthen their overall understanding of the notion of one-to-one. This proposal provides four vignettes that we found illustrative of the way students reasoned about one-to-one within and across the two contexts. More broadly we find the case of one-to-one as prototypical of the struggles students have in reasoning seeing similarities across contexts.

Literature and theoretical background

The nature of students' conceptions of function has a long history in the mathematics education research literature. This work includes Monk's (1992) pointwise versus across-time distinction, the APOS (action, process, object, scheme) view of function (e.g., Breidenbach, Dubinsky, Hawkes, & Nichols,1992; Dubinsky & McDonald, 2001), and Sfard's (1991, 1992) structural and operational conceptions of function. A comparison of these views may be found within Zandieh (2000). More recent work has focused on descriptions of function as covariational reasoning (e.g., Thompson, 1995; Carlson, Jacobs, Coe, Larsen & Hsu, 2002). A recent summary with a focus towards covariational reasoning is found in Oehrtman, Carlson, and Thompson (2008).

The work in linear algebra has tended to focus more on student difficulties (e.g., Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000; Harel, 1989; Hillel, 2000; Sierpinska, 2000). There have been a few studies on student understanding of linear transformation (Dreyfus, Hillel, & Sierpinska, 1998; Portnoy, Grundmeier, & Graham, 2006). However, we could not find studies that relate student understanding of function and linear transformation. In addition we did not yet find a study focused on the notion of one-to-one in either context.

In addition to work specifically on student conceptions of functions or linear transformation, we draw on the notion of concept image and the work done showing that students often have many aspects of their concept image that are not immediately compatible with their stated concept definition (Vinner & Dreyfus, 1989; Tall and Vinner, 1981). Our work is related but focuses more on how a student's concept image of a mathematical construct (in our case one-to-one) may be more or less compatible with their concept image of the same construct in another setting.

In addition to work that uses concept image as its framing, we find useful studies that (whether they refer to it by the term concept image or not) detail student concept images of mathematical constructs using the construct of a *conceptual metaphor* (e.g., Lakoff & Núñez, 2000; Oehrtman, 2009; Zandieh & Knapp, 2006). This follows from the earlier work in cognitive linguistics of Max Black (1977), Lakoff and Johnson (1980) and Lakoff (1987). Following from this work, our assessment is that a person's concept image of a particular mathematical idea will likely contain a number of metaphors as well as other structures.

We rely on metaphorical expressions to indicate when a conceptual metaphor is being employed. Lakoff and Johnson (1980) explain that, "Since metaphorical expressions in our language are tied to metaphorical concepts … we can use metaphorical linguistic expressions to study the nature of metaphorical concepts and to gain an understanding of the metaphorical nature of our activities (p. 456)." A metaphorical expression is an expression that uses metaphorical language, such as describing love as a journey: "they are in the fast lane to marriage"; "our relationship has come to a cross roads". Our framework includes clusters of metaphorical expressions that allow us to highlight the connections or discrepancies between student conceptions in the context of function and the context of linear transformations.

Methods

The data for this report comes from semi-structured interviews with 10 students who were just completing an undergraduate linear algebra course. The interviews were videotaped and transcribed and student written work was collected. In addition to the main interview questions like those listed below, students were often asked follow up questions to gain more insight into their thinking. The focus of the interview was to obtain information about students' concept image of function and their concept image of linear transformation and to see in what ways students saw these as the same or different. To this end we not only asked the students how they thought of a function or linear transformation, but also questions about characteristics that would be relevant to both functions and linear transformations such as one-to-one, onto, and invertibility. For the purposes of this paper, we draw on student responses to questions regarding the concept of one-to-one:

1. In the context of high school algebra, give an example of a function that is 1-1 and one that is not 1-1. Explain.
2. In the context of linear algebra, give an example of a linear transformation that is 1-1 and one that is not 1-1. Explain.
3. Please indicate, on a scale from 1-5, to what extent you agree with the following statement: "1-1 means the same thing in the context of functions and the context of linear transformations."

In order to identify the various ways students reasoned about function, linear transformation, and one-to-one in both contexts, we applied the theoretical framework developed by Zandieh, Ellis, and Rasmussen (2012). This framework details three main components of students' concept images of function and linear transformation: by drawing on *properties,* by drawing on *computations,* or by drawing on *metaphors*. The property ($P_{\text{property used}}$) category refers to student statements that do not delve into the inner workings of the function or transformation. Students also frequently drew upon computational language while reasoning through the interview tasks. We differentiated between computational language that described how a function or linear transformation behaves (labeled as C1) and side computations done involving the function or transformation (labeled as C2).

The metaphorical component consists of five related metaphors that share the common structure of a beginning entity, an ending entity, and a description about how these two are connected, as shown in Table 1. The first metaphor is the *input/ output* metaphor (IO), and involves an input, which goes into something, and an output, which comes out. The second metaphor is *traveling* (Tr), and involves a beginning location being sent or moving to an ending location. The third metaphor is *morphing* (Morph) and involves a beginning state of an entity that changes or is morphed into an ending state of the same entity. The fourth metaphor, *mapping* (Map), most closely resembles the formal Dirichlet-Bourbaki definition of function, and involves a beginning entity, an ending entity, and a relationship or correspondence between the two. The fifth and final metaphor is the *machine* metaphor (Mach), and includes a beginning entity or state, an ending entity or state, and a reference to a tool, machine or device *that causes* the entity to change from the beginning entity/state into the ending entity/state. Because these five metaphors share a common three-part structure, students often layer metaphors on top of one another. In the body of this paper, we will provide multiple examples of this layering of the metaphor clusters.

Table 1: Structure of the metaphor clusters.

Cluster	**Entity 1**	**Middle**	**Entity 2**
Input/Output (IO)	Input(s)	Entity 1 goes/is put into something and Entity 2 comes/is gotten out.	Output(s)
Traveling (Tr)	Beginning Location(s)	Entity 1 is in a location and moves into a (new) location where it is called Entity 2.	Ending Location(s)
Morphing (Mor)	Beginning State of the Entity(ies)	Entity 1 changes into Entity 2.	Ending State of the Entity(ies)
Mapping (Map)	First Entity	Entity 1 and Entity 2 are connected or described as being connected by a mapping (a description of which First entities are connected to which Second entities).	Second Entity

Machine (Mach)	Entity(ies) to be processed	Machine, tool, device acts on Entity 1 to get Entity 2.	Entity after being processed

In this study we are interested in to what degree students are able to reconcile their understandings of one-to-one in the contexts of function and linear transformation. In question 3 above we asked students directly to what extent they saw their understandings as compatible, and through follow up questions we asked them to show us how their understandings were compatible. The ten student interviews included students who clearly showed how the notion of one-to-one is compatible across the two contexts, and students whose notions of one-to-one were not as compatible across contexts. Below we present four vignettes that illustrate the primary variations in student thinking within our group of ten students. Table 2 provides summary information for each of the 10 students including the metaphors and properties that they referred to when answering the interview questions about one-to-one in the context of function (Column 2) and linear transformations (Column 3), their answer to question 3 (Column 4), and whether or not the student reconciled one-to-one across the two contexts (Column 5).

Table 2. Students' expressed understandings of one-to-one in the contexts of function and linear transformation.

Name	Function	Linear Transformation	Question 3 (scale 1-5)	Reconciled
Donna	IO P_{hlt}, $P_{shape\ of\ graph}$, P_{map}, P_{vlt}, $P_{no\ holes}$	IO $P_{different\ directions}$, $P_{infinite\ solutions}$, P_{map}, P_{line}, $P_{infinite\ inputs}$, P_{ld}, P_{li}	3	No
Nila	Map $P_{monotonic}$ $P_{shape\ of\ graph}$ P_{vlt}	IO Map Tr P_{ld}, P_{line}, P_{li}, P_{map}	1	No
Jerry	IO Map $P_{shape\ of\ graph}$	P_{ld} P_{li}	3	No
Josh	P_{hlt} $P_{shape\ of\ graph}$	P_{li}, P_{ld}, $P_{invertible}$, P_{line}, P_{diag}, P_{span}	3	No
Adam	Map $P_{shape\ of\ graph}$ P_{hlt}	IO Morph Tr C_1 P_{ld}, P_{line}, P_{li}, P_{span}	5	No
Gabe	IO Map	Map Tr P_{square}, P_{ld}, P_{li}, $P_{invertible}$	5	No
Lawson	IO Map $P_{shape\ of\ graph}$	Map Morph $P_{infinite\ solutions}$ P_{ld} P_{map}	5	No
Nigel	Map P_{hlt} $P_{shape\ of\ graph}$	Map Morph Tr	4	Yes
Randall	IO Map	IO Map	5	Yes
Brad	IO Map	IO Map Tr	5	Yes

Note. Students whose names are in italics represent their category in the vignettes below.

Vignettes

We found that students in our study fell into four categories with regard to their belief about and ability to reconcile the similarity of one-to-one across the two contexts. The first group gave low (1 or 3) scores to question 3 and did not reconcile; the second group gave scores of 5 to question 3 and did not reconcile; the third group gave a score of 4 and did reconcile; and the last group gave a score of 5 and reconciled. In the following section we provide a vignette from one student for each of these four categories. The first two vignettes show different ways that students struggled to show that their understandings of one-to-one were compatible across the two contexts, and did not reconcile their understandings. The last two vignettes show different ways that students were able to reconcile their understandings of one-to-one.

Vignette 1. The first vignette tells the story of a student whose understandings of one-to-one relied heavily on properties, which prevented her from seeing her descriptions of one-to-one as compatible. Donna's initial description of one-to-one in the context of function mentioned a mapping understanding, but she relied on the horizontal line test to determine when a function is or is not one-to-one. In the context of linear transformation she did not mention a mapping understanding and instead relied exclusively on linear (in)dependence. When asked if she was thinking of one-to-one the same way in both descriptions, Donna replied that she had not:

Donna: No, I don't think so, because I was thinking in terms of just simple, what I learned in high school, how the one-to-one function is something that has exactly one input for one output. And then in linear algebra, I was thinking in terms of linear dependency and independency and what we had learned prior in the class.

Donna's reliance on properties about one-to-one functions and one-to-one linear transformations prevented her from reconciling these notions.

Vignette 2. The second vignette tells the story of a student who strongly believed that one-to-one is that same in both contexts, but through his descriptions of one-to-one in each context did not completely recognize the underlying structure of one-to-one and eventually became uncertain about the strong similarity of one-to-one in the two contexts. Adam's initial description of one-to-one used properties and mapping language for function, and properties, computational, and input/output language for linear transformation. Specifically, he referenced the shape of a parabola for the reason x^2 is not one-to-one, and the linear independence of the column vectors of the 2 by 2 identity matrix as the reason why it is one-to-one. He strongly agreed that one-to-one meant the same thing in both contexts, and explained that he strongly agreed because "it just feels like the same thing, like if you put 1 in, it only comes out as only 1 possibility."

The interviewer then pointed out that the way he described one-to-one in both contexts "looked kind of different" and asked him to elaborate how he thought about them as the same. Adam then described how he saw linear (in)dependence related to one-to-one, drawing on input/output and traveling language. The interviewer asked him to elaborate examples of function and transformation that are not one-to-one, which he did by referencing properties for both, the mapping metaphor for functions, and computational language for linear transformation. When the interviewer pushed Adam to highlight how these two explanations were compatible, he realized that they were not as compatible as he had originally thought, concluding that his understandings of one-to-one the two contexts are "a little different."

Vignette 3. Nigel is a case of a student who did not initially see a connection between his ways of thinking about one-to-one in terms of function and linear transformation, but then realized a strong connection as he discussed his ideas with the interviewer. When asked question three, Nigel circled 4 and said, "I mean, I would agree, but at the same time, I don't really see a solid connection that I can explain myself."

His initial explanation of one-to-one for function used a property, but morphing ("transformed") and travelling language ("to that spot") for linear transformation:

> Nigel: I've learned of one-to-one means the horizontal line test. … But for linear transformations, I see it as, here's this vector, if it gets transformed by a one-to-one transformation, it's going to get plotted to its own specific new vector, and no other vector will be transformed to that spot.

When asked to reconcile his understandings in the two context, the following exchange occurred.

> Interviewer: So those 2 things sound pretty different on the surface at least, can you say more about the way in which it might be similar for the 2?
>
> Nigel: one-to-one is just for every y value, there's a unique x. … Maybe there's multiple y values for this x, ... so y = sin(x), so it looks something like that. For this, if you have a horizontal line, you just go down, you'll see across here there's … the same y values but for different x values. So, that's not one-to-one.

Nigel used mapping language to describe how the horizontal line test works. His language was more compatible with his description of one-to-one in the context of function, but he used different metaphors in the two contexts. In this interview Nigel never completely reconciled his notions in terms of using identical language but he did find some sense of understanding a compatibility that he had not recognized previously in that he said, "I never really explained it like that before!"

Vignette 4. The fourth vignette illustrates the case of students who can easily reconcile their understandings of one-to-one in the contexts of functions and linear transformations. Each of the two students in this category used similar language when describing examples in each context. For example, Brad discussed a one-to-one function by saying, "This is one output for every input." For a one-to-one transformation he said, "For every output there is one input to get there." In each context Brad used language from the mapping cluster, i.e., for every ___ there is one ___. He also uses input/output language in each context. A subtle difference was his inclusion of travelling language for linear transformation, with the phrase, "to get there." Both Brad and Randall were able to give examples of functions that are not one-to-one in each context and to show that the function or linear transformation is not one-to-one by finding two input values that map to the same output value.

Conclusion

The above four vignettes highlight various ways students understand one-to-one in the contexts of function and linear transformation, and how they explain the compatibilities of these understandings. We see that for the students who rely primarily on properties about one-to-one in the two contexts, such as a function is one-to-one if it passes the horizontal line test, it was difficult for them to identify the consistencies between one-to-one across the contexts. In contrast to this, students who drew on metaphorical language of one-to-one in the two contexts, such as mapping, morphing, or traveling language, the compatibility of the two understandings was more clear.

Each of the metaphorical clusters allow for the description of one-to-one in terms of the relationship between what we refer to in Table 1 as "Entity 1" and "Entity 2". The layering of the metaphorical clusters within one context and the compatibility of the metaphorical language across contexts allows for the recognition of similarities. In contrast, when a student thinks in terms of a property such as the horizontal line test or the linear independence of column vectors in a matrix, then the connections across contexts are more difficult to make. The horizontal line test is specific to the context of a function from the real numbers to the real numbers, and the column vector test is specific to a linear transformation that is defined in terms of matrix multiplication. Either of these properties could be unpacked in terms of one or more of the metaphor clusters (including in terms of the formal definition of one-to-one which fits in the mapping cluster). However, the tests in their most efficient, easy-to-apply format have been condensed or simplified in a way that hides the structural relationships that would allow students to compare across contexts.

This issue speaks to a broader goal of mathematics education: for students to be able to understand a construct, such as one-to-one, across a number of different contexts. One step to helping students develop these broader understandings is by identifying how they understand the construct within various contexts. Our theoretical framework provides such a tool. With this tool we were able to highlight similarities and differences across students understandings of one-to-one in each context. By making these comparisons, it became clear that a reliance on certain properties made developing a more context-free understanding of one-to-one difficult. This phenomenon is likely to occur for other mathematical constructs as well. We see this research as an illustrative example toward exploring this larger phenomenon.

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THE MERITS OF COLLABORATION BETWEEN MATHEMATICIANS AND MATHEMATICS EDUCATORS ON THE DESIGN AND IMPLEMENTATION OF AN UNDERGRADUATE COURSE ON MATHEMATICAL PROOF AND PROVING

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The goal of our study was to characterize the processes and to identify the ways in which different kinds of expertise (mathematics vs. mathematics education) unfolded in the planning and teaching of an undergraduate course on Mathematical Proof and Proving (MPP), which was co-taught by a professor of mathematics and a professor of mathematics education. The content of the course consisted of topics that were supposed to be familiar to the students, i.e., high school level algebra, geometry, and basic number theory. In particular, we looked at a case study describing the design and implementation of a particular task in order to help understand an instance in which each professor's expertise contributed to the course and complemented the other. The findings indicate that by co-teaching and constantly reflecting on their thinking and teaching, the instructors became aware of the added value of working together and the unique contribution each one had.

Key words: Mathematical Proof and Proving; Undergraduate Course; Problem-based Learning; Community of Practice

Setting the Stage: Context and goals

The focus of our study is on the design and implementation of a special undergraduate course on Mathematical Proof and Proving (MPP)[1]. Two common assumptions led to initiating this course: 1. Mathematical proof and proving are at the heart of mathematics; and 2. The notion of formal proof and the activity of mathematically proving are dauntingly difficult even for most good undergraduate students (Harel and Sowder, 2007).

The notion of proof is often incorporated into other mathematical courses and typically does not constitute the focal topic of one particular undergraduate course. There are many transitional courses in mathematics, most of which combine learning about proof with learning fundamental unfamiliar topics in mathematics. Consequently, the cognitive load on students is high and they encounter more difficulty than necessary since they need to deal with too many things at the same time: advanced mathematical ideas as well as proof and proving. The intention of the MPP course was to build on students' existing mathematical knowledge, and to draw on learning activities that involve familiar topics such as high school level algebra and geometry, and basic number theory (e.g., familiar properties of integers such as divisibility).

The challenge of attending to students' learning difficulties and at the same time maintaining an appropriate level of sound mathematics led to a collaboration between mathematicians and mathematics educators. Moreover, the MPP course was designed and co-taught by two instructors – a full professor of mathematics and a full professor of mathematics education. The initial goal of this collaboration was for the mathematics educator to bring her expertise on teaching mathematics (in general, and of proof and proving, in particular) and on students' difficulties in learning to prove, and for the mathematician to bring his expertise in the discipline of mathematics and the knowledge and understanding of MPP that students' need for successful participation in more advanced

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undergraduate mathematics content courses. This collaboration stemmed from a mutual respect for each other’s role and potential contribution, and the recognition that there is much to learn from each other. In reality, there were additional mathematicians and mathematics educators involved in various stages of this process.

While sharing the same concerns and long-term goals for the course, each instructor brought a different perspective on how students should be learning MPP and how to attend to their difficulties. From the outset it became clear that although the structure and syllabus of the course were pre-determined in full agreement between the two instructors, each instructor has his/her own views and interpretations, and that the joint efforts to produce an MPP course that would address the above concerns would require an ongoing professional dialogue and reflection. The big challenge was to bridge between the different perspectives and use these differences as a springboard to enhance the course.

The goal of our study was to characterize the processes and to identify the ways in which the different kinds of expertise (mathematics vs. mathematics education) unfolded in the actual planning and teaching of the MPP course; in particular, we looked for instances that would help understand how each expertise contributed to the course and complemented the other.

Conceptual Framework

Our study stems from two theoretical perspectives. One supported the design of the course. The other supported our approach to the design and study of the collaborative work between the two communities represented by the two instructors.

The following perspectives on learning and teaching guided the design of the MPP course: 1. Students' interactions and classroom discourse contribute to learning [to prove] (Yackel, Rasmussen, and King, 2000; Zaslavsky and Shir, 2005; Smith, Nicholas, Yoo, and Oehler, 2009); 2. Tasks play a significant role in learning (Henningsen and Stein, 1997); 3. Uncertainty promotes the need to prove (Dewey, 1933; Fischbein, 1987; Harel and Sowder, 2007, 2009; Zaslavsky, 2005; Zaslavsky et al, 2011); 4. Class discussions and activities should address students’ anticipated/manifested preconceptions and difficulties (Harel and Sowder, 2007, 2009; Weber, 2001; Reid, 2002; Buchbinder and Zaslavsky, 2009).

The decision to design and co-teach the course collaboratively, assigning two full professors as the MPP course instructors, is in a way a response to issues raised by Harel and Sowder (2009). Their study indicates that while mathematicians who teach undergraduate courses in mathematics have a broad and deep mathematical knowledge/understanding, many are not necessarily fully aware of students’ difficulties in learning to prove, or of effective ways to scaffold their learning. In our work, the team of mathematics educators and mathematicians is viewed as a community of practice. The team consisted of 3 mathematics educators – one full professor (Olga) and two doctoral students (Mark and Pola), and 3 mathematicians – two full professors (Jim and Frank) and one doctoral student (Sam). All the names in this paper are pseudonyms. Olga and Jim were the instructors of the course. Mark and Sam served as teaching assistants (TAs), Pola served as research assistant on the evaluation staff, and Frank was involved primarily in the planning sessions. The members varied with respect to their expertise and experience, as well as their roles, which is one of the characteristics that Roth (1998) considers essential to a community. Theories of communities of practices provide us with tools for analyzing the various kinds of learning of the members of the community as well as the contribution of each member to the shared goals of the community (Rogoff, 1990; Roth, 1998; Lave & Wenger, 1991). These theories consider knowledge as developing socially within communities of practice.

An integral characteristic of our community of practice is associated with the notion of reflective practice (Dewey, 1933; Schön, 1983). The notions of reflection on-action and reflection in-action have been recognized as effective components that can contribute to the growth of teachers' knowledge about their practice. In our study, reflection was a key issue for the development of the instructors' awareness and understandings related to teaching and learning to prove.

Data Sources and Analysis

The data for this study consisted of video-tapes and field notes of all the classes in the semester (13), audio-records and field notes of weekly meetings held a day after each class, and email conversations between the team members. In addition, students' written homework and TA's comments and grades were scanned and documented.

The methodology employed in the study followed a qualitative research paradigm in which the researcher is part of the community under investigation. It borrows from Strauss and Corbin's (1998) Grounded Theory, according to which the researcher's perspective crystallizes as the evidence, documents, and pieces of information accumulate in an inductive process from which a theory emerges. The researcher acts as a reflective practitioner (Schön, 1983) whose ongoing reflectiveness and interpretativeness are essential components (Erickson, 1986). In our case, the researchers were members of the community of practice that they investigated.

In the following section, we present an illustrative case that portrays the kinds of negotiations between and mutual contributions of the two instructors during a sequence in which they first discussed a particular task in a pre-class planning meeting, then implemented it in the classroom, and finally reflected on its implementation in a post-class meeting. The focus of this study is not on student learning, but rather on describing the mutual learning that occurred around this task by members the community. Through this case, we provide a glimpse at the ways in which two instructors reflect on their mutual understanding of how to teach proof, and work towards developing a shared understanding.

An Illustrative Case

The Cyclic Task

Based on the design principles of the MPP course, the following task was posed to students during the 5th week of the course as part of a sub-unit on direct proof. Olga started by choosing a 3-digit number: 814, which is divisible by 37. She then asked the students to check whether the numbers 148 and 481 (that are obtained by a cyclic change of 814) were also divisible by 37. Much to their surprise, they found that both 148 and 481 are also divisible by 37. Then she asked them to choose another 3-digit number that is divisible by 37 and to check whether any cyclic change of digits (i.e., the first permutation) is also divisible by 37. This way, students jointly tried out several cases that satisfied this property. At this point, Olga asked them: "*is this a coincidence?*" Although it seemed to work for the examples they chose, they were uncertain whether it would always work. Thus, this question led to the formulation of a conjecture and an attempt to prove or disprove it. The conjecture that was formulated was: "*If a 3-digit number is divisible by 37, then any 3-digit number that is obtained by a cyclic change of order of digits is also divisible by 37.*"

This task was selected for two main reasons. First, minimal knowledge about number theory and divisibility is required to form a conjecture, and both professors anticipated that the construction of a valid proof would be within students' abilities. Although the mathematical content of the cyclic task has its roots in elementary number theory, and can be explained by appealing to permutation theory and modular arithmetic, it is possible for students to construct a proof by appealing to the nature of the decimal system notation and the grouping of like-terms. For example, it is possible to show the implication of one cycle by expressing the number 'xyz' as $100x + 10y + z$, expressing 'yzx' in a similar form, and use substitution and grouping of like terms to show that if 'xyz' is divisible by 37, then 'yzx' can be represented as a sum of terms that are each divisible by 37. Second, the proof of the statement is not trivial. Tasks that evoke feelings of uncertainty in students have the potential to support meaningful learning situations (Zaslavsky, 2005; Zaslavsky et al, 2011), as well as to create a need for certainty that Harel & Sowder (2009) describe as one of the five elements that constitute an intellectual need particularly relevant to learning mathematical proof. Another affordance of this task is that it can be extended by asking students to reflect on their proof and determine if there are other numbers for which the cyclic pattern holds.

Pre-Class Planning of the Cyclic Task

During the pre-class instructors' meeting, Olga and Jim decided that they would give students substantial time to work on the cyclic task in class, corresponding to their goal of making the MPP course a problem-based course. Olga suggested that they give students one hour, as from her prior experiences giving the task to students she believed that discovering a pattern and then trying to prove it would not be easy. Jim, on the other hand, was hesitant about giving students this much time to work on their own: "*I am absolutely convinced the majority will not get it… it will be a very frustrating hour*." Olga convinced him that class time spent on the task would be interspersed with full class discussion and sharing, and it would not be the case that students would work on it for an hour in isolation.

Jim suggested that before proceeding to the cyclic task Olga review a problem from their previous homework on the "divisibility-by-3" rule. This homework was deliberately assigned to them a week earlier in order to prepare them for the cyclic task. They were asked to prove that: *If the sum of the digits of an integer n is divisible by 3, then n is divisible by 3*, for 4-digit integers.

Both Olga and Jim agreed that this was a good idea due to the conceptual similarity between this homework assignment and the cyclic task. They disagreed only in how to make the transition between the two. Jim suggested that after reviewing the "divisibility-by-3" rule "*we give this (cyclic) as an exercise and say can you use some of these ideas (from the homework) to show this*?" Olga argued against making this connection explicit. By the end of this pre-class planning meeting, each professor assumed that the other saw the merits of the task in a manner similar to their own.

The Unfolding the Cyclic Task in Class

As planned, during the first half hour of the lesson, Olga worked with students to construct a proof of the "divisibility-by-3" rule, as none of them had successively completed this homework assignment. Students learned how to represent a 3-digit number as a sum of powers of ten, and the affordances in the proof construction of regrouping this representation (i.e. $100x + 10y + z = (99x + x) + (9y + y) + z = (99x + 9y) + (x + y + z)$.

Olga introduced the cyclic task slowly, by having students verify that the claim was in fact accurate with two different examples. Students were instructed to work on the task in groups. During the group work, Jim sat with one group and listened to their discussion, while Olga walked around to check on different groups. After ten minutes, one group indicated that they made some progress and Olga asked them to share their work on the board. The group had discovered that the sum of the three cyclic changes of a general three-digit number was always divisible by 37, but did not know what to do with this information (Figure 1).

Jim raised a concern, asking one of the students – David, if his observation was based on the given statement (i.e. that $100x + 10y + z$ is divisible by 37), and when David replied that it was not, Jim noted: *"It could be any number, [this] line of thinking is going to tell you that oh, every number is divisible by thirty-seven."*

To get students back on track, Jim wrote on the board what needs to be proved (RTP). For this he used the cyclic change occurring in a counter-clockwise direction (Figure 2.a.), which was different from the clockwise direction in which the problem had been framed. Some students seemed puzzled by this choice. In response to a student's question of whether it was necessary to prove each of the two cyclic changes ($xyz \rightarrow yzx \rightarrow zxy$), Jim answered: *"If you prove one of them, you prove both of them."* At this point, Olga interrupted Jim and changed the notation on the board, to reduce students' confusion (Figure 2.b). In the post-class discussion, she explained her move.

Post Class Reflection

During the post-class meeting, Olga raised two issues about the lesson:

"[I] believe if you knew my thinking, you probably would have not jumped at some points. I suddenly realized, oh you are not aware why I am doing it".

Jim agreed: *"we should probably talk about these things before we go to the classroom to know where we are headed…"*

She explained that to reduce the complexity of the problem she wanted everyone to use at first the clockwise cyclic change "*because those [students] who started clockwise started using this notation, so I don't want to make it more complicated for people".* For Jim it made no difference whether to choose a clockwise or a counter-clockwise cyclic change. Clearly, from a mathematical point of view this is right. He was not aware of the pedagogical value of making this distinction.

Moreover, Olga planned to allow the students to prove that the first cyclic change is divisible by 37 and then move to the second one, in accordance with the "repeated reasoning" principle of the DNR (Harel and Sowder, 2009). Only after they did the proof again, for the second case, would she pose the question of whether the second proof was needed: *"You [Jim] would not even ask that question, since it is so obvious [to you]… I thought you raised this issue before they were prepared for that... Clearly you don't need to do it again, because it is like, the big idea of without the loss of generality.... and these are issues that are deep and I wanted them to think about them."*

It should be noted that in the pre and post-class meetings the entire team (of 6) participated. For brevity, we did not bring the full scope of the conversations surrounding the Cyclic Task.

Concluding Remarks: What is this a case of?

The conversation between Olga and Jim through planning and reflecting on this lesson is an example of how members of this unique community of practice discussed, negotiated meaning and came up with shared understandings and better informed ideas of each other's perspective about the practice of teaching proof to students.

In the pre-class meeting, neither Jim nor Olga described how the problem would unfold, as they took it as shared knowledge. In the actual implementation of the lesson, they both realized that although there is always room for spontaneous moves, they should be aware of each other's thinking. Nonetheless, each of them contributed to the lesson without prior coordination with the other. In some ways they complemented each other in a supporting way (e.g., the way Jim commented on the work that David presented (Figure 1), namely, his group's observation that the sum of any 3-digit integer and its two cyclic permutations is always divisible by 37). None of them anticipated this observation, which required spontaneous action. However, Jim's next spontaneous action, using the counter-clockwise cyclic change, before they were prepared for it, did not concur with Olga's pre-planned trajectory that aimed at addressing students' anticipated difficulties more gradually.

The illustrative case that we presented above captures ways in which the collaboration between these two experts made them conscious of each other's considerations and of the importance of questioning their assumptions and negotiating them. More generally, this study seized the opportunity to develop a community of practice that did not exist before, and to trace the process of exchanging expertise and learning from one another partly by apprenticeship and partly by reflecting in and on action and negotiating meaning.

Figures

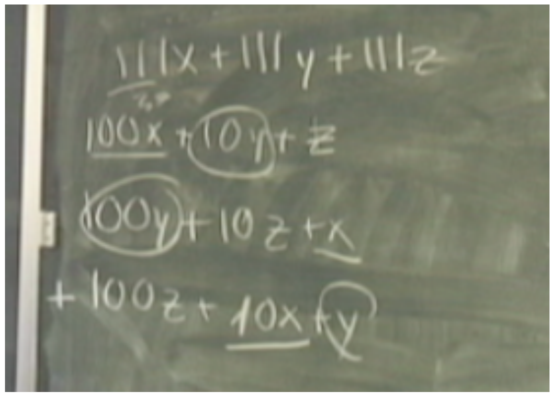

Figure 1. David's group work

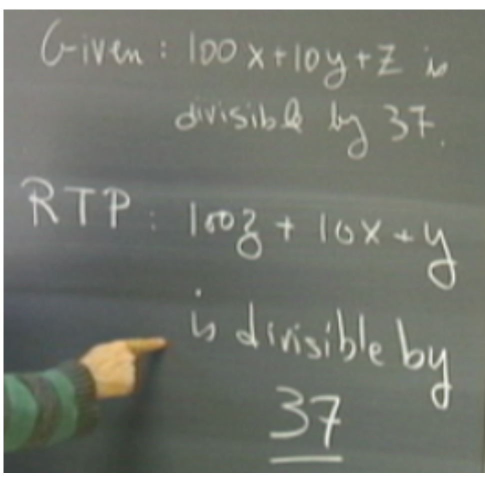

Figure 2.a. Jim's initial presentation - a counter-clockwise cyclic change

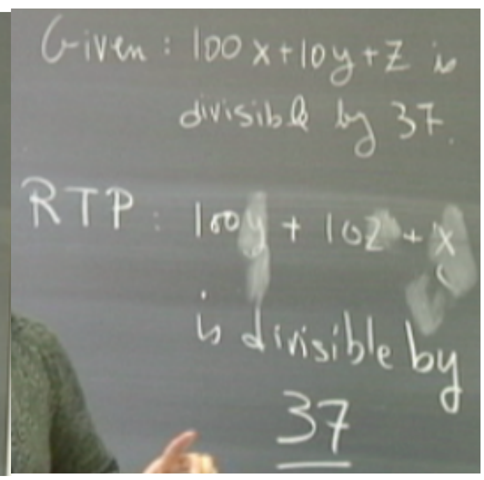

Figure 2.b. Olga's modified notation - a clockwise cyclic change

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ODD DIALOGUES ON ODD AND EVEN FUNCTIONS

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A group of prospective mathematics teachers was asked to imagine a conversation with a student centered on a particular proof regarding the derivative of even functions and produce a script of this imagined dialogue. These scripts provided insights into the script-writers' mathematical knowledge, as well as insights into what they perceive as potential difficulties for their students, and by extension difficulties they may have had themselves when learning the concepts. The paper focuses on the script-writers' understandings of derivative and of even/odd functions.

Key words: [Odd/even Functions, Proof-scripts, Calculus, Derivative]

There is an old adage that in order to truly understand something you have to teach it. In accord with this idea, the activity of teaching and thinking about teaching can reveal a lot about the teachers' own conceptions of the subject matter. In this study, a group of prospective secondary school mathematics teachers was invited to imagine their interaction with students about a given theorem and its proof, and describe it in a form of a script for a dialogue between a teacher and a student, referred to as a *proof-script*. Through writing a script the participants provided a glimpse into their own mathematical knowledge as well as into what they perceive as a potential difficulty for their students. In particular, I focus on participants' understanding of the concepts of derivative and of even/odd functions, as featured in their composed scripts.

Background

The research reported in this article concerns the theorem "The derivative of an even function is odd", and a particular proof of this result, which is accompanied by the diagram presented in Figure 1. This was studied using a proof-script methodology. As such, brief notes on scripting and on the concepts that appear in the theorem are provided below.

Scripting

The idea of scripting was inspired by the "lesson play" construct (Zazkis, Liljedahl & Sinclair, 2009, Zazkis, Sinclair & Liljedahl, 2013). Lesson play, as a method of preparing for instruction, presents a lesson or part of a lesson as a script of interaction between a teacher and students. Zazkis, Sinclair & Liljedahl reported on the use of this tool with prospective elementary school teachers. However, they suggested that the tool could be easily extended and adopted in other contexts. This report presents one such extension.

On Odd and Even Functions

There are several equivalent formulations of the definitions of odd functions and even functions. Analytically, even functions satisfy the property that $f(x)=f(-x)$. They can be defined graphically as functions that have a reflectional symmetry about the y-axis. Similarly, odd functions can be defined graphically as functions which have a 180º rotational symmetry about the origin, or analytically as functions for which $f(x)=-f(-x)$.

The reason odd and even functions are given the same names as subsets of the integers is tied to the properties of monomials of the form x^n where n is an odd/even number. But this does not tell the whole story, since the definition of odd and even functions are not restricted to monomials; it applies to all of functions that have the appropriate symmetry properties. In the case of polynomials any polynomial with only even exponents is an even function (e.g., $f(x)=3x^6-4x^2+x^0-x^{-4}$) and any polynomial with only odd exponents is an odd function (e.g., $g(x)= 3x^7+4x-x^{-3}$). This is also related to McLaurin series expansions. McLaurin series expansions of even and odd functions consist of only even and odd powers, respectively (Sinitsky, Zazkis, & Leikin, 2011).

Method

Fourteen prospective secondary school mathematics teachers participated in the study. They held degrees in either mathematics or science, and as such, all had several Calculus courses in their formal mathematics background. They were enrolled in their final semester of a teacher education program. The participants were asked to respond in writing to the Task presented in Figure 1.

The Task sates a theorem, “The derivative of an even function is an odd function”, and presents a proof for this theorem accompanied by a diagram. Both the proof and its accompanying diagram are borrowed from Raman (2001), who identified it as an intuitive but non-rigorous proof. The participants were invited to write a script for a dialogue with an imaginary student (or a group of students) in which the teacher examines the students’ understanding of the proof and explains, as necessary, issues that are potentially problematic.

I was interested in exploring the difficulties that perspective secondary teachers envision their students might have in understanding the given proof. Additionally, I explored what might be revealed by the script-writing method about participants’ personal understanding related to the concepts of derivative and odd/even functions.

Theoretical Framework

Hazzan's (1999) reducing abstraction framework serves as a theoretical lens for analyzing the data. This framework is based on three different interpretations of *levels of abstraction* discussed in the literature: (a) Abstraction level as the quality of the relationships between the object of thought and the thinking person, based on Wilensky’s (1991) assertion abstraction is a property of a person’s relationship with an object; (b) Abstraction level as a reflection of process-object duality, based on Sfard’s (1991) distinction between process and object conceptions of mathematical objects; and (c) Abstraction level as the degree of complexity of the concept of thought, based off the assumption that the more compound an entity is, the more abstract it is. These interpretations function as modes through which students can reduce the abstraction level of a mathematical object. It is important to note that they are neither mutually exclusive nor exhaustive. In terms of Hazzan’s framework that means that it may not always be possible to categorize every instance of reducing abstraction with exactly one of the above interpretations.

Results and Analysis

My analysis of the composed scripts focuses on the mathematical ideas that appear in the scripts, as related to even and odd functions and to the concept of derivative. These are attributed to either a teacher-character or a student-character. Throughout this analysis I use the terms ‘participants’ and ‘scrip-writers’ interchangeably to describe the authors of the scripts and use the terms ‘student’ and ‘teacher’ exclusively to refer to the fictional characters created by the scripts’ authors.

On the Meaning of ‘Odd’

The script-writers demonstrated an awareness of the fact that the terms ‘odd’ and ‘even,’ when attributed to a function, may cause some initial difficulty for students who try to connect these terms to their understanding of odd and even numbers. The following excerpts exemplify such awareness:

Excerpt 1 (Gina)

Teacher: Alright, so now that we have a good understanding of what an even function is does anyone have any ideas about odd functions?

Student: Are they functions that are not symmetrical about the y-axis?

Excerpt 2 (Carl)

Student: OK, well we know that an even function reflects over the y-axis and I know that even and odd are just opposites. So maybe that means that the function is reflected over the x-axis?

Excerpts 1 and 2 take place after the idea that even functions are symmetric about the y-axis has already been reiterated. The student character's conjecture that odd functions are "not symmetrical about the y-axis" (excerpt 1) is rooted in his knowledge of numbers, where an odd number is a number that is not even. This is explicitly mentioned in excerpt 2, where a student refers to even and odd as "just opposites". 'Opposite' is interpreted as a reflection over the x-axis.

This initial interpretation by students can be seen as a way of reducing abstraction, where a new and unfamiliar term, odd function, is thought of in terms of a familiar term, odd number. The teachers in these excerpts went on to correct the "symmetric about the x-axis" misconceptions and thus the misconception cannot be attributed to the script-writers. However, it is an indication that those participants deemed the terms odd/even when used to describe function to be misleading and likely to cause confusion with the number theoretic meaning of these terms.

On Examples and Definitions

Consider the following excerpts from the scripts.

Excerpt 3A (Gail)

Teacher: Well let's look at some examples. One of the examples of an even function that you gave was $f(x) = x^2$. What is the derivative of x^2?"

Excerpt 4A (Mike)

Teacher: Let's start calling the functions with the even polynomials EVEN functions and the odd polynomials ODD functions.

The common feature of these excerpts is their focus on a monomial function of the form $f(x) = x^n$. In excerpt 3A, $f(x) = x^2$ is treated as a particular example of an even function. However, in excerpt 4A, the monomial-based definition is introduced as an implicit agreement between the characters. From the discussed examples it is clear that the vaguely used term "even polynomials" is related to the parity of the exponent in $f(x) = x^n$.

Only two dialogues mentioned examples of an even function other than $f(x) = x^n$, for an even n: these were $f(x) = \cos(x)$ and $f(x) = |x|$. Others relied solely on $f(x) = x^n$, either as an example or as an informal definition of even and odd function, based on the parity of n. In other words, the invoked example space (Watson & Mason, 2005) for even and odd functions for the majority of participants was limited to monomial functions of the form $f(x) = x^n$. It may be the case that the diagram reminded participants of a parabola of the form $f(x) = x^2$, as is discussed in a next section.

Limiting the example space to monomial functions creates a convenient relationship in connecting the parity of the function to the parity of its derivative. Consider how the scripts in the above examples continue:

Excerpt 3B (Gail)

Teacher: Now can you tell me the rule you used to determine the derivative of x^2?

Student: Well you bring the exponent down in front of the x and then the new exponent becomes the old one minus one.

Teacher: We can generalize that in a formula $\frac{dy}{dx}x^n = nx^{(n-1)}$

If you were using this general rule, can you see why the derivative of an even function is always odd?

Student: I see, if n is an even number, the function is even. Then if you take the derivative n-1 will be an odd number so the derivative is an odd function.

Excerpt 4B (Mike)

Student: If I were to take the derivative of $f(x) = x^2$, then I would get $g(x) = 2x$; where g(x) is the derivative of f(x). […] This makes sense for all functions because you reduce your polynomial by a factor of one when you take the derivative, so any even number should be subtracted by one and will yield an odd number.

These excerpts explicitly mention how to find a derivative of polynomial functions and explicitly limit the example space to polynomials of a particular form. They also assume that function notation uses only a single exponent.

The script-writer's focus on a particular example of a mathematical notion rather than that notion itself is one of the ways of reducing abstraction. To reiterate, reducing abstraction is a mechanism of making sense by considering mathematics on a less abstract level than is suggested by an instructor or a text (Hazzan, 1999). Here the abstract concept of an even/odd function is replaced by a particular cluster of examples of such functions, functions of the form $f(x) = x^n$. Further, consideration of monomial functions can also be interpreted as an instance of reducing abstraction by focusing on familiar, or at least more familiar functions, both in terms of their graphs and in terms of the rules of finding a derivative. As mentioned, various forms of reducing abstraction are not mutually exclusive.

On The Use of a Diagram

The proof of the theorem provided in the Task for participants was accompanied by a diagram. However, while the diagram provides a helpful reference point, complete reliance on a diagram may lead to possible misconceptions, as exemplified by the following excerpts.

Excerpt 5 (Olga)

Teacher: Yes. Now looking at the diagram again, what sign will the slope have for f(-x)?

Student: f(-x) will have a negative slope.

Excerpt 6 (Beth)

Teacher: Great Jeff. Now I want you to look at the picture here. On the right side of the graph, is the slope positive or negative?

Jeff: Positive.

Teacher: Yes, can anyone tell me why? Ah… Laura.

Laura: Well both x_2 and y_2 are bigger that x_1 and y_1, so we will have a positive/positive, leaving using us with a positive slope.

Teacher: Perfect, now let's look at the slope at the point –x. Will the slope be positive or negative?

Laura: Negative.

Excerpts 5 and 6 explicitly reference the diagram, and identify the slope at (-x) as negative. This fails to capture the idea that the slopes get opposite values, regardless of which one is positive. Referring to the positive slope on the right side of the graph demonstrates a reliance on a diagram, which is not generalized. While the diagram assists in understanding the proof, it also limits the example space of functions that satisfy the condition of evenness. Reliance, or in this case over-reliance, on a diagram can be seen as yet another method of reducing abstraction. The diagram is seen as representing a class of functions that decrease for negative values of x and increase for the positive ones. As such, a particular example, or a class of examples, rather than a general case is being considered, which is in accord with interpretation (c) of the reducing abstraction framework.

Summary and Conclusions

The proof-script method of studying students' mathematical understanding evolved out of work on lesson plays (Zazkis et al., 2009, 2013). This paper contributes to an understanding of the utility of this method and adds to the literature regarding students' understandings of and approaches to odd/even functions and derivative.

In general, the script-writers showed evidence of being comfortable with both analytic and graphical modes of representing odd and even functions, drawing appropriate connections between the two. In several scripts student-characters showed initial confusion regarding the terms odd/even as related to functions, indicating that the script-writers were aware of the incongruence between the usage of these words in function context and in number theoretic contexts. However, the participants' invoked example spaces (Watson & Mason, 2005) triggered by the task of explaining the theorem seemed fairly limited. Most mentioned only monomials of the form $f(x)=x^n$ and only two of the 14 participants mentioned odd/even functions that were not polynomials. This also influenced the discussion of derivative. Most scripts noted the relationship of derivative and slope, however, the treatment of derivative was limited to that of monomial functions.

In several scripts it was mentioned that even functions are decreasing for negative values of x and increasing for positive values. I believe this is due to a combination of over-reliance on the provided diagram with limited sample space. How deeply rooted and systemic this issue is a topic for further research. It may be the case that some participants would be resistant to identifying even graphs which are increasing for negative values of x and increasing for positive values of x (e.g. $f(x)= -x^2$) as even functions.

Instances of reducing abstraction were evident in all of the proof-scripts. These instances arose with both student and teacher-characters. When attributed to student characters these can be interpreted as the script-writers' awareness of the difficulties students may face in learning the concepts and strategies those students may use to cope with these difficulties. These may be influenced by artifacts left from script-writers' own learning experiences. However, the instances of reducing abstraction either introduced or reinforced by teacher-characters can be seen as indicative the script-writers' own approaches and strategies. The concretization of odd/even achieved through limiting example space to the set of monomials was the most ubiquitous example of this tendency.

The method of engaging participants' in script-writing appeared useful in investigating their understanding of mathematical concepts. While the scripts also provide interesting information about the envisioned pedagogical approaches, these were mentioned only in passing and could serve as a focus of future research.

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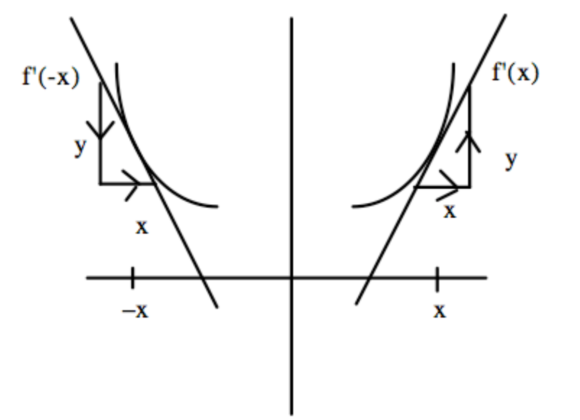
Theorem: The derivative of an even function is an odd function.

Consider the following proof of this theorem:

If f(x) is an even function it is symmetric over the y-axis.

So the slope at any point x is the opposite of the slope at (-x)

In other words, f'(-x) = - f'(x) , which means the derivative of the function is odd.



Imagine that you are working with a student and testing his/her understanding of different aspects of this proof.

What would you ask? What would s/he answer if her understanding is incomplete? How would you guide this student towards enhanced understanding? Identify several issues in this proof that may not be completely understood by a student and consider how you could address such difficulties. In your submission:

(a) Write a paragraph on what you believe could be a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.
(b) Write a scripted dialogue between teacher and student that shows how the hypothetical problematic points you highlighted in part (a) could be worked out (THIS IS THE MAIN PART OF THE TASK).
(c) Add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers.

Figure 1: The Task

ON MATHEMATICS MAJORS' SUCCESS AND FAILURE AT TRANSFORMING INFORMAL ARGUMENTS INTO FORMAL PROOFS

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In this paper, we examine 26 instances in which mathematics majors attempted to write a proof based on an informal explanation. In each of these instances, we represent students' informal explanations using Toulmin's (1958) scheme, we use Stylianides' (2007) conception of proof to identify what one would need to accomplish to transform the informal explanation into a proof. We then compare this to the actions that the participant took in attempting to make this transformation. The results of our study are categories of actions that led students to successfully construct valid proofs and actions that may have hindered proof construction.

Key words: Proof; Proof construction; Informal argument

Introduction and Research Questions

Proving is central to the practice of mathematics. Consequently, a goal of most upper-level mathematics classes is to improve mathematics majors' abilities to construct proofs. Unfortunately, numerous studies have documented mathematics majors' difficulties with writing proofs (e.g., Alcock & Weber, 2010; Hart, 1994; Moore, 1994). Research in this area has identified particular difficulties that students have with proof writing, such as a limited understanding of the mathematical concepts being studied (Hart, 1994) and not knowing how to begin when asked to write a proof (Moore, 1994). However, exactly how undergraduates can and should write proofs remain important questions in undergraduate mathematics education.

The proofs that undergraduates are asked to write in their advanced mathematics courses are required to be formal. That is, these proofs are expected to begin with definitions, axioms, and/or appropriate assumptions and proceed deductively to reach a desired conclusion, often while employing logical syntax. However, although a proof that is produced is required to be formal, the process of producing this proof may be far less rigorous. Numerous mathematics educators advocate that mathematics majors should base at least some of their proofs on informal arguments (e.g., Garuti et al, 1996; Raman, 2003; Weber & Alcock, 2004). For instance, although it is not valid to infer a property about a concept by the inspection of a single example or a diagram of the concept, the insights gained from studying a diagram or example can suggest properties that may be true and useful for constructing a valid proof.

In recent years, the literature on mathematics education has moved beyond simply recommending that students base proofs off of informal explanations and has begun to analyze the *types* of informal arguments that students can and cannot formalize into formal proofs (e.g., Alcock & Weber, 2010; Pedemonte, 2007; Pedemonte & Reid, 2011). The research questions in this proposal focus on the cognitive actions an individual can take to formalize an argument. In particular, we investigate the following:

(i) When mathematics majors give an informal explanation for why an assertion is true but are unable to prove this assertion, why were they unable to do so?

(ii) When mathematics majors successfully transform informal explanation into a valid proof, what actions did they take that enabled them to do so?

Related Literature

Garuti et al (1996) introduced the construct of *cognitive unity* to describe a continuity between the informal reasons that students have for believing a mathematical assertion is true and the proof that they produce of that assertion, arguing that it is desirable that these two be linked in students' proof productions. This is consistent with Weber and Alcock (2004) citing the benefits of semantic proof productions (i.e., proofs based upon informal representations of concepts) and Raman's (2003) call for students to produce proofs based upon a key idea, meaning that students should translate personal intuitive arguments into public formal proofs.

There are two empirical findings that support these recommendations. The first is that these are consistent with the practice of many mathematicians, who also base their formal proofs on informal explanations (e.g., Burton, 2004; Raman, 2003). Second, case studies reveal that *some* students who engage in proof writing in this way can be highly successful (e.g., Alcock & Weber, 2010; Gibson, 1998). However, there are also two reasons to question the viability of these findings. First, as Duval (2007) emphasizes, the structure of informal explanation and proofs differs greatly, notably with respect to the epistemological status of assertions within these arguments. Others (e.g., Alcock, 2010) argue the transition between an informal argument and a formal proof is cognitively difficult. Second, there are also many instances in the literature of students not being able to prove a statement despite seeing why an assertion is true (e.g., Alcock & Weber, 2010; Pedemonte, 2007). This emphasizes the need for a greater understanding of the process of basing a formal proof off of an informal argument.

Recently, Pedemonte (2007) and Pedemonte and Reid (2011) advanced the discussion of this issue by introducing the construct of the *structural distance* between an informal explanation and a formal proof, measuring how easy or difficult it would be to translate the former to the latter based on the type of warrants (in the sense of Toulmin, 1958) that were employed. Pedemonte (2007) argued that informal explanations based on process-based generalizations or abductive inferences had a shorter structural distance to proof than those based on results-based generalizations; consequently, students who generated the former types of explanations had more success at proof writing.

Theoretical Framework

In this study, we follow Pedemonte (2007) and Pedemonte and Reid (2011) in using Toulmin's (1958) framework to analyze students' informal explanations and formal proofs. Furthermore, we adapt Stylianides' (2007) characterization of proof as a normative framework to highlight the gap between explanation and proof and distinguish between valid and invalid proofs. We view an argument as consisting of a series of inferences. Applying Toulmin's framework to each inference, we say that an inference consists of a *claim* (the conclusion that is being advanced), *data* (the facts that support the claim), and a *warrant* (the reason that the claim is necessitated by the data). We note that as arguments are based on a series of inferences, the data or warrant for a new inference may have been the conclusion of a previous inference.

Stylianides (2007) argued that arguments should meet three standards to qualify as a proof; the argument should (i) use inference methods that are valid, (ii) be based upon facts that are true and acceptable, and (iii) use representations that are appropriate, both to the audience who is observing the proof and to the larger mathematical community. Within our Toulmin perspective, for an argument to be a proof, the warrants must be deductive, the initial assertions (i.e., the initial data) being used must be definitions and established facts, and the claims in the proof should be expressed in standard mathematical syntax. We note that informal explanations may not satisfy these standards. Starting points of mathematical argumentation need not be definitions, but may be other representations of mathematical concepts, such as a diagram.

Warrants may not be deductive, but may be perceptual (e.g., “f is increasing because it looks that way on the graph”) or inductive (e.g., “the first four perfect numbers are even, so all perfect numbers are even”). Finally, in informal arguments, claims may be expressed informally in everyday language (e.g., “goes up” instead of “increasing”).

To illustrate, consider a hypothetical student who was informally justifying why $4x^3 - x^4 = 30$ has no solutions. She might say, “the graph of $4x^3 - x^4$ seems to have a maximum value of 27. Therefore, it will never reach 30 and no solution can be obtained”. In the first inference, the claim, “$4x^3 - x^4$ seems to have a maximum value at 27” is inferred from the data (the graph of the function $f(x)=4x^3 - x^4$) through a perceptual warrant (it “seems” that way from the graph). Transforming this argument into a proof would involve stating the claim more rigorously (e.g., “$4x^3 - x^4 \leq 27$ for all real x”), basing it on acceptable data (e.g., the symbolic representation of $f(x)$ rather than a graphical representation), and using a deductive rather than a perceptual warrant (e.g., using calculus to find global maxima of functions). In this study, we investigate what types of cognitive processes might help or hinder students to make these translations.

Methods

Participants. Twelve recent graduates who had majored in mathematics at a large public university in northeastern United States agreed to participate in this study. Students had all taken an introductory proof course at this university.

Materials. Participants were asked to construct seven proofs in calculus and seven proofs in linear algebra. These tasks were chosen so that they could be successfully completed either by syntactic or semantic reasoning, in the sense of Weber and Alcock (2004). That is, we believed it was plausible that students could construct a proof either by symbolic manipulation and logical deduction or by translating an informal explanation into a proof.

Procedure. Participants met individually with a member of our research team for two task-based interviews each lasting approximately 100 minutes. Participants were videotaped as they completed the proofs. They worked on each proof, one at a time, either until they produced what they believed was a proof, they felt they could make no further progress, or ten minutes had elapsed. At any time, participants were allowed to ask for a sheet containing the formal definition and an example object of any concept involved in the study. They were also given access to graphing software. After each proof attempt, participants were asked to describe their thought process as they worked on the proofs.

Analysis. We first flagged each of the 168 collective proof attempts (12 students each attempting 14 proofs) for instances of an informal argument. We defined an informal argument as an argument containing at least two inferences with at least one of the inferences being based on a warrant that was non-deductive. There were 26 such informal explanations. We analyzed each informal explanation and the corresponding proof (when a proof was written) using Toulmin’s scheme. We coded each warrant as an instance of logical deduction, perceptual reasoning, results-based generalization, process-based generalization (cf. Harel, 2002), abductive inference (cf., Pedemonte, 2007), or an abductive warrant. An abductive warrant involved where participants conjectured a principle that might explain why data implied a conclusion (e.g., upon observing that $\sin x$ and $\sin^3 x$ were odd functions, guessing that the product of odd functions was odd). We then coded each informal explanation as being correct (i.e., each inference claim was true) or incorrect and each proof as being valid or invalid. Our qualitative analysis focused on the ways that participants failed or succeeded to transform their informal explanations into valid proofs by focusing on how they:

(i) transformed their initial data from unacceptable facts (e.g., unjustified assertions, informal representations such as graphs) to acceptable facts (e.g., formal definitions),
(ii) transformed arguments based on non-deductive warrants to deductive ones,
(iii) and expressed the assertions in the informal explanation more rigorously.

Results

In the presentation, we will present a list of categories for how students successfully and unsuccessfully attempted to transform informal explanations into proofs. To illustrate the type of analyses that we will discuss, we present one interesting case in detail here. P1 was attempting to prove that $x^3 + 5x = 3x^2 + \sin x$ only had a solution at $x = 0$. After several false starts, P1 graphs $f(x) = x^3 + 5x - 3x^2$ using the computer graphing software and says:

"Yeah... [the graph] doesn't have a bump so I guess it's going to go through that region really and then only going to be between zero and one in a really small area and I guess I just need to prove that it doesn't cross more than once in that area. Oh, and that in that area it's going to be strictly increasing and that sine is also going to be strictly increasing, and that it can only cross once."

Based on these and subsequent comments, we interpreted this argument as saying that a solution occurs when $f(x) = \sin x$. Since the range of sin x is [-1, 1] a solution can only occur for x values where $-1 \leq f(x) \leq 1$. She further notes that $f(x)$ is strictly increasing in this area (or "doesn't have a bump"). She concludes that the only solution to $f(x) = \sin x$ is $x = 0$, inferring that increasing functions can only intersect once. We coded the argument as follows:

Inference 1: Claim: $f(x)$ is between -1 and 1 for a small region.
Data: The graph of $f(x)$.
Warrant: Perceptual. $f(x)$ appears to be between -1 and 1 only briefly based on its graph.

Inference 2: Claim: A solution to $f(x)=\sin x$ can only occur when $f(x)$ is between -1 and 1.
Data: (inferred) The range of sine is [-1, 1].
Warrant: (inferred) Algebraic-deductive. If $f(x)>1$ and $g(x)\leq 1$, then $f(x)\neq g(x)$.

Inference 3: Claim: $f(x)$ is strictly increasing in the region (described in Inference 1)
Data: The graph of $f(x)$.
Warrant: Perceptual. The graph of $f(x)$ is increasing.

Inference 4: Claim: $f(x)$ and $\sin x$ only intersect once in that region (described in Inference 1)
Data: $f(x)$ is increasing (from Inference 3) and $\sin x$ is increasing in this region.
Warrant: (inferred) Abductive. Increasing functions can only intersect once in a region.

We coded the warrant in Inference 4 to be an abductive warrant, or in Pedemonte and Reid's words, a "creative warrant". That is, we believe that in trying to determine why $\sin x$ and $f(x)$ only intersected once in that region, P1 made the inference that increasing functions could only intersect once. This inference, and consequently the informal explanation in its entirety, is not correct (e.g., $f(x)=2x$ and $g(x)=2x + \sin x$ are increasing functions that intersect infinitely often). Transforming this argument into a proof would require (a) expressing the ideas contained in the claim more clearly (e.g., specifying precisely what the "region" in the argument referred to), (b) basing the argument on the algebraic rather than the graphical formulation of $f(x)$, (c) providing a deductive, rather than graphical, warrant to support Inference 1 and Inference 3, and (d) recognizing the warrant in Inference 4 was invalid and offering an alternative justification.

P1 accomplished (a), (b), and (c). She justified Inference 3, noting that $f'(x)=3x^2 - 6x + 5 = 3(x+1)^2 + 2$, which was strictly positive. This illustrates that some students are able to provide deductive backing for non-deductive warrants if they are aware of proving schema to establish the claims in question. (Note here the schema is one establishing $f(x)$ is increasing by showing $f'(x)>0$). She also expressed Inference 2 more rigorously. In her proof, after justifying why it was sufficient to show that $f(x)$ = sin x only had one solution and demonstrating that $f(x)$ was strictly increasing, she wrote, "Note f(-π/2)=-14.32<-1 and f(π/2)=4.276>1. Since -1≤sin x≤1, the solution can only have a real solution in the range –π/2<x<π/2". Note that P1 changed the range of her original argument which would have required computing $f^{1}(-1)$ and $f^{1}(1)$. This modification also obviated the need to justify Inference 1.

However, P1 did not reconsider the invalid abductive warrant for Inference 4. As this warrant was not in the established theory of calculus (and indeed was false), the resulting proof was not valid. We note that the inference of an abductive warrant, even an invalid one, need not cause a proof to be invalid. We have documented other cases in which participants assessed the plausibility of their inferred warrants with examples, observed that they were false, and successfully reformulated their argument. Hence, challenging one's abductive inferences and attempting to provide deductive backing for these inferences are important for proving success.

Summary and Significance

In this study, we have identified several reasons that participants are unable to transform informal explanations into valid proofs, including not investigating the veracity of one's abductive inferences (as discussed above) and simply changing the representation of the argument (i.e., using more formal language) without addressing the warrants by which the inferences in the argument were based (a fairly common occurrence that we did not discuss due to space limitations). We have also identified several ways that participants were able to base proofs on valid arguments, including identifying deductive schemas that could be used to justify inferences based on perceptual or results-based generalizations (as illustrated above). If we expect students to successfully use informal arguments as the basis for proving, as numerous authors suggest (e.g., Garuti et al, 1996; Raman, 2003; Vinner, 1991; Weber & Alcock, 2004), then it is incumbent upon the instructors to have students develop strategies such as these that allow them to do so.

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THEORY BUILDING IN THE MATHEMATICS CURRICULUM?

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Abstract

Mathematicians distinguish two modes of their practice – problem solving and theory building. While problem solving has a robust presence in the mathematics curriculum, it is less clear whether theory building does, or should, have such a place. I will report on a curricular design to support a kind of simulation of mathematical theory building. It is based on the notion of a "common structure problem set" (CSPS). This is a small set of mathematical problems with a two-part assignment: I. Solve the problems; and II. Find, articulate, and demonstrate a mathematical structure common to all of them. Some examples will be presented and analyzed. Relations of this construct to earlier ideas in the literature will be presented, in particular to the notion of isomorphic problems, and cognitive transfer. Designing effective instructional enactments of a CSPS is still very much in an experimental stage, and feedback about this would be welcome.

Key words: Theory Building, Mathematical Structure, Mathematical Practices, Transfer

Introduction

Calls for the practices of the discipline to be featured in the teaching and learning of mathematics are a recurring refrain: the Common Core Mathematical Practice Standards (2010); the NCTM Process Standards (2000); the *Adding It Up* strands of mathematical proficiency *(NRC, 2001)*; the EDC Habits of Mind (2011); etc. Mathematical practices refer broadly to the kinds of things one does when doing mathematics, and the above sources do indeed capture important aspects of this. But if one asks a professional mathematician, a disciplinary practitioner, what doing mathematics involves, the first order response often features something absent from all of these sources, illustrated by this quote from Timothy Gowers (2000): "The "two cultures" I wish to discuss will be familiar to all professional mathematicians. . . . I mean the distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories."

These two modes of mathematical work, *problem solving* and *theory building*, though symbiotic, and not always separated by clear boundaries, are importantly distinct. Are they manifested in the mathematics curriculum? If so, where and how? In fact problem solving has a fairly robust presence, across many grades, and supported by the writings of Polya (1957), Schoenfeld (1994), and others. But theory building? *What activities in the mathematics curriculum could reasonably be construed to be opportunities to learn theory building practices?* That is the central question that animated the work I shall describe here. This question begs for clarification of the underlying concepts, but that clarification itself becomes an important theoretical part of the research.

In the mathematics curriculum theories typically first appear in high school geometry, and in university level courses (calculus, linear algebra, real analysis, abstract algebra, etc.). What such courses generally offer are aspects of a mature, well-articulated theory, and what students are expected to learn is how to manipulate the ideas and techniques of that ready-made theory for problem solving and applications (sometimes routine, sometimes ambitious). What these courses do not (even pretend to) do is have students experience the processes that led to the creation of the fundamental mathematical structures of the theory. In fact it would likely be impractical to incorporate this goal together with the more applications oriented design of such courses in the same time frame.

Mathematical theories commonly arise when a variety of different problems are solved by methods that seem to have something fundamental in common. By distilling, and articulating that common underlying *mathematical structure*, one can create a conceptual frame that simultaneously resolves, or at least illuminates, all of the problems as different specializations and contextualizations of one, more abstract, problem. This general formulation then has the potential to help solve many more such problems (those that can be modeled by the theory) as well. Thus, while theory building might first seem to be an idea somewhat remote from the school curriculum, we see that it is closely linked to ideas – like mathematical structure, connections, generalization – that are more commonly evoked in the mathematics education literature.

I have thus chosen to formulate and operationalize one notion of theory building activity as *seeking, using, and developing mathematical structure*. This formulation can be viewed as an elaboration of the Common Core mathematical practice – *finding and using mathematical structure*. For example, our system of place value notation for numbers is a (very important) *developed*, or *built* mathematical structure, whereas the Mandelbrodt Set is one *sought* and *found*. One virtue of the above formulation is that it works at many grain sizes, and does not require some grand edifice. For example, when a kindergartner characterizes a repeating pattern as "same, same, different, same, same, different, . . ." she is abstracting a mathematical structure that she could then reconstruct with other materials, sounds, gestures, etc.

Common structure problem sets (CSPS)

I offer here a curricular design that is intended to simulate aspects of the theory building practice of the discipline, as characterized above. The design is conceptually a template that can be applied across different mathematical domains, and I have constructed several instantiations, at different levels, and in different subject areas. While the mathematical construct is well formulated, its instructional enactment, and success in achieving important learning goals, is still very much experimental.

The basic construct is what I call a *common structure problem set* (CSPS). This consists of a collection of (say 4-10) mathematics problems, with a two part assignment: I. Solve the problems; and II. Find, articulate, and demonstrate a mathematical structure that is common to all of these problems. The two parts of the assignment correspond to: (I) problem solving; and (II) theory building. In order for this to authentically approximate the mathematical practice of theory building, it is important that the presence of a structure common to all of the problems not be transparent, or evident on the face of things. Moreover, the problems might be quite varied, for example asking very different kinds of questions about a common structure. Two illustrative examples are given and analyzed below. For each CSPS, I provide an analysis (mathematical, pedagogical, cognitive demand) of each task, and an explication of a (not "the") common mathematical structure.

Links to the literature

The mathematics education literature is replete with references to mathematical structure. For example, the "New Math" reforms featured a Bourbaki style version. Dienes and Jeeves (1965) used the mathematical notion of a group. They designed instruction for 10 year olds to learn the structure of some small groups – the group with two elements, and the Klein four-group. The group elements were designated by letters, but the children had to discover the law of composition, and group properties. In an interesting and somewhat analogous design, Simpson et al. (2006) describe work with a pre-service secondary teacher who was given an encrypted version of the modular ring $\mathbf{Z}_{99}$, and guided to discover its commutative ring structure.

The concept of a *multiple-solution connecting task* was introduced by Leikin et al. (2007). It refers to a single mathematical task that can be used to exhibit mathematical connections, such as multiple representations of a concept, or links between different concepts in the same domain, or even links between different domains. A related idea is that of an *interconnecting problem*, developed by Kondratieva (2011). Such a problem (1) allows a simple formulation (without specialized mathematical terms and notions); (2) enables various solutions at both elementary and advance levels; (3) may be solved by various mathematical tools from distinct mathematical branches, and finding multiple solutions; and (4) is used in different grades and courses and can be understood in various contexts. She illustrates these features with the following example: Given a point inside an angle, draw a circle tangent to the sides of the angle and passing through the given point. The notion of "multiple-solution connecting task" is greatly elaborated in the book of Sally and Sally (2007). While these ideas all emphasize mathematical connections, they are distinct from the idea of different problems with common structure, discussed here.

Silver (1979), inspired by Polya's heuristic, "think of a related problem," conducted a study of students' perceptions of relatedness of families of word problems, designed to vary on two dimensions of relatedness: structure; and context. Among his findings, students with high proficiency levels, by a variety of measures, tended to sort problems by structure, while those with lower levels focused more on context. One might then ask whether, reciprocally, nurturing sensitivity to structure could support broader kinds of proficiency.

"Isomorphism" of two problems is a special case of the notion of common structure discussed here. Roughly speaking, two problems A and B are said to be *isomorphic* if there is a correspondence between the elements (objects and operations) of A and B so that a solution process for problem A translates directly into a solution process for problem B (and conversely). Isomorphic problems have the same underlying structure, but common structure (as used here) does not imply isomorphism; it is instead more general. Here is an example of a pair (A,B) of isomorphic problems, taken from Greer and Harel (1998):

A. A straight angle is partitioned into four angles, α_1, α_2, β_1, and β_2, with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Find the measure of $\alpha_1 + \beta_2$.
B. You and your sister had 180 dollars altogether. Your sister gave me half of what she had, and you gave me half of what you had. How much money do you have left between you?

The isomorphism of these problem pairs is often not seen by problem solvers.

The notion of isomorphic problems seems first to have been taken up mainly in cognitive science, in connection with the question of (cognitive) transfer: To what extent does student understanding of how to solve problem A transfer to the ability to solve an isomorphic problem B, believed to differ from A only in "surface features." For example Simon and Hayes (1976), after initiating subjects to the Towers of Hanoi problem, presented them with a contextualized problem isomorphic to it. In another similar study Siegler (1977) defines two problems to be isomorphic if "they are formally identical but differ in their surface features." He compares children's strategies in a 20-question game, to "guess my number" (among 1 through 24) and, in an isomorphic problem, "guess my letter" (among A through X). In general, transfer effects were found to be weak. Lave (1988) published an influential critique of the theoretical roots and culture of such transfer research. Lobato et al. (2002) further questioned the surface/structure distinction. They noted that, “a surface feature can present conceptual complexities for students that are more structural in nature than previously understood.” They illustrate this with a problem about finding the slope of a line, and another, with common structure, about the steepness of a wheelchair ramp.

Janackova and Janacek (2006) analyzed the different solution strategies used by a high school student on four isomorphic problems in combinatorics. Their problems are in fact equivalent to a subset of the “Pascal” CSPS considered below. But they did not present the student with the task of identifying, articulating, and demonstrating the structure common to all of these problems.

More recently, the CME curriculum on combinatorics (Educational Development Center, 2011) features the notion of isomorphic problems. Moreover, Lin et al. (2011) and others have used isomorphic problems to study analogical reasoning across domains, for example mathematics and physics.

Maher et al. (2010) report on one of the few longitudinal studies of children’s development of high level mathematical practices, including development of schemas for connecting structurally related problems. Powell et al. (2009) identified a set of conditions in this project that seemed to shape students’ success with connections among the problems. First, students were given *challenging tasks* (in combinatorics) to work on, tasks that were accessible but for which they had no previously developed strategies. Second, students were asked to work on “*strands*” *of problems*, superficially different but which shared the same mathematical structure (in fact were often isomorphic), and researchers encouraged the students to try to relate new problems to previously solved ones. A third feature of the work was that students were given *sufficient time* to explore the problems and to revisit the problems they had explored. Finally, students developed effective *heuristics*, such as first scaling a problem down to an easier version, or generating data and looking for patterns, etc. The authors note also that extended time was important for students to develop skills for detecting connections.

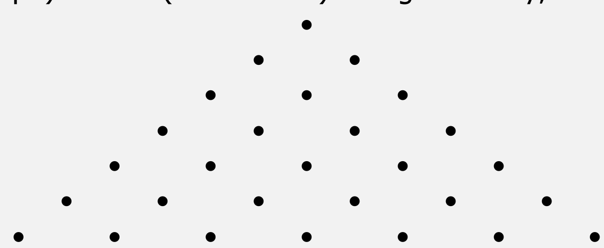
Powell et al. (2009, p. 135) further point out that, “. . . expert problem solvers have and use schemas – or abstract knowledge about the underlying, similar mathematical structure of common classes of problems.” They “can categorize problems into types based on their underlying mathematical structure.” This is consistent with the work of Silver (1979) cited above. Evidently this sensitivity to structure, and the notion of a “strand of problems” (the second condition above) is close to what I am calling a common structure problem set. In fact, the strand illustrated in Powell et al. (2009) is a subset of the “Pascal” CSPS that I present below. What distinguishes this approach from the one that I propose is not in the nature of the mathematical problems, but rather in the charge to the students. With a common structure problem set, the students are explicitly challenged to *identify, articulate, and demonstrate* a mathematical structure common to all of the problems. In other words, what is often left cognitively tacit, and hoped to be assimilated by students, I propose to make an explicit goal of the instruction.

In 1968 Zal Usiskin published, in The Mathematics Teacher, a brief paper, “Six nontrivial equivalent problems.” In fact, his six problems constitute an early example of a common structure problem set. His paper was addressed to teachers, but there was no discussion then of the possible use of such problem sets to challenge students to find and demonstrate common structure.

The Pascal CSPS

1. (Taxi cab geometry). A taxi wants to drive (efficiently) from one corner to another that is 10 blocks north, and 4 blocks east. How many possible routes are there to do this?

2. (Triangular graph) In the (unbounded) triangular array,



. . . continued . . .

connect each dot by an edge to the two nearest dots just below it. At each dot, write the number of "edge-paths" downward to it from the top dot. What is the number in 15^{th} row, the 5^{th} dot from the left?

3. (Walk on the line) On the number line, starting at 0, you are to take 14 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -6. How many ways are there to do this?

4. (Unifix towers) Using 10 white and 4 red unifix cubes, how many different 14-cube towers can you make?

5. (Soccer score progression) The home team won a soccer game 10 to 4. What are all the possible sequences of scoring as the game progressed?

6. (Choosing a team) In a class of 14 students, you need to select a 4-student team. How many different ways are there to do this?

7. (Balls in two bins) What are all ways of putting 14 balls into two bins so that 10 balls are in bin A and 4 balls in bin B.

8. (Cutting a ribbon) You are to cut a 15-inch ribbon into five pieces, each of length a whole number of inches. How many ways are there to do this?

9. (Binomial theorem) In the polynomial $(1 + y)^{14}$, what is the coefficient of y^4?

10. (Higher derivatives of a product) Let Df = f′, denote the derivative of a real function f. If f and g are two differentiable functions, then $D^{14}(f\bullet g)$ will be a linear combination of the functions $(D^p f)\bullet(D^{14-p} g)$ $(0 \le p \le 14)$. What is the coefficient of $(D^4 f)\bullet(D^{10} g)$?

Some will quickly recognize that that these problems lead to the binomial coefficient, 14-choose-4 = 14•13•12•11/4! = 7•11•13 = 1,001, the number of ways of choosing 4 things from a group of 14 things. Nonetheless, problems 1-8 can be approached without much prior knowledge of such combinatorics. Specifically, the solution space for each of these problems can be modeled by what I shall provisionally call, "binary sequences," i.e. sequences of terms which have only two possible values. More precisely, we define a (14,4)-(X,Y)-sequence to be a sequence of 14 terms, each term being either an X or a Y, and exactly 4 of the terms are Ys. The number of these is 14-choose-4, since it is determined by the 4 (out of the 14) places where the Y terms appear. I will demonstrate now that the set of such binary sequences is a structure that models the solution space of each of problems 1-8.

1. Taxi cab geometry. The taxi must drive 14 blocks, 4 of them east (E) and 10 of them north (N), so the set of routes is represented by (14,4)-(N,E)-sequences.
2. Triangular graph. An edge path downward from the top dot can be represented by a sequence of right (R) and left (L), since those are the direction choices when descending from any vertex. To arrive at row 15, we need a sequence of length 14 such choices. To arrive at the 5^{th} dot from the left, exactly 4 of the terms must be R. Thus the set of edge paths in question is modeled by (14,4)-(L,R)-sequences. (An alternative perspective is to view the image of problem 2 as like a 135^{o} rotation of the image of problem 1.)
3. Walk on the line. A 14-step walk on the line is modeled by a (-1,+1)-sequence of length 14, with +1 = a step to the right, and -1 = a step to the left. The number at which this walk arrives is simply the sum of the terms of the sequence. In order to arrive at -6 one needs 10 (-1)s and 4 (+1)s. Thus, the solution space here is represented by (14,4)-(-1,+1)-sequences.
4. Unifix towers. Clearly the solution space here is represented, with W = white, and R = red, by (14,4)-(W,R)-sequences.
5. Soccer score progressions. The solution space here is represented by (14,4)-(H,V)-sequences, where H denotes a point scored by the home team, and V a point scored by the visitors.
6. Choosing a team. Labeling the students 1, 2, . . . , 14, the solution space is represented by (14,4)-(0,1)-sequences, where 1 (or 0) in position j signifies that student j is on, (or off) the team.
7. Balls in two bins. If we label the balls 1, 2, . . . , 14, then the solution space is represented by (14,4)-(A,B)-sequences, with (A in position j) signifying (put ball j in bin A), and similarly for B.
8. Cutting a ribbon. Put inch markers on the 15 inch ribbon. There are 14 of these strictly between the two ends. To cut the ribbon into 5 pieces as in problem 7 is to cut the ribbon at 4 of the above 14 inch-markers. Thus the solution space consists of all ways of choosing 4 cuts out of 14 places, so this is isomorphic to the situation in problem 5, for example.

Note that, even though we see a common structure for (the solution space of) each of these 8 problems, we have not indicated how to count the size (14-choose-4) of these solution spaces. However, the common structure shows that this computation need be done only once, and not separately for each problem.

Problems 9 and 10 require more background knowledge than do problems 1-8. Specifically, problem 9 directly invokes the Binomial Theorem. Here is a treatment that explicitly links the binomial coefficients to binary sequences, as above. First consider a product of n binomials, $P = (a_1+b_1)(a_2+b_2)\bullet\bullet\bullet(a_n+ b_n)$. Multiple use of the distributive law shows that P is the sum of all products $c_1c_2\bullet\bullet\bullet c_n$ where each c_j is either a_j or b_j. Now suppose that $a_j = a$ and $b_j = b$ for all j, so that $P = (a + b)^n$. Then $c_1c_2\bullet\bullet\bullet c_n$ above can be viewed as an (n,p)-(a,b)-sequence, where $c_j = b$ for p of the values of j = 1, 2, . . . , n, and then $c_1c_2\bullet\bullet\bullet c_n = a^{n-p}b^p$. Thus, the coefficient of $a^{n-p}b^p$ in $(a + b)^n$ is the number of (n,p)-(a,b)-sequences. When n = 14 and a = 1, the coefficient of b^4 is the number of (14,4)-(1,b)-sequences.

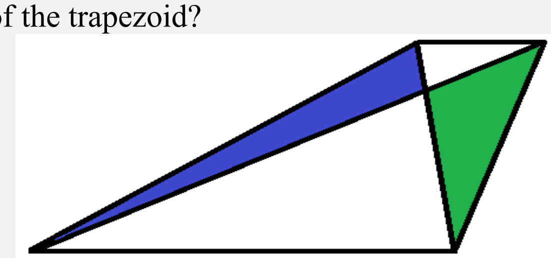
Problem 10 involves a more sophisticated occurrence of the Binomial Theorem:

$$(*) \qquad D^n(f\bullet g) \quad = \quad \Sigma_{0\leq p\leq n}\ (n\text{-choose-}p)\bullet(D^{n-p}f)\bullet(D^p g)$$

This can be proved by induction, using the usual product rule (n = 1), plus the Pascal relation for binomial coefficients. However, it naturally tempts one to see if (*) is in fact a special case of the Binomial Theorem. To see that this is indeed the case, see Appendix A. While problem 10 does not directly involve a structure in common with problems 1-8, it is perhaps worth exhibiting as a complement to the other problems to show the widespread and sometimes unexpected manifestations of the Binomial Theorem.

An instructional design for this common structure problem set might reasonably begin with a small subset of problems 1-8, leading to combinatorial formulas for n-choose-p, and then progressively introduce the remaining problems with the suggestion to relate them to prior problems. Problems 9 and 10 of course would first require a treatment of the Binomial Theorem.

The Measure exchange CSPS

1. (Tea & wine) I have a barrel of wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Question: Is there now more wine in the teacup than there is tea in the wine barrel, or is it the other way around?
2. (Heads up) I place on the table a collection of pennies. I invite you to randomly select a set of these coins, as many as there were heads showing in the whole group. Next I ask you to turn over each coin in the set that you have chosen. Then I tell you: The number of heads now showing in your group is the same as the number of heads in the complementary group. Question: How do I know this?
3. (Faces up) I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. Your challenge, *while still blindfolded*, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.
4. (Triangle medians) In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A' be the mid-point of AC, B' the mid-point of BC, and D the intersection of AB' and BA'. How are the areas of AA'D and BB'D related?
5. (Trapezoid diagonals) The diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non parallel sides) of the trapezoid? 

This problem set is unusual in several respects. While problems 4 and 5 fit comfortably in the geometry curriculum, problems 1-3 are essentially puzzles, and it is far from clear that these problems share a common structure. Problem 1 involves comparison of quantities of liquid. Problem 2 (resp. 3) involves comparison of numbers of pennies (resp., cards). And Problems 4 and 5 involve comparisons of areas. Thus each problem involves some species of *measurement*: liquid volume, numbers of cards or pennies, and area. I will argue that the structure these problems have in common is a simple (and self evident) principle of measurement: *If two quantities have equal measure, and you remove from each what they have in common, then what remains of each of them still have equal measure.* Let me formalize this as what I shall call the "Measure Exchange Lemma."

Consider some measurable object M. It might be a volume of liquid, a collection of pennies, a deck of cards, or a plane region. If X is a part of M, let m(X) be its measure (volume, cardinal, area, as the case may be). Let W and W' be subsets of M of the same measure:

(1) $m(W) = m(W')$. Then: (2) $m(W \setminus W') = m(W' \setminus W)$[1].

Proof that (1) => (2): It follows from (1) that: $m(W \setminus W') = m(W) - m(W \cap W') = m(W') - m(W \cap W') = m(W' \setminus W)$, whence (2).

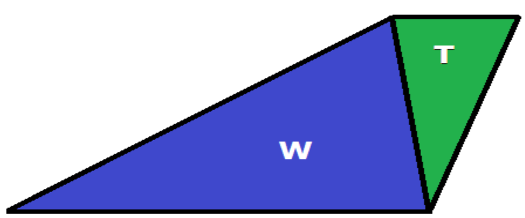
Now let $T = M \setminus W$ and $T' = M \setminus W'$, the complements in M of W and W', respectively. Then $W \setminus W' = W \cap T'$ and $W' \setminus W = W' \cap T$. So we can restate (2) as:

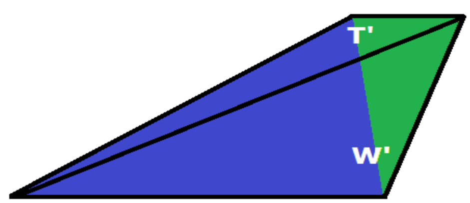
(3) $m(W \cap T') = m(W \cap T')$.

I will now show that this Measure Exchange Lemma is a mathematical structure common to each of the Measure Exchange Problems.

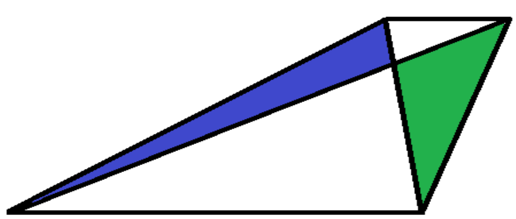
Application to trapezoid diagonals (#5): Each of the two diagonals of a trapezoid M decomposes M into two triangles, call them T and W for one diagonal and T' and W' for the other, and so that T and T' share a common base, and likewise for W and W'. Since all four triangles have the same height (the distance between the parallel sides of M), it follows that

$$m(T') = m(T) \quad \text{and} \quad m(W') = m(W)$$





It follows then from (statement (3) of) the Measure Exchange Lemma that $m(W' \cap T) = m(W \cap T')$.



Application to the triangle medians (#4): This is just an application of #5 to the quadrilateral AA'B'B, that is easily seen to be a trapezoid, with parallel sides AB and A'B'.
Application to the tea and wine problem (#1): Let M be the combined totality of the tea, T, in the teacup and the wine, W, in the wine barrel. After the two teaspoon exchanges, let T' be the mixture in the teacup, and W' the mixture in the wine barrel. Clearly we have the conditions of the Measure Exchange Lemma, and so we have
m(tea in the wine barrel) $=$ m($W' \cap T$) $=$ m($W \cap T'$) $=$ m(wine in the teacup)
Application to the Heads Up problem (#2): Let M be the collection of pennies on the table, T those with tails up, and W those with heads up. Let W' be the coins that you choose, and T' those you left behind. Again, we have the conditions of the Measure Exchange lemma, and so #($W' \cap T$) $=$ #($W \cap T'$). In other words, the number of tails (T) showing in your group (W') = The number of heads (W) showing in what you left behind (T'). After you turn your coins (W') over, the two groups have the same number of heads showing.
Application to the Faces Up in the Deck (#3): Let M be the deck of cards, W the 13 cards that are face up, and T the complement. While blindfolded, you choose a set W' of any 13 of the cards and put them aside, without any of the cards being turned over; let T' be the complement of W'. The Measure Exchange Lemma tells us that #($W' \cap T$) $=$ #($W \cap T'$). In other words, the number of face down cards (T) among your chosen cards (W') = the number of face up cards (W) among those you did not choose (T').
After turning your chosen cards over there will be the same number of face up cards in each group.

As observed above, problem 4 is easily reduced to problem 5. Also problems 2 and 3 are fairly easily seen to involve common structure (pennies, heads, tails) corresponding to (cards, face up, face down). On the other hand these problems ask very different questions about this structure. In a sense, problem 2 gives you a theorem and asks you to prove it. On the other hand, problem 3 expects you to guess the theorem, and apply it. In this sense, I consider problem 3 to be more challenging than problem 2, especially if it were presented in the absence of problem 2.

I learned Problem 1 from Vladimir Arnold, who said that this kind of problem was presented by Russian parents to young children, prior to their formal mathematical training. He claimed that such children solved the problem more easily, and more simply, than mathematicians. This problem seems at first unlike problems 2 and 3 since it involves continuous rather than discrete measurement. Moreover the analysis above shows that the details of the teaspoon exchanges are mostly irrelevant. In fact one could mix the tea and wine arbitrarily together and then divide the mixture into a teacup and a barrel, and the mathematics of the problem would be unchanged.

This common structure problem set has some attractive features. The problems require very little background, except for some elementary geometry in problems 4 and 5. Moreover, it is far from obvious that they share a common mathematical structure, and it is somewhat subtle to identify and articulate such a structure. In this sense, work on this CSPS provides a rich opportunity to engage in mathematical structure practices. It seems well suited for small group work. Moreover, the puzzle problems (1-3) are interesting to share with friends and family.

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Developing an Explication Analytical Lens for Proof-oriented Mathematical Activity

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Sjogren (2010) suggested that formal proof could be understood as an explication (Carnap, 1950) of informal proof. Explication describes the supplanting of an intuitive or unscientific concept by a scientific or formal concept. I clarify and extend Sjogren's claim by applying Carnap's criteria for explication (similarity, exactness, and fruitfulness) to definitions, theorems, axioms, and proofs. I synthesize a range of proof-oriented research constructs into one overarching framework for representing and analyzing students' proving activity. I also explain how the analytical framework is useful for understanding student difficulties by outlining some results from an undergraduate, neutral axiomatic geometry course. I argue that mathematical contexts like geometry in which students have strong spatial and experiential intuitions may require successful semantic style reasoning. This demands that students' construct rich ties between different representation systems (verbal, symbolic, logical, imagistic) justifying explication as a reasonable analytical lens for this and similar proof-oriented courses.

Key words: Explication, proof, axiomatic geometry

In the course of researching the teaching and learning of proof, mathematics educators are continually faced with the need to clarify what it means to understand proving and which aspects of proving students must learn. Thus, researchers adopt the task of developing operational definitions of mathematical proof, at least as it relates to the classroom (Stylianides, 2007). These attempts have included clarifying the purpose(s) of proof (Hanna & Jahnke, 1993; Hersh, 1993), delineating the boundaries of acceptable proof (Weber & Alcock, 2009), and relating proof and argumentation (Pedemonte, 2007). One common theme to many of these discussions is the relationship between more and less formal mathematical notions and proofs. Sjogren (2010) claimed that formal proof could be understood as the *explication* (Carnap, 1950) of informal proof. Carnap defined explication as the process of replacing informal or unscientific concepts with formal or scientific ones. In this paper I explore how Carnap's criteria for explication provide a lens for analyzing student's proof-oriented mathematical activity, particularly in a mathematical context like geometry in which students have rich sources of intuitive and pre-formalized meanings. In so doing, I elaborate and specify Sjogren's claim regarding aspects of the relationship between more formal and less formal proof in advanced mathematics.

The main contribution of this paper is the presentation and elaboration of the framework for using explication to analyze students' proof-oriented activity. I present some basic findings regarding students' learning in undergraduate, neutral axiomatic geometry to illustrate the utility of the analytical lens. I do not fully endorse Sjogren's (2010) claim that *all* formal proof should be viewed as explication of informal proof. Rather, in ways I shall make more explicit, I view explication as a useful lens for particular advanced mathematical contexts. Axiomatic geometry appears exceptionally appropriate for analysis in terms of explication because:

1. students often enter an undergraduate geometry course with more spatial, experiential, and semi-formalized knowledge than in any other advanced mathematics course (i.e., intuitive and less formal conceptions in need of explication),
2. the course which housed these investigations defines and constructs proofs related to very basic experiential concepts such as distance, spatial arrangement, rays, and lines, and

3. it is difficult to understand and construct many geometric proofs in a completely syntactic style (Weber & Alcock, 2004, 2009) without spatial or imagistic interpretation (i.e.- semantic reasoning is often preferable in geometry).

Carnap's (1950) criteria for explication

Carnap was a proponent of logical positivism in the sense that he believed in the advancement of knowledge toward objective truth via the expansion of logical and scientific analysis. As such, he viewed explication as an historical phenomenon by which unscientific concepts were replaced by concepts ready for scientific treatment. He claimed a formal concept explicates a less formal one whenever it has *similarity*, *exactness*, and *fruitfulness*. Carnap (1950) provides the example of how temperature explicates warmness. Warmness is an experiential concept that is relative to the observer (the same room may be warm when stepping in from the snow and not warm when stepping out of a sauna). Temperature is *similar* to warmness in that most experiences of the latter can be expressed and explained in terms of the former. Temperature however differs in that it is an *exact* concept embedding warmness in a scientific body of theory, namely that of numerical measurement or quantity. Finally, temperature explicates warmness because it is *fruitful* for the construction of further scientific theory such as the precise numerical relationships among pressure, temperature, and volume (PV=nRT). Fruitfulness is a relative property of any explication that would primarily function to distinguish various possible explications (were several available).

My use of *explication* differs from Carnap's (1950) in the sense that I am interested in conceptual shifts in the understanding of individual students, a focus that Carnap explicitly distances himself from (calling it "psychologism," p. 41). As such, I may refer to the processes I analyze in this paper as *psychological explication*, the process by which an individual supplants or corresponds an intuitive or less formal concept with a formalizable concept appropriate for the construction of mathematical proof.

Explication in a body of geometric theory

To illustrate how this relates to formal mathematical concepts, I provide several examples of mathematical explications from the body of theory developed in the classroom from which I draw the data presented in this paper. I claim that any element of formal mathematical theory (definition, axiom, theorem, or proof) may at times be a psychological explication of an intuitive or less formal concept. Definitions often explicate informal categories or properties (Alcock & Simpson, 2002; De Villiers, 1998); axioms explicate basic intuitions of number or space; theorems explicate pre-formalized patterns and mathematical phenomena (Stylianides & Silver, 2009); proofs can explicate less formal arguments. The literature on cognitive unity (Antonini & Mariotti, 2008; Pedemont, 2007) explores the question of exactly when arguments and formal proofs correspond more thoroughly than I shall in this paper.

To illustrate mathematical explications, consider how the text (Blau, 2008) used in the axiomatic geometry course I studied defines basic geometric objects. Any *plane* as a collection (**P**, **L**, *d*, [ω], μ) where **P** is the set of points, **L** is the set of subsets of **P** called lines, *d*:**P**x**P** → □ a real-valued distance function (AB=*d*(A,B)), ω the sup of the set of all distances (called the diameter of the plane), and μ the angular distance function. Table 1 presents how other geometric concepts are then defined (explicated) from these elements of any plane.

Concept	Intuitive meaning	Notation	Definition (Blau, 2008)
Point B lies between points A and C (called a betweenness relation)	B lies on the direct path from A to C or B is spatially between A and C	A-B-C	Points A, B, C are distinct and collinear such that AB+BC=AC

Line segment with endpoints A and B	The section of a line between and including A and B	$\overline{AB}$	Given that $0<AB<\omega$, $\overline{AB} = \{A,B\} \cup \{X\mid A\text{-}X\text{-}B\}$
The ray with endpoint A through point B	Half-line or all points beginning with A in the direction of B	$\overrightarrow{AB}$	Given that $0<AB<\omega$, $\overrightarrow{AB} = \{A,B\} \cup \{X\mid A\text{-}X\text{-}B\} \cup \{X\mid A\text{-}B\text{-}X\}$

Table 1. Blau's (2008) definitions for basic geometric properties and objects in the plane.

I claim that each of these formal definitions represents an explication of the associated intuitive concept. For instance, the definition of ray is *similar* in the sense that the set defined above coincides with rays as they are commonly defined on the Euclidean plane. Blau (2008) defines spherical geometry where every point on the sphere is distinct, lines are great circles, and distances correspond to the length of the minor arc between points. As such, the above definition of a ray on the sphere designates a "half-line" in the sense that it has two endpoints that are antipodes (see Figure 1). The definition of ray is *exact* because it embeds the spatial relations in a precise body of theory: numerical equations and set theory. The definition is *fruitful* since the author proved theorems about rays based on this definition and the associated axioms. This final criterion assessing the definition based on its ability to yield formal proof relates closely to Mariotti et al.'s (1997; Antonini & Mariotti, 2008) notion of "theorems" as a triad of statement, proof, and associated body of theory. A definition's quality as an explication of a concept is intimately related to the associated body of theory and proofs built upon that definition.

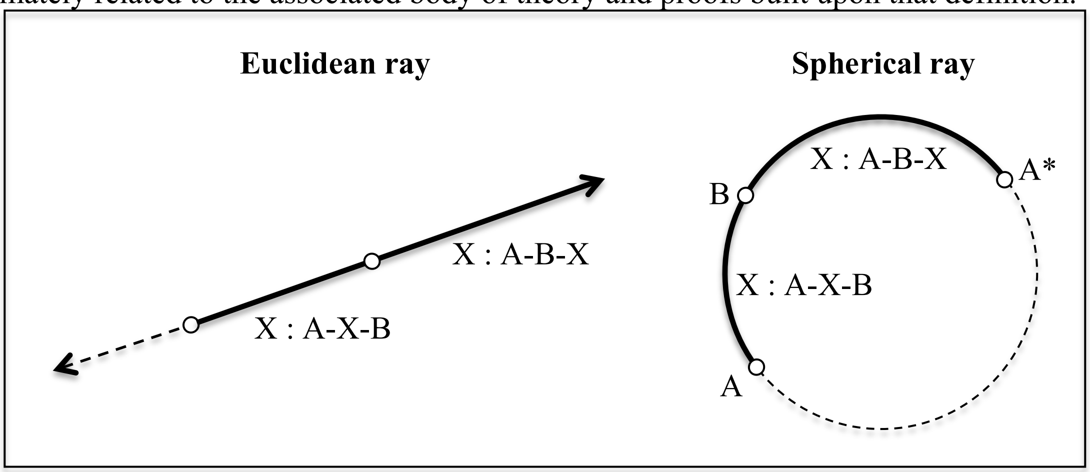


Figure 1. Imagistic representations of rays in the Euclidean and spherical planes.

These definitions are by no means similar in every way to the intuitive concepts they explicate. Distance intuitively corresponds to the space between two points or the distance travelled along a path, but Blau (2008) simply defines distance as a function that obeys certain axioms (positivity "AB≥0", symmetry "AB=BA", definiteness "AB=0 iff A=B", additivity "If AB+BC≤ ω, then either AB+BC=AC, BC+CA=BA, or CA+AB=CB"). Rays generally carry a notion of "direction" that is absent from the set-theoretical definition. The second point "B" in a ray is arbitrary, but the definition appears to distinguish it from other points in the ray. The theorem "Given $C\in\overrightarrow{AB}$ such that $0<AC<\omega$, $\overrightarrow{AB}=\overrightarrow{AC}$" explicates the arbitrarity of the second point in the ray. However, the provability of such theorems simultaneously display the fruitfulness of the explication and restore similarity between formal and less formal concepts (what Weber, 2002, called "proofs that justify axiomatic structure").

An Explication Framework for Proof-oriented Activity

The lens of explication assumes a relationship between less formal and more formal conceptions. However, in traditional advanced mathematics classrooms formal theory is often presented as a self-sustaining structure with varying levels of connection to informal or intuitive meanings (Weber, 2004). As such, explications (noun) are presented to students without seeking to engage students in explication (verb) leaving the level of coordination between formal and informal meanings up to the student (what Freudenthal, 1973, 1991, would call an anti-didactical inversion). Thus I shall apply the psychological explication criteria as metrics for measuring cognitive distances between students' formal and informal conceptions within a conceptual "space". To be clear, students may lack any conceptions or schemes for activity that correspond to "formal conceptions," in which case the metric assesses the distance between student conceptions and expert (teacher) conceptions embodied in expectations upon student learning. I draw upon a number of extant constructs to define this "space" for psychological explication.

First, as Weber and Alcock (2004, 2009) used Goldin's (1998) theory of representation systems to distinguish syntactic and semantic reasoning, I shall generally define students' formal and less formal mathematical conceptions according to the representation system in which they exist. For instance, according to Weber and Alcock (2009), any use of diagrams or images for proving constitutes a shift from the *representation system of mathematical proof* (RSMP) into an imagistic representation system. This indicates that the student corresponds contents formal statements with less formal instantiations in a diagram. For the definitions involved, this generally involves a shift from the symbolic notation or formal definition of a concept to an example represented by a diagram (generic or particular). In the language of Tall and Vinner (1981) the student has shifted from the concept definition to some element of the concept image. Regarding theorems, a student may seek to prove a theorem by reasoning about an intuitive paraphrase of the statement claim (Dawkins, 2012b). This entails translating the statement from the RSMP into a less formal verbal representation system (like paraphrasing the above theorem to say "The second point in a ray is arbitrary"). Regarding proof, when a student develops an argument or informal proof (I use the term "thought experiment" after Lakatos, 1976) toward formally proving a claim, they often do so based on less formal interpretations without thorough warrants or details. As such, thought experiments may draw from a number of tools outside of the RSMP before it is translated into a form compatible with the RSMP like formal proofs.

Figure 2 visually represents this "space" relating the formal and less formal elements in their respective representation systems. The left column represents definitions, the middle column represents theorems or axioms, and the right column represents proof. The upper row represents the explications for the lower row's less formal elements. The upper elements are related to their neighbor below by *similarity*. The upper elements are distinguished from the entry below them by their *exactness*. Finally, the *fruitfulness* of definitions, axioms, and theorems depends upon their ability to yield appropriately explicated terms to their right in the array (definitions must help articulate theorems that must be provable based on those definitions). I distinguish between the extent to which a formal element matches its informal counterpart (*similarity*) and the extent to which the informal conception or representation matches its formal counterpart (*co-similarity*).

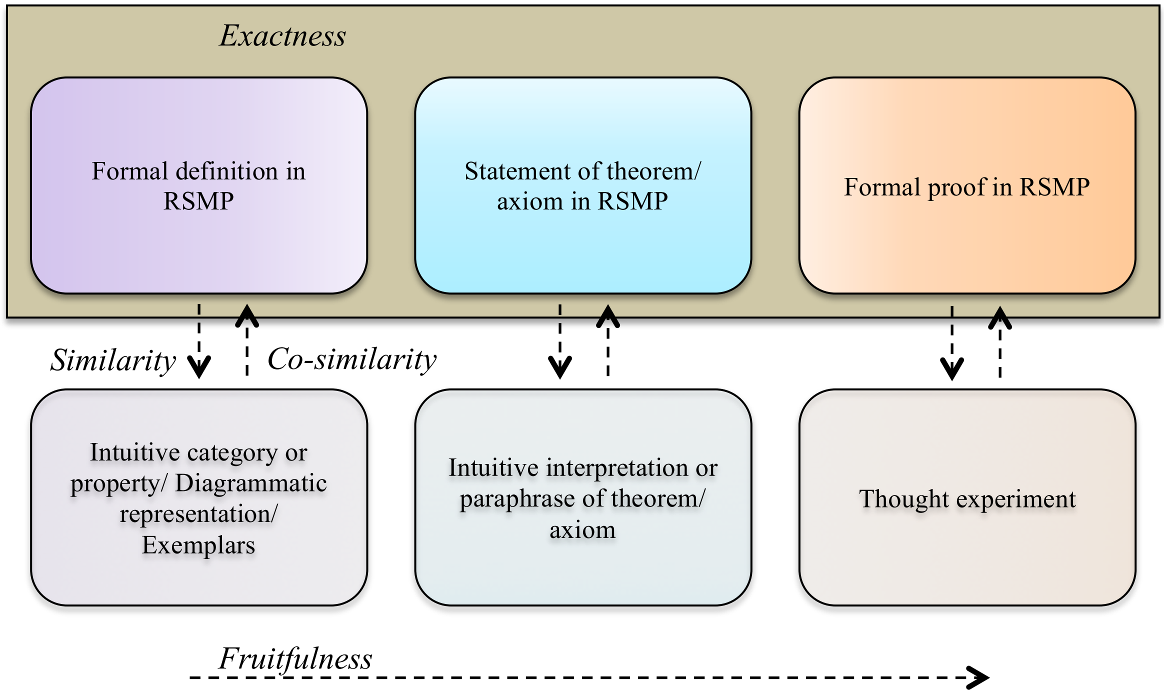


Figure 2. The space of cognitive elements involved in proof-oriented mathematical thinking.

Connections to Prior Research on Proof Production

To show the theoretical utility of this representation, it is helpful to illustrate the number of prior theoretical tools and research that informed my construction of it. I use Goldin's (1998) notion of representation systems reflecting my framework's conceptual ancestry in the semantic/syntactic dichotomy identified by Weber and Alcock (2004; 2009). They distinguish proof attempts according to whether they stay in the RSMP (the upper row of Fig 2) during their proving activity (called syntactic proof activity) or whether they shift into an alternative representation system (the lower row) to guide their proving via less formal reasoning (called semantic proof activity). Thus, mapping out students proving actions in the space represented in Fig 2 entails identifying which of these two types of proof actions students use (this diagram is a more elaborate transpose of Alcock & Inglis's, 2009, diagrams for semantic/syntactic proving).

The process of psychological explication is strongly related to the Realistic Mathematics Education (Freudenthal, 1973, 1991; Gravemeijer, 1994) emphasis on engaging students in the processes of mathematical activity as a means of mathematical learning. In particular, Zandieh and Rasmussen (2010) adapted Gravemeijer's (1999) stages of mathematical modeling to the case of students developing a mathematical definition. This process involves students initially forming a concept definition out of a concept image (represented in Fig 2 as movement up in the first column). This stage establishes basic similarity and exactness. Next students reorganize or develop their concept image in light of the new concept definition. This stage entails the creation of a concept image of their concept definition (which I locate in the upper left representation system) and the reorganization of less formal elements of the concept image (in the lower left representation systems). This reorganization of the less formal concept image helps establish co-similarity. As the definition is extended to new disparate mathematical tasks and contexts, the concept image and concept definition are continually modified to accommodate a broader scope of mathematical activity (Dawkins, 2012a). These latter stages of generalization and formal activity yield the fruitfulness of the explicated definition.

Dawkins (2012b) displays how one students' consistent interpretation of formal statements of theorems and axioms (upper middle) in terms of dissimilar paraphrases (lower center) inhibited his ability to form logically valid proof within the RSMP. The logic of this student's reasoning appeared inherently descriptive of quasi-empirical phenomena in his mental diagrams (lower left). This fostered a thought experiment that convinced the student, but his reasoning subtly changed the logical structure of the statements to be proven. As a result, his thought experiment lacked exactness because it conflicted with the formal logic of conditional statements in which the theorem to be proven was embedded. Though he produced a thought experiment in a semantic style, the lack of co-similarity in his reasoning undermined his success.

Finally, several tools for analyzing students' proving activity relate thought experiments to formal proofs. As mentioned above, *cognitive unity* (Mariotti et al., 1997; Pedemonte, 2007) was coined to describe similarity in this regard. Raman (2003) defined *key ideas* as intuitive or strategic insights that allowed students to explicate a thought experiment. Proof schemes (Harel & Sowder, 1998) distinguish means of convincing that do (deductive) and do not (inductive, by authority) satisfy the exactness criterion for the RSMP.

Analyzing Proving with an Explication Lens

This analytical lens emerged through analysis of a series of task-based interviews in conjunction with a neutral axiomatic geometry course. Each interview was video-recorded and all written notes were captured with a Livescribe pen. All dialogue was transcribed and qualitatively coded in several stages. First, each interview was "chunked" according to the task students worked on and then into individual proving actions taken by the students. Each proving action was assigned (1) a description summarizing the mathematical activity, (2) codes noting the resources the students drew upon for proving, and (3) codes denoting how the proving action related to the four explication criteria. This analysis allowed me to analyze correspondences between proving actions and the explication criteria. Mapping students' proving actions within the framework space above provides a visual means of mapping students' progressions of reasoning during their proof production. Each of the four primary criteria (similarity, co-similarity, exactness, and fruitfulness) has helped assess students' successful or unsuccessful attempts to prove claims. The ongoing analysis indicates that certain types of proof actions tend to correspond with particular explication criteria. For instance, Kirk often paraphrases formal statements into an informal register (paraphrasing actions) subtly shifting the meaning of the statements leading to a lack of co-similarity (loss of semantic content of the claim) or a lack of exactness (shifts in logical structure of the claim). This sometimes frustrated his proof attempts because the statements he reasoned about or cited as warrants differed non-trivially from the formal statements in the body of theory (Dawkins, 2012b).

Oren, Kirk's interview partner, instead displayed strong intuition and logical reasoning skills, but rarely learned the statements of theorems. He shows a strong preference for diagrammatic reasoning, but often could not translate his visually-produced thought experiments into proofs in the RSMP. This largely resulted from his lack of awareness of available warrants to justify his claims. His diagrammatic reasoning also led him to metonymize related mathematical objects such as points A,B (undefined), distance AB (a number or function value), and ray $\overrightarrow{AB}$ (a set). Table 2 presents how static diagrams of geometric objects lack wholly distinct representations of related objects whose explications differ significantly. The superficial similarity of the symbolic notations exacerbates students' tendency to metonymize or conflate these objects. Oren's visual reasoning led him to consistently metonymize geometric objects and to construct thought experiments based on spatial intuition, rather than axioms, definitions, or theorems from the body of theory. As such, his reasoning often lacked exactness (appropriate mathematical structure) and fruitfulness (finding appropriate warrants within the class' body of theory).

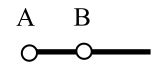
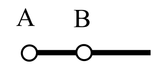
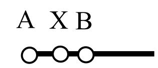
Object	Explicated Structure	Symbolic Representation	Imagistic Representation
Points A, X, and B	Undefined Terms	A, X, B	
Distance between A and B	Function value d(A,B)	d(AB)=AB	
Segment with endpoints A and B	Set of points universally quantified over a betweenness relation	$\overline{AB}$	
Betweenness A-X-B	Property of triplet of collinear points given AX+XB=AB	A-X-B	
Ray with endpoint A through point B	Set of points universally quantified over two betweenness relations	$\overrightarrow{AB}$	

Table 2. Distinct but related mathematical objects represented in multiple registers.

Future Directions

The Explication analytical lens has helped in articulating, representing, and analyzing Kirk and Oren's proving activity in neutral, axiomatic geometry. It also revealed correspondences between the proving actions common to those two students and how they often rendered their proving activity ineffective. The overarching goal of the interviews was to provide the researcher with pedagogically actionable insight into how students act in relation to the body of theory in this course. In light of Kirk and Oren's challenges and successes, the author piloted a redesigned geometry course focusing on helping students view the formal elements of theory as explications of intuitive geometric concepts. As such, the class engaged in mini-reinvention cycles exploring how the axioms could "model" the known geometric spaces (Euclidean, spherical, and hyperbolic). The author was guided in his teaching method by trying to support students in establishing similarity, co-similarity, exactness, and fruitfulness regarding formal and informal conceptions of key course concepts. Thus, my ongoing research explores how the explication lens can foster insight into students' proving and how semantic and syntactic elements can be thereby balanced and integrated in proof-oriented instruction (Weber & Alcock, 2009).

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DEVELOPING HYPOTHETICAL LEARNING TRAJECTORIES FOR TEACHERS' DEVELOPING KNOWLEDGE OF THE TEST STATISTIC IN HYPOTHESIS TESTING

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In the past decade, educators and statisticians have made new suggestions for teaching undergraduate statistics. In light of these new recommendations it is important to (re)evaluate how individuals come to understand statistical concepts and how such research should impact curricular efforts. One concept that plays a major role in introductory statistics is hypothesis testing and the computation of the test statistic to draw conclusions in a hypothesis test. This proposal presents a theoretical approach through the development of a hypothetical learning trajectory of hypothesis testing by utilizing sampling distributions as the building block to understand statistical inference. In addition, this proposal presents how this hypothetical learning trajectory may support the development of research-based curricula that foster an understanding of the test statistic and its role in hypothesis testing.

Key words: Hypothesis tests, Test statistic, Sampling distributions, Guided reinvention

1. Introduction

Hypothesis testing has been used as a research tool in the fields of science, business, psychology, and education, among others. Because of the widespread use of hypothesis testing in research, it is no surprise that it has found its place in introductory statistics curriculum. The Guidelines for Assessment and Instruction in Statistics Education (GAISE) report for collegiate curriculum was released outlining changes needed in introductory statistics courses (Aliaga et al, 2010). The report pushed for changes to instruction that fostered an understanding of statistics, including improving the education of hypothesis testing. If the goal of educators is to improve students understanding of hypothesis testing, then teacher's must help students view hypothesis testing as more than a procedure. Teachers must also have a robust understanding of the concepts of hypothesis testing because their knowledge has a direct impact on student learning. Robust knowledge of hypothesis testing includes understanding concepts like the level of significance, the p-value, the test statistic, and the sampling distribution.

In traditional hypothesis testing, the computation of the sample test statistic plays a crucial role. Surprisingly, introductory textbooks provide vague definitions of the test statistic. For example, Bluman (2012) defines the test statistic as "the numerical value obtained from a statistical test, computed from (observed value – expected value) / standard error" (p. 812). Levine and Stephan (2005) define the test statistic as "the statistic used to determine whether to reject the null hypothesis" (p. 274). These definitions do very little in outlining the importance of the test statistic or its relationship to other statistical concepts. In hypothesis testing, the test statistic is compared to a critical value or used to find the p-value to generate an inference. The test statistic is also crucial because the theoretical sampling distribution in traditional hypothesis testing is a result of a collection of many sample test statistics. If the goal of teachers is to foster student understanding of hypothesis testing then it is important that teachers understand the role of the test statistic and how test statistic formulas are generated. Thus, the goal of this research paper is to answer the following question: "What would a hypothetical learning trajectory look like for developing teacher understanding of test statistic formulas in hypothesis testing?"

To answer this question, the proposal begins with a review of the literature on hypothesis testing. Focusing on trends in the research and what factors are needed to improve the training of teachers. Then a hypothetical learning trajectory (HLT) designed to support in-service and pre-service teachers' (IPST) learning of the role of a test statistic in hypothesis testing is proposed. Finally, the proposal ends with a discussion about the benefits and plans for future research in statistics education.

2. Literature Review

A review of the literature uncovered extensive research related to concepts of hypothesis testing (Batanero, 2000; Castro Sotos et al, 2007; Falk, 1986; Haller & Krauss, 2002; Thompson, Liu, and Saldahna, 2007; Vallecillos, 2002; Vallecillos & Batanero, 1997). Vallecillos and Batanero (1997) revealed that students have difficulties identifying the null and alternative hypotheses. Researchers have also found that students and teachers struggle with interpreting the level of significance and p-value (e.g. Batanero, 2000; Castro Sotos et al, 2007; Garfield & Ben-Zvi, 2008; Haller & Krauss, 2002; Falk, 1986; Liu, 2005; Liu & Thompson, 2005; Vallecillos, 2002; Vallecillos & Batanero, 1997) and fail to understand the role of a sampling distribution in hypothesis testing (e.g. Thompson, Liu, & Saldahna, 2007).

Research conducted by Thompson, Liu, and Saldahna (2007) discovered that teachers have difficulty in seeing the role of sampling distributions in hypothesis testing and understanding the logic of hypothesis testing. They conducted a professional development seminar with eight teachers who had extensive coursework in statistics. The results of the research revealed "some of the teachers' conceptions of probability were not grounded in the concept of distribution which hindered their thinking about distributions of sample statistics and the probability that a given statistic is within a given range of the center of the distribution" (p. 228). The fact that the relationship between distribution and probability is problematic for teachers is troubling because these ideas lie at the heart of statistical inference. Thompson et al. argued that instruction for teachers should focus on developing their understanding of sampling distributions. In the same study, Thompson et al. also investigated the teachers' understanding of unusualness of samples in the context of statistical inference through sampling distributions. A sample statistic would be considered rare or unusual if it fell in a region of the sampling distribution that had small occurrences of other sample statistics. The importance of the sampling distributions in hypothesis testing (and statistical inference in general) is a view shared by many statistics education researchers (e.g. Garfield & Ben-Zvi, 2008; Lipson, 2003; Rubin et al, 1990; Saldanha & Thompson 2002; Thompson, 2004; Watson & Moritz, 2000).

There has been extensive research covering concepts of level of significance, p-value, sampling distributions and null hypothesis, but an exhaustive search of the literature revealed no research investigating teachers' or students' understanding of the test statistic. This lack of research testing is quite troubling because of the prominent role test statistics play in introductory statistics textbooks' treatments of hypothesis testing. However, research pertaining to the concepts of unusualness and sampling distributions could play a key role in generating methods for developing IPSTs' understanding of test statistics.

3. Theoretical Perspective

Many researchers recommend that sampling distributions be central in the teaching of statistical inference (Garfield & Ben-Zvi, 2008; Lipson, 2003; Rubin et al, 1990; Saldanha &

Thompson 2002; Thompson, 2004; Thompson, Liu, and Saldahna 2007; Watson & Moritz, 2000). For example, Thompson et al. (2007) state "we suspect that teachers who value distributional reasoning in probability and who imagine a statistic as having a distribution of values will be better positioned to help students reason probabilistically about statistical claims" (p. 229). Thus, if one uses suggestions by researchers to simply use a sampling distribution of sample proportions (or means), then making a statistical inference must rely on utilizing the sampling distribution in a hypothesis test. To illustrate this, consider the following problem:

Suppose a researcher wanted to determine whether a college population has more males than females. He surveys a group of people and finds that 70% of them are male. Is this sufficient evidence to claim there are more males than females?

To perform a hypothesis test we begin by first assuming the population is equally proportioned between male and female. This identifies the null hypothesis as population proportion of males being 50% (i.e. H_0: p = 0.50) with an alternative hypothesis being that there are more males in the population (i.e. H_1: p > 0.50). This produces a hypothetical population distribution of 50% males. Using a computer simulation, one could generate multiple samples from the assumed population of 50% males. A sampling distribution of proportions could then be produced from the simulated samples. An individual could determine the unusualness of the *observed* sample proportion (i.e. $\hat{p} = 0.70$) by locating its position in the sampling distribution (Figure 1). If the observed sample fell in a region of the sampling distribution where other hypothetical null sample proportions are unlikely to fall, then the observed sample is considered to be unusual. A person can then claim with statistical significance that 50% is not likely to be the true population proportion of males in the college population.

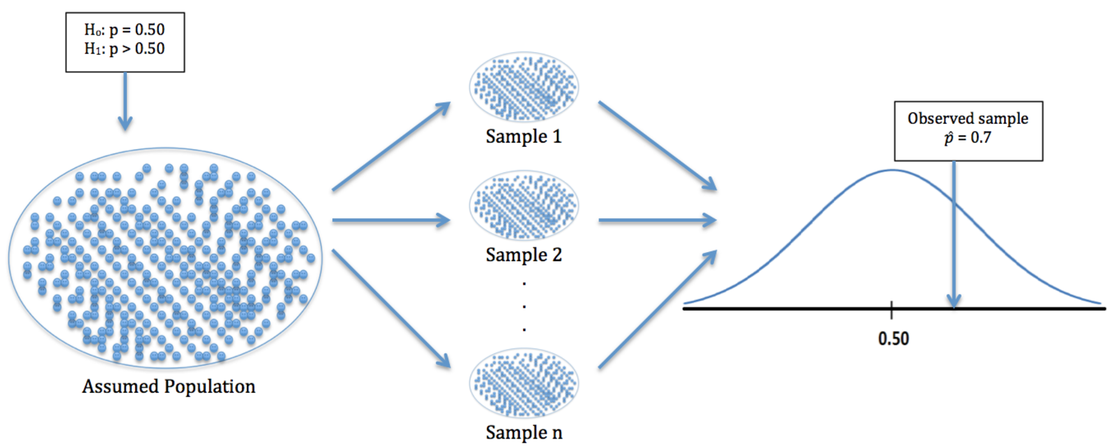

Figure 1. Hypothesis testing approach utilizing a sampling distribution.

This approach places the sampling distribution at the center of the hypothesis testing argument. Furthermore, the null hypothesis is prominent in this approach because it requires IPSTs to generate samples from an assumed null population. Finally, it allows IPSTs to realize that a sample's unusualness is a result of a *frequentist* approach to probability. That is, the probability of a sample proportion occurring is a result of a long-term stochastic process of sampling many times from the null population. The approach described above has many benefits, but problems may arise when dealing with complex situations.

Teaching hypothesis testing utilizing the above approach is valid if we study a single proportion. If the problem were to include multiple proportions, then one could speculate that the approach of hypothesis testing described above should easily transition to multiple proportions. Let us consider the following hypothesis test problem.

Suppose a researcher wanted to determine whether there was a difference between the proportion of freshmen, sophomore, juniors, and seniors in a college population. He surveys a sample of students from the college and the sample contains 40% freshmen, 30% sophomore, 20% juniors, and 10% seniors. Is this sufficient evidence to claim distribution of freshmen, sophomore, juniors, and seniors are not equal?

In this example, we begin with the assumption that freshmen, sophomores, juniors, and seniors are equally distributed (i.e. 25% freshman, 25% sophomores, 25% juniors and 25% seniors). Following similar logic as above, this would generate an assumed population that is equally distributed between the different categories. Once again, an individual could use computerized simulations to produce an empirical sampling distribution based on many samples from the assumed population. This begs the question, "How do we create a sampling distribution to represent the null assumption and determine the rarity of an observed sample?" One option is to generate multiple sampling distributions for each category. That is, generate a sampling distribution for the percentage of freshmen, sophomores, juniors, and seniors with 0.25 as the population proportion (center) for each distribution. A second option is to use the chi-squared test statistic formula (i.e. $\chi^2 = \sum \frac{(Observed-Expected)^2}{Expected}$) to generate a single sampling distribution (Figure 2).

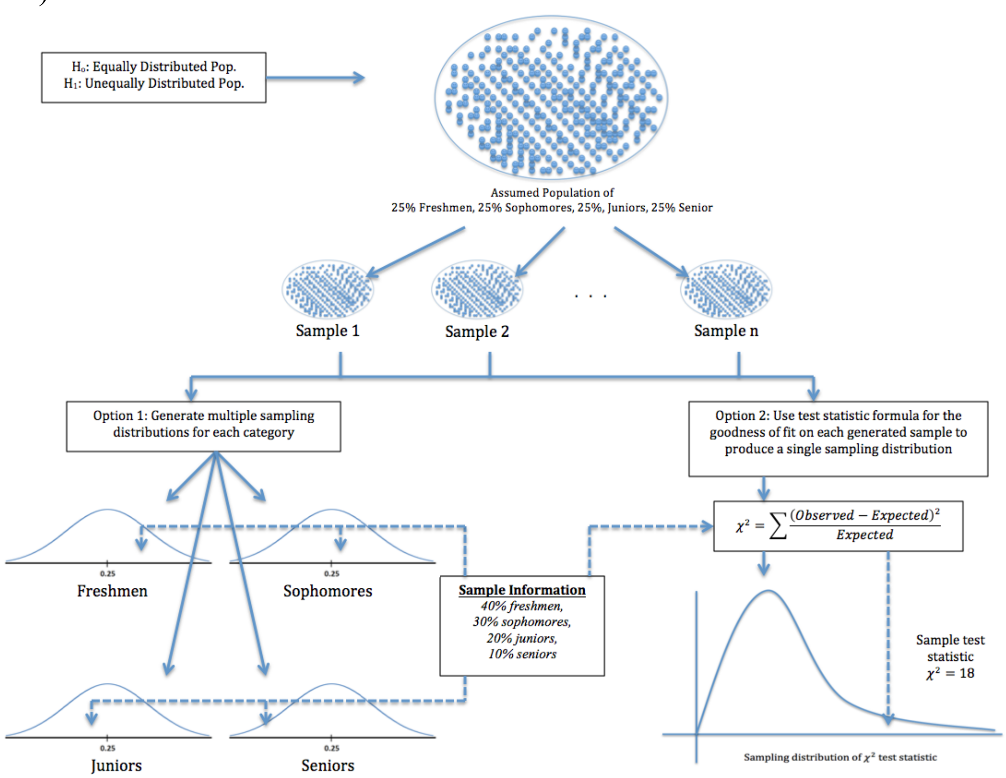

Figure 2. Two approaches of hypothesis testing for multiple proportions.

Option 1, or the multiple sampling distribution approach (MSDA) may be a logical choice for someone who has already used sampling distributions as outlined in the first example (Figure 1) as a means for studying hypothesis testing. One could determine the likelihood of an observed sample by considering where the observed proportion from each category fell with respect to that category's empirical sampling distribution. For instance, seeing where 40% freshmen fell in the sampling distribution of freshmen based on the 25% null assumption. This approach might seem intuitive at first, but an investigation of this approach uncovers significant problems. One such problem arises when considering the following question: What if two of the categories were considered unusual, but two were not? For example, sophomores and juniors are not unusual in their respective sampling distribution because 30% and 20% are not unusually far from the center of 25%, but freshmen (40%) and seniors (10%) are far from the center of their respective distributions. This would mean developing additional criteria to determine a sample's unusualness. This problem increases in difficulty as additional categories are added. A second problem with this approach is that by generating multiple sampling distributions, the probability of generating an incorrect inference increases. Comparing an observed sample with multiple proportions across multiple sampling distributions compounds Type-I errors. If it is decided that unusual is a sample that has a 5% chance of occurring for each category, then this approach is not really comparing unusual at the level of 5%.

Option 2, or the single sampling distribution approach (SSDA), as the name implies, uses a single sampling distribution to discuss unusualness. The chi-squared test statistic formula generates a sampling distribution of sample chi-squared test statistics. Unusualness of a sample could once again be determined by locating where the observed sample's chi-squared test statistic falls within the null sampling distribution. SSDA is the approach we want IPSTs to know when multiple categories are being investigated. SSDA is directly related to the traditional approach found in statistical textbooks, which is formally called the chi-squared goodness-of-fit test. Another example of a SSDA is IPSTs might generate a test statistic formula where they sum the absolute deviations (i.e. $\sum |Observed - Expected|$) to construct a sampling distribution. Thus, there are student-generated approaches (SGA) within SSDA and MSDA to testing multiple proportions. Currently we can only speculate what SGA of hypothesis testing might be, but it would be beneficial to analyze how SGA can be used to leverage IPSTs towards traditional hypothesis tests.

The examples above provide motivation towards developing a hypothetical learning trajectory (HLT) from which to study teachers' development of test statistics for more complicated hypothesis tests. I conjecture that an approach to understanding a test statistic must encompass a relationship between the observed sample information, unusualness, null hypothesis, and sampling distribution. If the new approach towards hypothesis testing is to base decisions on the sampling distributions, it is important for teachers to also understand the meaning of the points used to generate the sampling distribution. These points are the direct results of the test statistic formula being applied to samples. Therefore, the test statistic formula provides a numerical summarization of sample information. The motivation for developing a test statistic that generates a single numerical value is to generate a single sampling distribution rather than multiple sampling distributions. Furthermore, the sampling distribution we wish to generate must express the unusualness of samples in light of the null hypothesis. In other words, developing a test statistic formula should be viewed as a way to quantify unusualness of an observed sample under the null assumption. Viewing the test statistic formula through this perspective could be useful in generating tasks where IPSTs reinvent the test statistic formula. One such approach to teaching where IPSTs reinvent mathematical concepts is through *guided reinvention* (Gravemeijer, 2004).

Guided reinvention is part of the theoretical framework of realistic mathematical education (RME). The basis of RME is that mathematics should be learned naturally through discovery and discussion, as students are involved in solving mathematical problems realistic to his/her perspective. In short, mathematical knowledge is developed by an individual through experiences. Rather than a traditional lecture, students learn through instructional tasks and discussion. Students' shared ideas play a central role of the learning while the teacher serves as a mediator to ensure discussions are directed toward a learning goal. The goal is outlined through a *hypothetical learning trajectory* (HLT). "The notion of a hypothetical learning trajectory entails that the teacher has to envision how the thinking and learning, in which the students might engage as they participate in certain instructional activities, relate to the chosen learning goal" (Gravemeijer, 2004, p. 8). The HLT consists of three components: (1) establishing learning goals, (2) envisioning students mental process, (3) instructional design (Gravemeijer, 2004).

In order to generate activities with the goal of building understanding of the test statistic, a careful description of the HLT is needed. Prior to working with tasks on developing a test statistic formula for multiple proportions, IPSTs should already have an understanding of hypothesis testing using single proportions (i.e. Table 1). This way, when IPSTs are presented the task of multiple proportions they are already motivated to generate a sampling distribution(s) to make decisions about the null assumption. The goal of the HLT for test statistic activities is to move IPSTs towards SSDA above, where they begin to understand the role of the test statistic as a numerical quantification of unusualness leading towards the development of a single sampling distribution.

When generating a task around the goodness-of-fit test to develop a test statistic formula, one approach is to have IPSTs compare unusualness of samples against other samples in light of an assumption. One such task could be the ranking task below (Figure 3). The goal of the task is for IPSTs to generate a method to numerically measure unusualness of a sample in light of the null assumption.

The Dean of Admissions wanted a researcher to investigate the distribution of students in different colleges. The dean expected that undergraduate students were equally distributed in their respective credit years. After conducting 8 surveys, the following results were presented. Generate a method to rank the unusualness of each sample using the dean's assumption.

Sample	Freshmen (%)	Sophomores (%)	Juniors (%)	Seniors (%)
1	23	27	25	25
2	18	24	29	29
3	18	23	38	31
4	14	27	29	30
5	20	14	36	30
6	25	24	16	35
7	10	15	23	52
8	16	25	23	36

Figure 3. Ranking task to developing a test statistic formula for multiple proportions

I conjecture that IPSTs would intuitively see how an observed sample differs from the expectation for each category. The goal of the task is to lead IPSTs towards developing a chi-squared test statistic formula in order to apply a SSDA for a hypothesis test problem. Ideally, we would want IPSTs to generate the chi-squared test statistic. It is possible that IPSTs will not generate this test statistic at first. For instance, IPSTs might develop a formula where they sum the absolute deviations. In this case, moving towards the chi-squared test statistic would require additional tasks and discussions. This task also helps IPSTs connect the relationship of the null assumption with the test statistic formula and build an understanding that unusualness is based on comparing samples. Following this task, IPSTs can attempt a hypothesis test utilizing their constructed test statistic formula to generate a sampling distribution. Discussion regarding properties of the test statistic formula and its role in the hypothesis testing procedure can follow.

4. Discussions

A review of the literature has uncovered a lack of research on student or teacher understanding of the test statistic. This paper presents a methodology for developing tasks that would foster an understanding of the test statistic formula. By utilizing the suggestions of researchers, the methodology offered here supports the importance of sampling distributions as a major part of instruction on hypothesis testing by extending sampling distributions to encompass complex samples. I also offer a perspective of the test statistic formula by viewing it as a tool to quantify the unusualness of a sample in light of the null hypothesis. Further, I present a possible HLT that could be utilized in order to build understanding of a test statistic for a goodness-of-fit test. Plans for future research will focus on actual implementation of a teaching experiment using the prescribed HLT. During the teaching experiment, it would be worthwhile to also examine other approaches that might differ from the MSDA and SSDA described above. Finally, this research focuses on samples with multiple proportions but it would be worthwhile to consider tasks where IPSTs develop test statistics formula for the various hypothesis tests. The goal of this paper was to produce a methodology towards understanding the test statistic formula. In the process, I have also uncovered a new view of hypothesis testing that could be useful for the future of statistic education research.

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THE ACTION, PROCESS, OBJECT, AND SCHEMA THEORY FOR SAMPLING

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This paper puts forth a new theoretical perspective for students' understanding of sampling. The Action, Process, Object, and Schema Theory for Sampling serves as a potential bridge between Saldanha's and Thompson's Multiplicative Conception of Sampling and APOS Theory. This theoretical perspective provides one potential way to describe the development of a student's conception of sampling. Additionally this perspective differs from most other perspectives in that it does not focus on the sample size the student uses or the sampling method, but rather how the student understands sampling in terms of a sampling distribution.

Key words: Sampling, APOS Theory, theoretical perspective, statistics, multiplicative conception of sampling

Introduction

Statistics is one of the most ubiquitous branches of mathematics in everyday life and it is also arguably one of the least understood areas of mathematics. One important aspect of Statistics is that of statistical inference. While students are taught how to calculate a statistic and then perform an appropriate test on that statistic, often the meanings that these students have is a procedural one (Lipson, 2003). Students rarely understand that when making inferences from a statistic, they are making judgments from a sample distribution (theoretical or experimental). Sampling and sampling distributions are thus an integral part of statistical inference. The National Council of Teachers of Mathematics (NCTM, 2000) noted that students should be able to construct and use sampling distributions in regards to statistical inference. Thus, it is important to examine how a student might understand sampling.

While there are several articles on students' understandings of sampling, the work of Saldanha and Thompson (2002) was highly influential in the development of this theoretical perspective of a student's understanding of sampling. Saldanha and Thompson's multiplicative conception of sampling (MCS) serves as a basis for building a more nuanced framework for student thinking about sampling. APOS theory's applicability to a wide range of mathematical concepts including, but not limited to functions, infinity, limits, mean, standard deviation, and the Central Limit Theorem, suggests that it could provide a beneficial framework to refine MCS. In essence, this proposed theoretical perspective serves as a link between MCS and APOS and provides a useful way to characterize a student's understanding of sample with an eye towards the theoretical development of that understanding.

To layout the APOS Theory for Sampling (APOS-Sampling), a brief description of the constructs involved in APOS-Sampling, APOS theory in general, and the multiplicative conception of sampling will be presented. Following this will be a presentation of the APOS-Sampling theoretical perspective with examples. Finally, a comparison to other theoretical perspectives will be made.

Constructs and Background Theories

Given that a construct is an idea existing in the mind of an observer about how an individual understands a concept, a catalytic construct serves as an observer's model of how an individual's understanding might advance within a given framework. Within the APOS framework, there are three catalytic constructs. As described by Sfard (1992), these catalytic constructs are interiorization, condensation, and reification[1]. Interiorization may be thought of as an individual's mental "process performed on already familiar objects" (Sfard, 1992, p. 64). In addition, Sfard (1992) defines condensation as taking the mental process used in interiorization and creating "a more compact, self-contained whole" (p. 64). The final catalytic construct used here is reification; a qualitative leap in how an individual thinks. To help explain these catalytic constructs, a brief example will be given.

When APOS theory is applied to a variety of mathematical contexts, an action is thought of as a transformation on familiar objects to obtain new objects (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Clark, Kraut, Mathews, & Wimbish, 2003; Dubinsky & McDonald, 2002; Dubinsky, Weller, McDonald, & Brown, 2005a, 2005b; Mathews & Clark, 2007; Sfard, 1992). Using APOS-Function[2] as a backdrop, consider the already familiar objects of numbers to a student. As the student repeats carrying out manipulations on numbers, effectively interiorizing mental processes, he/she develops an Action conception of function. Essentially, Sfard's interiorization is the catalyst that allows for a student to move from an understanding of function where "function" is meaningless to the Action conception of function. Similarly, as the student continues to work with functions more and more, he/she begins to condense the processes involved in functions. These processes settle into what Thompson (1994) describes as "self-evaluating expressions" (p. 6). When this happens we say that the student has the Process conception of function. Finally, when the student reifies (or engages in reflective abstraction) function from a set of processes into an object akin to a number, the student has made qualitative leap in his/her understanding, reaching the Object conception of function.

A brief note should be made about the use of the word "process". "Process" has been used multiple times with multiple meanings. In particular, "process" is used within the description of interiorization as well as within the Process conception. While the meanings here are complimentary, one should not view the meanings as interchangeable. There is a subtle distinction between the two meanings that centers on the vantage point of the individual. The meaning of "process" for interiorization places the student *inside* the process, while the meaning of "process" within the Process conception places the student *outside* of the process. For additional clarity, consider the following function, $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x^2-4$. When a student is *inside* the process, he/she focuses on the individual components of the process even if he/she sees the whole process. Thus, this individual would see $f(2)$ as the following sequence: square 2, get 4; multiply 4 by 2, get 8; subtract four from 8, get 4; the result is 4. When an individual has a vantage point of *outside* the process, the individual sees the process as one thing, i.e., $f(2)$ is what you get out of f when you put 2 into f.

Within the APOS literature, schema is most often expressed as a student's "collection of actions, processes, objects, and other schemas which are linked" (Dubinsky & McDonald, 2002, p. 3). Dubinsky and McDonald (2002) go on to note that the usage of the term "schema" within APOS is consistent with Tall and Vinner's (1981) term "concept image" ("the total cognitive structure associated with a concept" (p. 152)). For the APOS-Sampling perspective, schema will have the same meaning as expressed here.

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[1] These catalytic constructs are akin to Piaget's constructs of internalization, interiorization, and reflective abstraction, each of which can be considered a catalytic construct.

[2] The author uses APOS-(Concept) as a shorthand reference to the body of work where APOS theory is applied to a specific concept.

Saldanha and Thompson (2002) characterize a student having a MCS as the student "having conceived a sample as a quasi-proportional mini version of the sampled population, where the 'quasi-proportionality' image emerges in anticipating a bounded variety of outcomes, were one to repeat the sampling process" (p. 266). They propose the following three levels to MCS. The first level requires that an individual visualize a proportional[3] relationship between the sample and the population. The second level occurs when the individual begins to see a sampling distribution take shape as he/she looks at the same statistic across multiple samples. Finally, the third level of understanding supports the individual making inference statements about the expectation of a particular sampling outcome and the outcome's representativeness. It should be noted that the use of the word 'multiplicative' does not entail multiplication, but rather an individual keeping in mind multiple attributes of a quantity (or quantities) at the same time.

Action, Process, Object Conceptions of Sampling

Using APOS Theory and the above catalytic constructs, I will describe the generation of APOS-Sampling. As with APOS-Function (see Breidenbach et al., 1992; Sfard, 1992; Thompson, 1994), APOS-Mean (see Clark et al., 2003; Mathews & Clark, 2007), etc., we must first start with an object familiar to the student. However, unlike the Action conception of function where a student needs only numbers and operations, sampling requires the student to have in mind more than numbers and operations. In the case of sampling, the familiar object starts with the student's conception of the population but also requires the student to have some conception of a stochastic process, as well as a conception of a statistic of interest. Additionally, the student must also have at least level one of the MCS (a proportional relationship between the sample and the population). Consider the following excerpt from Saldanha and Thompson (2002):

> D: Ok. It's asking…the question is…like "do you like Garth Brooks?". You're gonna go out and ask 30 people, it's[4] gonna ask 30 people 4500 times if they like Garth Brooks. The uh…(talks to himself) what's this? Let's see…the actual…like the amount of people who actually like Garth Brooks are…or 3 out of 10 people actually prefer like Garth Brook's music. (p. 262)

This excerpt demonstrates the Action conception of sampling. The population of familiar objects is people (who like or don't like Garth Brook's music). D wants a count of people who like Brook's music, thus the mental process he (or rather he has the computer perform) is a count of a number of people who like Brook's music out of a sample of 30 people. The stochastic process that D has in mind here references what Saldanha and Thompson (2002) refer to as the first two phases of the sampling process. D takes a sample of thirty people from the population, counts those who like Garth Brooks (phase one) and then continues this process 4499 more times (phase two). Thus the Action conception of sampling involves interiorizing (in Sfard's usage) phase one and two of the sampling process in relation to a collection of familiar objects and getting a collection of statistics[5].

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[3] "Proportional" here should be thought of as 'quasi-proportional' which stems from envisioning a bounded variation of outcomes if the sampling process is repeated.

[4] The "it" that D is referring to is a computer program that will draw multiple samples where the population is set to have 30% of people prefer Garth Brooks.

[5] It should be noted that the generation of a statistic of interest pre-supposes that the statistic is an embedded object of the selected sample as well as another set of mental processes.

Just as with APOS-Function, the distinction between the Action conception of Sampling and the Process conception of Sampling involves a difference of vantage points. For Sampling, when a student is inside the process, the student appears to focus on the process of generating the multiple sample statistics. These sample statistics are simply results of a process. However, when the student's vantage point is outside of the process, his/her attention moves away from the process that yields the sample statistics to the statistics. Recall that condensation refers to turning a mental process into a self-contained whole. As the student continues to interiorize phases one and two of the sampling process, the phases, the statistic, and population begin to condense into a sophisticated meaning for the sample statistics. It is at this point that the student has moved to the Process conception of Sampling. The condensation shifts the student's focus from inside to outside of the process and allows for the student to engage in the third phase of sampling process as described by Saldanha and Thompson (2002). The third phase is a rather sophisticated scheme of ideas. Here the student creates categories that each sample statistic can fit into. This shift of focus to the sample statistic is consistent with the student progressing to the second level of MCS. The creation of categories is driven by the student's conception of a desired judgment he/she wants to make about a statistic. An example would beneficial here.

> D: When you go out and take one sample of 30 people, the cut off fraction means that if you're gonna count, you're gonna count that sample, if like 37% of the 30 people preferred Garth Brooks. And then it's going to tally up how many of the samples had 37% people that preferred Garth Brooks. (Saldanha & Thompson, 2002, p. 262)

Here D is exemplifying the process conception of sampling. In particular, D is focused not on the process that generates the sample statistic, but rather manipulating those statistics by comparing each proportion to a cut-score (the 37%) through the computer. Thus, D has created two categories, those sample statistics above or below the cut-score. Just as with other instances of APOS Theory, just because a student is able to speak/work in a way that is consistent with the Action conception to an observer, does not mean that the student is locked at that level. D is clearly able to talk about sampling at the Action conception, however, he is also able to move beyond this level of understanding and speak at the Process conception level of Sampling.

While there is some sense of variation within the Action conception of Sampling, a student at the Process conception must realize that variation of the sample statistic is not just possible but inevitable. At the Action level, a student is much more focused on the process that yields the sample statistic and is thus not likely to focus on differences in the values of the statistic. If he/she does, he/she may attribute the variation to causal features in sense of Konold's (1989) outcome approach. However, the variation of the sample statistic is an integral part of the Process conception of Sampling. Without the expectation of variation amongst the sample statistics, the student cannot create categories of the statistics.

As a student continues to partition the collection of sample statistics, the student begins to develop a distribution of the sample statistic. The student can start seeing the sampling distribution as an object in its own right. When the sampling distribution is reified by the student, he/she could now be said to be at the Object conception of Sampling. In APOS-Function, for example, when a student is at the Object conception, he/she is able to take a function (and that student's conception of the encapsulated process) and manipulate that function as if it were a familiar object, like a number. The student can add two functions, multiply, or compose two functions together. The student gets another function out of this manipulation.

However, in APOS-Sampling, the manipulations take a different form. Instead of having a pair of similar objects (two functions), the student has a sampling distribution (the reification of multiple sampling processes) and a sample statistic. The manipulation that the student now uses is a comparison between the sampling distribution and the sample statistic. This manipulation of the sampling distribution allows for the student to get a sense of what to expect when taking a sample of size *n* from the population prior to sampling. In addition, the comparison allows the student to decide how representative a sample of size *n* is of the population after sampling. The result of this manipulation/comparison is not necessarily another object but rather a characterization/judgment of samples and the population. The Object conception of Sampling is consistent with the third level of MCS. Consider the following excerpt:

> D: If like…if you represent—if you give it like the split of the population and then you run it through the how—number of samples or whatever it'll give you the same results as if—because in real life the population like of America actually has a split on whatever, on Pepsi, so it'll give you the same results as if you actually went out, did a survey with people of that split.
> I: Ok, now. What do you mean by "same results"? On an particular survey at all—you'll get exactly what it—?
> D: No, no. Each sample won't be the same but it's a…it'd be…could be close, closer…
> I: What's the "it" that would be close?
> D: If you get…if you take a sample…then the uh…the number of the like whatever, the number of "yes's" would be close to the actual population split of what it should be.
> I: Are you guaranteed that?
> D: You're not guaranteed, but if you do it enough times you can say it's with like 1 or 2% error depending upon uh how many times—I think—how many times you did it.
> (Saldanha & Thompson, 2002, pp. 265–266)

The above excerpt shows that D is progressing into the Object conception of Sampling. At times it is a little murky about what exactly D refers to when he says "it", however it is clear that D is talking about the purpose of resampling. While D knew the actual population proportion, there is evidence that D thinks about the sampling distribution ("Each sample won't be the same."). In the last line, D talks about the representativeness of a sample based off of the sampling distribution he is imagining.

Comparisons to Other Perspectives of Sampling

There has been some work done focusing on the (mis-) conceptions that adults have on sampling and other statistical topics (e.g., Kahneman's, Solvic's, and Tversky's 1982 book <u>Judgment Under Uncertainty: Heuristics and Biases</u>) as well as some work on children's conceptions (Lajoie, Jacobs, & Lavigne, 1995) of statistical topics. With reference to children's understanding of sampling, there have been several articles that focus on different aspects of sampling than this proposed perspective. Watson and Moritz (2000a, 2000b) looked at a collection of Australian students at various grade levels. Using Watson's statistical literacy framework, they developed six categories of individuals understanding of sampling. However, Watson and Mortiz's six categories are based on the size and type of sample that the student constructed. When sampling was examined within the frame of expectation, variation, and decision making, the student's conception of sample size was used as the point of reference for the student's understanding of sample (Watson & Kelly, 2006). Jacobs (1997) also looked children's understanding of sampling; however Jacobs focused on how students (fourth and fifth graders) evaluated different sampling methods and how those students drew conclusions based upon presented samples (with sampling methods). The APOS-Sampling theoretical perspective sets aside the size of the sample that the student constructs and instead focuses on what understandings the student has for sampling.

This theoretical perspective is related to that of Saldanha and Thompson (2002, 2007) in that this perspective takes a distributional approach to sampling. This is to say that the student's understanding of sampling is placed within his/her ability to keep in mind a sampling distribution. In fact, the work of Saldanha and Thompson greatly influenced this theoretical perspective. While Saldanha and Thompson laid out a three phase sampling process and a multiplicative conception of sampling, this perspective serves as an attempt to extend and reframe their work. The original MCS defines a hierarchy of levels but does not address how a student might move from one level to another. APOS-Sampling serves as one perspective that allows for a development framework to extend the MCS and suggest one way to get students to develop a richer understanding of sampling.

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ILLUSTRATING A THEORY OF PEDAGOGICAL CONTENT KNOWLEDGE FOR SECONDARY AND POST-SECONDARY MATHEMATICS INSTRUCTION

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Abstract. The accepted framing of pedagogical content knowledge (PCK) as mathematical knowledge for teaching has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach mathematics? Many have worked to address this question, particularly among K-8 teachers. What about teachers with broader mathematics knowledge (e.g., from algebra to proof-based understandings of topics in advanced mathematics)? There is a need for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics. This theory development piece offers background and examples for an extended theory of PCK as the interplay among conceptually-rich mathematical understandings, experience of teaching, and multiple culturally-mediated classroom interactions.

Keywords: Pedagogical content knowledge, Discourse, Intercultural awareness

Since Shulman's (1986) seminal work, a rich collection of theories and measures of mathematics pedagogical content knowledge (PCK) continues to grow (e.g., Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). As PCK has become widely utilized in research on early grades (K-8) teacher development, a model based on mathematical knowledge for teaching (MKT) has emerged (Hill, Blunk, et al., 2008). Most framing of MKT for early grades includes little in the way of intermediate and advanced algebra or of proof-based understandings, such as are found in college mathematics. The existing K-8 work is valuable in thinking about PCK in secondary and post-secondary settings and to build on it, there is a need for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics.

The common framing of mathematical knowledge for teaching has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach mathematics? Many have worked to develop measures to address this question, most notably Ball and colleagues (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). In their work they have defined three types of PCK: knowledge of curriculum, knowledge of content and students (KCS), knowledge of content and teaching (KCT). Even with this carefully developed model, challenges exist in identifying and measuring PCK (Hill, et al., 2008).

Other researchers have offered a supplement to the K-8 view, emergent from radical constructivist perspectives (i.e., Piagetian). It is the idea that for some, PCK is "predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum." (Silverman & Thompson, 2008, p. 502). In this framing, PCK grows when a teacher gets better at the transformation of personal and intimate forms of mathematical knowing. Our purpose in building theory is to describe and illustrate an unpacking of this idea while also attending to the reality of culturally heterogeneous classroom contexts.

Here we report on our efforts to develop an expanded theory and model of PCK that attends to a key aspect of Shulman's framing of PCK that is absent in existing models: a fourth component of mathematical knowledge for teaching, *discourse knowledge*. This brings to PCK the mathematical semiotics that was part of Shulman's original description. Ultimately, we seek to develop a theory and measurement tools/guidelines that allow exploration of such questions as: *What is the interplay among mathematical understandings, experience of teaching, and culturally-mediated communication in defining and growing algebra PCK? ...proof PCK?*

We start with brief definitions associated with "discourse," describe our model with additional PCK constructs, and make a foray into some key ideas in intercultural orientation. Over the last 10 years, the authors have been involved in a variety of ways in research and professional development with college and university faculty, in-service secondary mathematics teachers, and their students. In that work, we are regularly asked by mathematically-trained stakeholders for examples and non-examples of PCK in use. To provide a compact and relatively simple contextualized illustration, we conclude with two classroom examples. Vignette 1 represents Teacher Pat in the third year of teaching experience; Vignette 2 represents Pat's classroom again, after three more years that included professional work related to responsive noticing of student thinking for generating and sustaining conceptually-focused discourse during instruction. The two vignettes and brief analyses of them are presented to illustrate the theorized PCK constructs. These illustrations are *not* definitions. They are offered as anchors for discussion. For this initial report we include a pair of algebra-based vignettes. At the RUME session and in the final long paper, we will share similar snapshots and analyses for at least one advanced topic (e.g., linear algebra or group theory). Our proposed framework relies on three existing theories related to human interaction in mathematics teaching and learning: for discourse, for PCK, and for intercultural sensitivity development.

Background on Discourse

A classroom culture is a set of values, beliefs, behaviors, and norms shared by the teacher and students that can be reshaped by the people in the room (Hammer, 2009). Though not everyone in the classroom may describe the culture in the same way, there would be a general center of agreement about a set of classroom norms, values, beliefs, and behaviors. Gee (1996) distinguishes between Discourse and discourse. The "little d" discourse is about language-in-use (this may include connected stretches of utterances and other agreed-upon ways of communicating mathematics such as symbolic statements or graphs). Discourse ("big D") includes little d discourse but also includes other types of communication that happen in the classroom (e.g., gestures, tone, pitch, volume, and preferred ways of presenting information). Big D Discourse also includes syntactic knowledge, an aspect of PCK introduced by Shulman and colleagues but not regularly or explicitly tackled in current K-8 focused approaches to describing and measuring PCK; understandings about how to conduct mathematical inquiry. Gee notes:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artifacts', of thinking, feeling, believing, valuing, and acting that can be used to identify oneself as a member of a socially meaningful group or 'social network', or to signal (that one is playing) a socially meaningful 'role' (p. 131).

The forms of communication in discourse are usually explicit and observable, while the culturally embedded nature of inquiry in Discourse is largely implicit. That is, as part of PCK, this is knowledge for working effectively in the classroom with the multiplicity of Discourses students bring into the classroom. In particular, each Discourse includes a cultural context. Discourses may differ from person to person or group to group. The ways that teachers and learners are aware of and respond to these multiple cultures is a consequence of their orientation towards cultural difference, their *intercultural orientation* (individually and collectively). A teacher with rich knowledge of multiple Discourses, whose orientation towards cultural difference is developmentally advanced, can juggle and balance and engage with myriad cultures in-the-moment to support effective communication among all in the room. We come back to intercultural orientation, below, after unpacking what we mean by Discourse a bit more.

The “big D” Discourse of academic mathematics values particular kinds of “little d” discourse. Valued inscriptions are figural (e.g., in representations such as graphs of functions) and logico-deductive (e.g., proofs). Especially valued in advanced mathematics are explanation, justification, and validation (Arcavi, Kessel, Meira, & Smith, 1998; DeFranco, 1996; Weber, 2004). As in other fields, instructors ask questions to evaluate what students know and to elicit what students think. For instance, a model of classroom interaction common in the U.S. is the dialogic pattern where teacher *initiates* – student *responds* - and teacher *evaluates* (Mehan, 1979). More recent work has led to a more broadly defined *initiation –response –follow-up* or *I•R•F* structure (Wells, 1993). In college classrooms, this is most often initiated by teachers, but not exclusively so, and the (implicit) rules for how initiating, responding, and following-up will happen are worked out by the people in the room (Nickerson & Bowers, 2008). In his ethnographic work, Mehan identified four types of teacher questions (see Table 1).

Table 1. Initiate – Respond – Follow-up (*I•R•F*) question types and anticipated response types.

Evaluate what students know	***Elicit what students think***
Choices – response constrained to agreeing or not with a statement (e.g., Did you get 21?) ***Products*** – response is a fact (e.g., What did you get?)	***Processes*** – response is an interpretation or opinion (e.g., Why does 21 make sense here?) ***Metaprocesses*** – response involves reflection on connecting question, context, and response (e.g., What does the 21 represent? How do you know?)

Research suggests that U.S. mathematics instructional practice lives largely to the left of Table 1 (Stigler & Hiebert, 2004; Wood, 1994). The unfortunate aspect here is not necessarily the fact that evaluative questions are common but that the eliciting *processes* and *metaprocesses*, in the right column, are not. These more complex spurs for discourse can lead to iterative patterns that cycle through and revisit the frame of reference “in ways that situate it in a larger context of mathematical concepts” and foster “mathematical meaning- making” (Truxaw & DeFranco, 2008, p. 514). The use of process and metaprocess questions, for example as *follow-up (F)*, readily expands discourse into the “reflective toss” realm of comparing and contrasting different ways of thinking (with justification but without judgment), monitoring of a discussion itself, as well as attending to the evolution of one’s own thinking (van Zee & Minstrell, 1997).

Four Component Model of Pedagogical Content Knowledge

While Hill, et al. (2008) acknowledge the importance of teacher knowledge of standard and non-standard mathematical representations and communication, discourse knowledge as we construe it – composed of both discourse and Discourse understandings – does not appear explicitly in their model of pedagogical content knowledge. One way of visualizing our extension is as a tetrahedron whose base is the MKT model with apex of discourse knowledge (see Figure 1, next page). As indicated in Figure 1, our attention has focused on discourse knowledge and the three “edges” connecting it to the components in the MKT model (Hauk, Jackson, & Noblet, 2010). These edges, though labeled in Figure 1 and discussed here as kinds of “knowledge,” might more appropriately be labeled as “ways of thinking,” with the aspects of the MKT model taken as “(ways of) understanding” (Harel, 2008). The distinction is still an area of theory development for the authors and will form part of the conference session discussion.

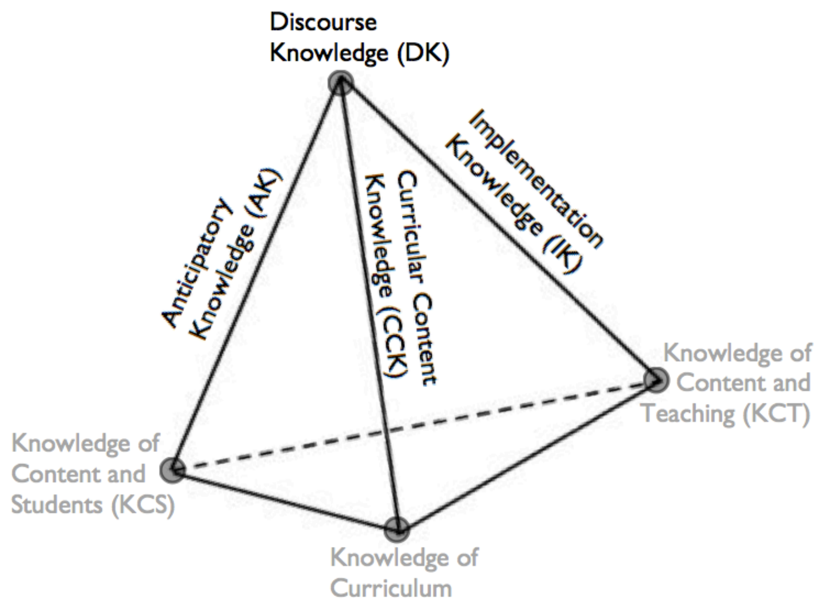

Figure 1. Tetrahedron to visualize relationship among PCK model components. Corners of the base are the aspects of MKT PCK articulated in Hill, Ball, and Schilling (2008).

As indicated above, ***Discourse knowledge*** (**DK**) is d/Discourse knowledge about the culturally embedded nature of inquiry and forms of communication in mathematics (both in and out of educational settings). This collection of understandings includes syntactic knowledge, "of how to conduct inquiry in the discipline" (Grossman, Wilson, & Shulman, 1989, p. 29).

Curricular content knowledge (**CCK**) is substantive knowledge about topics, procedures, and concepts along with a comprehension of the relationships among them and conventions for reading, writing, and speaking them in school curricula. In its most robust form, this part of PCK contributes to what Ma (1999) called "profound understanding of mathematics" (p. 120). In combination, curricular content and discourse knowledge are the home of Simon's (2006) "key developmental understandings."

Anticipatory knowledge (**AK**) is an awareness of, and responsiveness to, the diverse ways in which learners may engage with content, processes, and concepts. Part of anticipatory knowledge growth involves what Piaget called "decentering" – building skill in shifting from an ego-centric to an ego-relative view for seeing and communicating about a mathematical idea or way of thinking from the perspective of another (e.g., eliciting, noticing, and responding to student thinking; Carlson, Moore, Bowling, & Ortiz, 2007). Teachers with rich anticipatory knowledge know how to manage the tensions among their own instrumental and relational understandings of mathematics and its learning and those of their students (Skemp, 1976). Such perspective-shifting is deeply connected to Discourse through the awareness of "other" as different from "self." We return to this idea, below, in discussion of intercultural orientation.

Implementation knowledge (**IK**) is about how to enact teaching intentions in the classroom. Moreover, for us, it includes how to adapt teaching according to content and socio-cultural context and act on decisions informed by discourse as well as curricular content and anticipatory ways of thinking. We do not argue for an intention to enculturate in the sense of Kirshner's (2002) "teaching as enculturation" (i.e., to identify a reference culture and then target instruction for students to acquire particular dispositions). Nor do we propose his alternate framings (habituation, construction) or any other preference for implementation knowledge paradigm.

Intercultural Orientation

The construct of "big D" Discourse as part of mathematics PCK pivots on the idea of intercultural orientation. Our referent framework is the *Developmental Model of Intercultural Sensitivity* (Bennett & Bennett, 2004). The developmental continuum of orientations towards awareness of cultural difference, of "other," runs from a monocultural or ethnocentric "denial" of difference based in the assumption "Everybody is like me" to an intercultural and ethnorelative "adaptation" to difference. The development from denial to the "polarization" orientation comes with the recognition of difference, of light and dark in viewing a situation (e.g., Figure 2a).

Figure 2a.

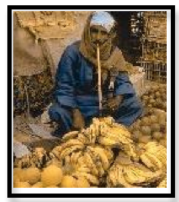

Figure 2b.

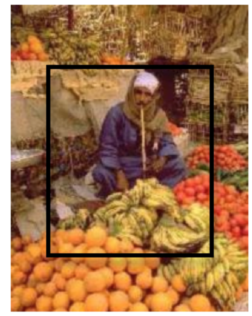

Figure 2c.

Denial ⇨ Polarization ⇨ Minimization ⇨ Acceptance ⇨ Adaptation

Figure 2. Intercultural orientations and developmental continuum.

The polarization orientation is driven by the assumption "Everybody should be like me/my group" and is an orientation that views difference in terms of "us" and "them." Evaluative prompts about student thinking are more likely (left side of Table 1) for this orientation. Moving along the continuum towards ethno-relative perspectives leads to a minimizing of difference, focusing on similarities, commonality, and presumed universals (e.g., biological similarities – we all have human brains so we all learn math essentially the same way; and values – we all know the difference between right and wrong and naturally will seek right). This is the "minimization" orientation. A person with this orientation will be blind to recognition and appreciation of subtleties in difference (e.g., Figure 2b, a "colorblind" view). The minimization orientation tends to take the form of ignoring fine detail in how people might have differing ways of thinking. For example, efforts at eliciting d/Discourse (right side of Table 1) may take the form of listening for particular ways of thinking. Transition from a minimization orientation to the "acceptance" orientation involves attention to nuance and a growing awareness of self and others as having culture and belonging to cultures (plural) that may differ in both obvious and subtle ways. While aware of difference and the importance of relative context, how to respond and what to respond *in the moment of interaction* is still elusive. From this orientation, classroom d/Discourse may include process and metaprocess prompts, but sustained cycles of such interactions can be challenging to maintain in the immediacy of dynamic classroom conversation. The transition to "adaptation" involves developing frameworks for perception, and responsive skills, that attend to a spectrum of detail in an interaction (e.g., the detailed and contextualized view in Figure 2c). Adaptation is an orientation where one may shift cultural perspective, without violating one's authentic self, and adjust communication and behavior in culturally and contextually appropriate ways. There is an instrument for measuring developmental orientation (see idiinventory.com). One area of ongoing work for us is the relationship between intercultural orientation and what orientation(s) may be necessary, if not sufficient, for rich d/Discourse knowledge.

Vignettes and Discussion

Vignette 1–Snapshot of a Classroom

ALGEBRA CLASSROOM*

PAT (teacher) stands in front of whiteboard, 36 students in 6 rows of 6 face the front in small desk-chairs. Problem on the board reads: *At some time in the future John will be 38 years old. At that time he will be 3 times as old as Sally. Sally is now 7 years old. How old is John now?* Shuffling of paper and scratching of pencils but no voices as students work.

Pat: Okay. Let's talk about this. What did you get? How is it that you thought about it?

Lee: I divided 38 by 3. Then I subtracted 7 from 12 2/3 and got 5 2/3. Then I subtracted that from 38 and got 32 1/3. *(Pause)* John is 32 1/3.

Pat: Right. *(Pause)* Why did you divide 38 by 3?

Lee: *(Appearing puzzled by the question, Lee looks back at her work. She looks again at the original problem.)* Because John is 3 times older.

Pat: Okay. *(Looks around the room)*. Any questions? *(Turns and erases the board).*

Jackie (quietly to herself): Isn't the answer 21?

Pat: Hum? *(Turns to face the room)* No. If he was 21 he'd be three times as old as Sally is *now*.

Jackie: It says that he is 3 times as old as Sally, and Sally is 7.

Pat: Well, the problem says John is 3 times as old when John is 38, at some time in the future. *(Pause)* Do you understand?

Jackie: (shrugging): Okay.

Pat: Okay. Does anyone have any more questions? Okay. Now you try one, number 19 on page 33.

Problem 19. *At some time in the past Luana was 24 years old. At that time she was 4 times as old as Rodney. Rodney is now 12 years old. When will Luana be 40?*

*This and the content of Vignette 2 are derived from Thompson, Philip, Thompson, & Boyd (1994).

Figure 3. Vignette representing Teacher Pat's classroom instruction in third year of teaching.

Curricular Content Knowledge. In Vignette 1, Teacher Pat identifies Lee's work as correct. However, the vignette does not offer detail on Pat's *curricular* content knowledge. In his inference that it is similar to the newly assigned item, Pat only implicitly connects the problem to other mathematics relevant to the students in the local context of their learning,. In fact, there are important differences between the mathematical thinking needed for the John and Sally problem and the level-appropriate ideas likely to be called on to tackle the new Luana and Rodney item. Vignette 2 (next page) provides more insight into what the teacher notices about student thinking. Pat demonstrates knowledge of the mathematical requirements appropriate to the curricular focus of the class. Pat's attention to the multiple problem solving approaches and acting as a guide through the discussion are evidence that the particular concepts, and the use of particular tools (e.g., the table) are curricular-level-appropriate for the class.

Anticipatory Knowledge. In Vignette 1, Pat demonstrates anticipatory knowledge (and approval) of a correct student solution path when expressed as procedural knowledge. A moment later, though, Pat appears unaware of the origin or nature of Jackie's confusion. That is, in Vignette 1, Teacher Pat does not appear to anticipate common student struggles (unpacking mathematical relationships from densely worded problems and organizing information from a word problem context). As we see in Vignette 2, such anticipation could be a valuable resource for enhancing students' understanding of mathematics. In the second vignette, Teacher Pat anticipates that students may focus on steps to the right answer and inserts a purposeful halt. Pat aims for a socio-mathematical norm in which explanation for sense making regarding how and why of doing mathematics is valued. In addition, Pat elicits an intellectual need for considering the potential mismatch of the information from students through a display of multiple representations – table and number sentences. In the second scenario, Teacher Pat has a richer anticipatory knowledge. It leads to broader student contribution, allows for the teacher to make sense of students' current thinking, and helps sustain engagement of students in the lesson.

Vignette 2 - Snapshot of a Classroom
ALGEBRA CLASSROOM

PAT stands in front of whiteboard, 36 students in 6 rows of 6 face the front in small desk-chairs. John and Sally problem is on the board. Shuffling of paper, scratching of pencils, whispers as students work.

Pat: Let's talk about this problem a bit. How did you think about the information in it?
Sam: Well, you gotta start by dividing 38 by 3. Then take away-
Pat: *(Interrupting)* Wait! Before you tell us about the calculations you did, explain to us why you did what you did. What were you trying to find?
Sam: Well, you know that John is 3 times as old, so you divide 38 by 3 to find out how old Sally is.
Pat: Do you all agree with Sam's thinking?
Several students say, "Yes"; others nod their heads.
Lia: That's not going tell you how old Sally is *now*. It'll tell you how old Sally is when John is 38.
Pat: Is that what you had in mind, Sam?
Sam: Yes.
Pat: *(To the class)* What does the 38 stand for?
Lia: John's age in the future.
Pat: So 38 is not how old John is now. It's how old John will be in the future. *(Pat starts a table on the board, as discussion continues, he adds to it)*

	Future	
John	38	

The problem says that when John gets to be 38 he will be 3 times as old as Sally. Does that mean "3 times as old as Sally is *now*" or "3 times as old as Sally will be when John is 38?"
Several students respond, "When John is 38."
Jonah: Couldn't you just say John is 21? *(Pause)* Couldn't you just multiply 3 times 7?
Pat: What will that give you?
Jonah: Twenty-one!
Pat: But what would 21 represent? What is it that's 21?
Jonah: That's how old John is now. Isn't that what we want to find?
Maura: No! *(Pause)* I mean, yes! That's what we want to find, but that's not right!
Pat: What is it that is not right, Maura? We do want to find out how old John is now, don't we?
Maura: Right. But see, he's not 3 times older than Sally now. He'll be 3 times older than Sally *when he is 38*. You have to keep track of what's true now and what's so in the future.*(Pat adds to the table)*
Pat: Okay, so how are we going to use the information that John will be 3 times as old as Sally when he gets to be 38? *(Pause)* Who can explain?
Sam: You can divide 38 by 3 and get 12.66....
Pat: Remember to tell us what your numbers stand for. What does the 12.66… stand for?
Gina: That's how old Sally will be.
Pat: When? *(Several respond, "When John is 38.")*
Pat: (Looking around) Let's keep going. Nasir?
Nasir: Okay, you can say that Sally will be 12ish. So if you subtract 7 from that you get 5. Then you take away 5 from 38 and get John is 33. Done!
Pat: Wait a minute, too fast! Explain your reasoning.
Nasir: (Patiently) You know Sally will be 12 and something, and you know that she is 7 now. So that means that there are 5 years between now and then. Actually a little more than 5 years.
Pat: So 5 years is how much time there is between now and the time in the future when John is 38?
Nasir: Yes. So if you take 5 away from 38, that's how old John is now.
Pat: Did everyone follow that? (Pause) Who will recap the solution we've just been through?

Figure 4. Vignette representing Teacher Pat's classroom instruction in sixth year of teaching.

Discourse Knowledge. In Vignette 1, Teacher Pat foregrounds the correct answer and a single path to that answer. That is, the primary discourse (little "d") in the classroom is largely univocal: Pat's utterances to identify a correct procedure. Discourse (big D) is also centered with the teacher, as the explanations valued in the classroom are Pat's. In Vignette 2, Pat asks students to "explain to us why you did what you did, what were you trying to find?" To be able to participate in discourse (little "d"), responding students have been asked explicitly to offer their own thinking to provide a convincing argument. Such eliciting questions by Pat are evidence of an intention to build a particular socio-mathematical norm. An aspect of the Discourse, then, is that engaging in deep explanation is an expectation of all in the classroom. Further evidence is Pat's contribution of a table to organize information as well as in the final question asked. While Pat's voice is first to offer the table, the utterances in the room are dialogic, not univocal.

Implementation Knowledge. In Vignette 1, Teacher Pat implements *choice* and *product* questions. If these questions dominate a teacher's contributions to discourse, then multiple disconnected *I•R•F* interactions can yield a teacher-regulated kind of interaction that does not include deep participation by students. This can be true even in inquiry-based instruction (Nassaji & Wells, 2000; Wertsch, 1998). This is evident from Pat's responses and questions in which the focus is on the steps of the computation rather than reasoning. There is no student-to-student interaction and when Pat overhears Jackie's question to herself, Pat responds by correcting (staying to the left of Table 1). In the second vignette, Pat elicits student thinking. The environment of the classroom is interactive with students sharing their reasoning and questioning each other as Pat encourages them to make sense of each other's ideas. As students present their ideas, Pat emphasizes the process rather than the product. Pat also uses multiple modes of discourse, including a table and confirming questions in order to support the needs of various students. Pat ensures that the students use mathematical terminology and language as they present their solution and share their understanding.

Acknowledgement

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IMPLICATIONS OF REALISTIC MATHEMATICS EDUCATION FOR ANALYZING STUDENT LEARNING

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The primary goal of this work is to articulate a theoretical foundation based on Realistic Mathematics Education (RME) that can support the analysis of student learning, both individual and collective, by documenting changes in local activity. To do so, I will build on previous work on the analytic implications of the Emergent Perspective, specifically Rasmussen and Stephan's (2008) analytic approach to documenting the establishment of classroom mathematical practices. The Emergent Perspective is broadly consistent with RME, but the existing analytic methods related to the Emergent Perspective fail to draw on the theoretical constructs provided by RME. For instance, current analytic methods fail to draw on the RME Emergent Models heuristic to inform the analysis of the development of mathematical practices related to models of/for student mathematical activity. Here I will be explicitly considering the roles that RME constructs could play in analytic processes consistent with the Emergent Perspective.

Key Words: Realistic Mathematics Education, Learning, Mathematizing, Emergent Models, Analytic methods

As I worked to investigate the implementation of an inquiry-oriented abstract algebra curriculum (Johnson & Larsen, 2012; Johnson, 2012) I became increasingly aware of the need for a comprehensive approach to making sense of student learning in this context. In particular, since the curriculum was designed based on Realistic Mathematics Education (RME), it was apparent that analyses of student learning should draw on the theoretical constructs that comprise RME. However, the existing formal analytic approaches do not do so, and the current formulations of RME are not articulated in a way to support the development of such approaches.

In order to articulate RME in such a way that supports new analytic techniques, I will coordinate the various RME constructs involving levels (of generality) of activity. For example, the Emergent Models design heuristic features a transition from what is called referential activity to general activity (Gravemeijer, 1999). I will look at the role of vertical and horizontal mathematizing (Gravemeijer & Doorman, 1999) in such a transition and consider how chains of signification (Gravemeijer, 1999) can facilitate this transition on the local scale.

Based on the Emergent Perspective (Cobb, 2000), Rasmussen and Stephan (2008) have developed an analytic scheme to document the development of classroom mathematical practices. Here I will consider how such an analytic scheme may be refined or expanded by drawing explicitly on various RME design heuristics related to shifts in levels of activity.

An RME Characterization of Student Learning

From an RME perspective it makes sense to conceive of learning as the creation of new mathematical realities. RME is an instructional design theory that is grounded in the belief that formal mathematics can be developed by engaging in mathematical actives, where these activities serve to progressively expand students' common sense. As described by Gravemeijer (1999), "what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality" (p. 156).

The students' experiential reality includes what the students can access on a "commonsensical level".

> "Real" is not intended here to be understood ontologically (whatever ontology may mean), therefore neither metaphysically (Plato) nor physically (Aristotle); not even, I would even say, psychologically, but instead commonsensically as … meant by the one who uses the term unreflectingly. It is not bound to the space-time world. It includes mental objects and mental activities. (Fredunethal, 1991, p. 17)

Therefore, the problem context for RME based curriculum need not be "real" in the sense that the students would access such scenarios in their everyday life. Instead, the students only need to be able to access the problem context on an intuitive level. In this way a magic carpet can be understood as experientially real, even though it is not physically real. As such, a context based on the movements of a magic carpet may form the foundation for the reinvention of formal mathematics (Wawro et al., 2012).

Within an experientially real context, RME based curriculum presents instructional tasks that promote mathematizing the problem context. This activity of mathematizing, "which stands for organizing from a mathematical perspective" (Gravemeijer & Doorman, 1999, p. 116), is a central mathematical activity in RME based curriculum and can be used to explain the learning process.

> In this view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. (Gravemeijer, 1999, p. 158)

It is through this cycle of progressive mathematizing, between mathematizing reality and mathematizing mathematical activity, that students reinvent mathematics by expanding their mathematical reality.

Initially, as students mathematize their experiential reality, they are engaging in horizontal mathematizing. Horizontal mathematizing includes activities such as translating, describing, and organizing aspect of problem context into mathematical terms (Gravemeijer & Doorman, 1999, p. 116-117). For instance, students may be asked to describe and devise a set of symbols for the symmetries of an equilateral triangle (Larsen, 2012). In this way, "horizontal mathematisation leads from the world of *life* to the world of *symbols*" (Fredunethal, 1991, p. 41). The artifacts of horizontal mathematizing include inscriptions, symbols, and procedures.

While horizontal mathematizing is a crucial step in the reinvention process, reinvention "demands that the students mathematize their own mathematical activity as well" (Gravemeijer & Doorman, 1999, p. 116-177). Vertical mathematizing characterizes activities through which students mathematize their own mathematical activity and may include generalizing, defining, and algorithmatizing (Rasmussen et al., 2005). For instance, for an RME inspired differential equations course, Rasmussen et al. (2005) describe a scenario in which students are first asked to approximate the number of fish in a pond given various initial input values. This task resulted in the students generating inscriptions during their initial horizontal mathematizing (i.e. tables and graphs that recorded their results). The students were then asked to reflect on their previous work in order to generate an algorithm by which approximations may be found regardless of the initial constraints. This task necessitates that the students engage in vertical mathematizing, as the students needed to mathematize their previous mathematical activity. Further, the algorithm that was generated through this vertical mathematization is now available to the students for further mathematization. In that sense, an artifact of vertical mathematization can become part of the students' expanding mathematical reality.

Therefore, student learning can be understood as the incorporation of new mathematics into the student's expanding mathematical reality, where this reality expands through the process of progressive mathematization. As Gravemeijer and Doorman (1999) state, "it is in the process of progressive mathematization - which comprises both the horizontal and vertical component - that the students construct (new) mathematics" (p. 116 – 117).

Notice that, as students' activity shifts from horizontal to vertical mathematizing, there is a shift in the generality of the student activity. Initially, horizontal mathematizing is limited to the specific problem context. As students transition to vertical mathematizing, this specific problem context is no longer the focus of the activity, rather the students mathematize their own mathematical activity to support their reasoning in a different or more general situation. In this way, vertical mathematizing may involve activities such as abstracting, generalizing, and formalizing (Rasmussen et al., 2005). One way that the RME guides the design of instruction intended to support these shifts in generality is through the emergent models heuristic.

Emergent Models

The RME Emergent Models instructional design heuristic is meant to promote the evolution of formal mathematics from students' informal understandings through the development of models (Gravemeijer, 1999). Models are defined as "student-generated ways of organizing their activity with observable and mental tools" (Zandieh & Rasmussen, 2010, p. 58).

Concepts that first emerge as a *models-of* student activity become *models-for* more formal activity. Gravemeijer (1999) describes four layers of activity. Initially student activity is restricted to the *task setting*, where their work is dependent on their understanding of the problem setting. *Referential* activity develops as students construct models that refer to their work in the task setting. *General activity* is reached when these models are no longer tied to the task setting. Finally, *formal activity* no longer relies on models. In regards to these four levels of activity, the shift from *model-of* to *model-for* occurs as students shift from *referential activity* to *general activity*.

It is at this shift between referential and general activity that the model transitions from the result of mathematizing the problem context into an object which itself can be the basis for further mathematizing. On the global scale, transitioning between any two levels of activity can be interpreted as the result of vertical mathematizing. (Whereas activity within a single level of activity can be understood as horizontal mathematizing.) In particular, the activity that supports the transition between a *model-of* to a *model-for* is a particularly significant example of vertical mathematizing as the activity shifts from referential to general.

Gravemeijer (1999) concedes that while the "model" is a global overarching concept, in practice the "model manifests itself in various symbolic representations" (p. 170). The construct of a *chain of signification* provides one way to describe changes in the symbolic representation of the model during the instructional sequence. Central to the chain of signification construct is the idea of a sign, which is made up of a signifier (a name or symbol) and the signified (that which the signifier is referencing, such as the students' activity). As the chain builds, previous signs can become the signified in subsequent signs. In this way, student activity can become an object that a signifier references. In this way, a chain of signification accounts for reification on a local scale. These local changes then support the reification of the global model.

While a chain of signification looks at reification on a local scale, record-of/tool-for serves as a way to understand how result of an activity (a record-of) is used in further mathematics (tool-for). As described by Larsen (2004), an inscription representing students' mathematical activity transitions from a *record-of* to a *tool-for* when the students use the notational record to achieve subsequent mathematical goals. Therefore, instead of focusing on the relationships between the students emerging symbols and notations (as with chains of signification), the record-of / tool-for construct focuses on changes in how the emerging symbols and notations are used.

Gravemeijer (1999) notes that, " the shift from model of to model for is reflexively related to the creation of a new mathematical reality" (p. 175). On the one hand, the transition from a model-of to a model-for reflects a transition of the model as a product of student activity to an instrument for supporting more formal mathematical reasoning. Therefore, the of/for transition serves to expand the mathematical reality. On the other hand, the creation of a new mathematical reality (as understood as local shifts from record-of activity to tool-for further mathematizing) aids in the transition of the global model. As a result, the emergent model construct offers a promising starting point when trying to document the development of a new mathematical reality.

Implications for Analyzing Student Learning

If an instructional sequence were designed to promote the reinvention of a concept by way of an emergent models transition, then one would want an analysis of students' learning to draw on the emergent models construct. So if the goal is to promote a shift from model-of to model-for, then an analysis of students' activity should explicitly draw on theoretical constructs related to such shifts. Here I will consider Rasmussen and Stephan's (2008) analytic framework for documenting the development of classroom mathematical practices (Cobb, 2000) before presenting a revised methodology that explicitly draws on the emergent model construct.

The Emergent Perspective is a framework for analyzing individual and collective mathematical activity in classroom settings. Cobb (2000) describes the Emergent Perspective as an "interactionist perspective on communal classroom processes and a psychological constructivist perspective on individual students' activity as they participate in and contribute to the development of these collective processes "(p. 321). Generally speaking the Emergent Perspective and RME are consistent, as "both content that mathematics is a creative human activity and that mathematical learning occurs as students develop effective ways to solve problems and cope with situations. Further, both propose that mathematical development involves the bringing forth of a mathematical reality" (p. 317). One analytic methodology situated within the Emergent Perspective, and used to analyze student learning in RME based contexts, is Rasmussen and Stephan's (2008) process for documenting of the development of classroom mathematical practices.

Rasmussen and Stephan (2008) define a classroom mathematical practice as "a collection of as-if-shared ideas that are integral to the development of a more general mathematical activity" (p. 201). The example presented by the authors was the classroom mathematical practice of "creating and organizing collections of solution functions" (p. 201), which entails four related taken-as-if-shared ways of reasoning about graphs and functions. Rasmussen and Stephan's (2008) methodology for documenting the development of classroom mathematical practices can be understood as a process for identifying changes over extended classroom sessions, where these long term changes are identified by looking for local shifts in the classroom's normative ways of reasoning.

To document the development of such a classroom mathematical practices (and by necessity the normative ways of reasoning that comprise them), Rasmussen and Stephan outlined a three-phase approach based on the idea that "learning is created in argumentation" (p. 197). Accordingly, Rasmussen and Stephan's methodology documents the evolution of collective argumentation by tracking the claims, data, warrants, and backings provided during classroom discussions.

Rasmussen and Stephan (2008) identified two local shifts in classroom argumentation that signify that a mathematical idea has become a normative way of reasoning. First, if arguments no longer require warrants or backing by the community, then the mathematical idea is considered to be taken-as-if-shared. Second, if any part of an argument (data, claim, warrant, backing) is used in a different way in a new argument and is unchallenged, then the mathematical idea that shifted positions is considered to be taken-as-if-shared.

I propose that, when documenting the development of classroom mathematical practices related to *models of/for* student mathematical activity, one can similarly look for local changes in the students emerging symbols and notations. These local changes can either be 1) in the relationships between the students emerging symbols and notations, as described by the chains of signification construct, or 2) in how the emerging symbols and notations are used, as described by the record-of/tool-for construct. Here I will briefly present an example of each by drawing on an inquiry-oriented abstract algebra curriculum (Larsen, Johnson, & Bartlo, 2012) and discuss how each can be seen an analogous to aspects of Rasmussen and Stephan's (2008) methodology.

The abstract algebra curriculum launches in the context of symmetries of an equilateral triangle. Initially, the students begin by physically moving a triangle in order to identify the six symmetries of an equilateral triangle. The students are then asked to represent these six symmetries with a diagram, a written description, and a symbol (see figure 1). This set of inscriptions can be thought of as a signifier that signifies the students' activity of manipulating the triangle. The students are then asked to generate a new set of symbols, this time representing each symmetry in terms of a vertical flip, F, and a 120° clockwise rotation, R. This new set of symbols represents the next step in the chain of signification, with the earlier sign "sliding under" this subsequent sign. The original sign, which was comprised of both the students' initial signifier (i.e. their initial inscriptions) and the original signified activity (i.e. physically manipulating the triangle), is now signified by this new set of symbols. So as the chain of signification builds, students no longer need to directly consider the original activity of manipulating the triangle. For example, when working with symbols expressed in terms of F and R, students may no longer need to keep in mind that they refer to motions of a triangle. In this way, one sign "sliding under" to become signified in a subsequent sign can be seen as analogous to the dropping of warrants and backings in Rasmussen and Stephan's (2008) argumentation methodology – as in both cases an idea/inscription that used to be seen as necessary drops away (but can be retrieved if needed).

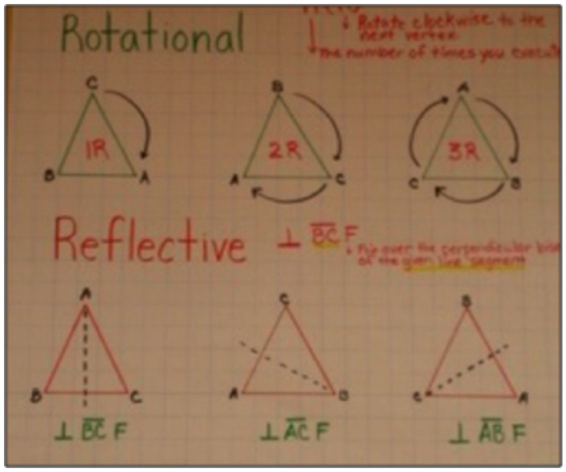

Figure 1. Pictures and Initial Symbols for the Symmetries of an Equilateral Triangle

Once the classroom develops a common set of symbols using *F* and *R*, the students are asked to consider any combination of two symmetries. An operation table initially emerges as a *record-of* the students' activity. Later, as the students argue that the identity element of a group must be unique, some students draw on the operation table as a *tool-for* constructing a proof (for a full description see Larsen, 2009). So, an inscription that first served to record the students' mathematical activity later served as a tool for subsequent mathematical activity. I argue that this shift is analogous to that represented by a claim from one argument becoming a warrant in a later argument (Rasmussen and Staphan's (2008) second criteria). In each case, the role of the idea/inscription changes in an important way – specifically the role of the idea/inscription shifts from being a product (e.g., the claim of an argument or a record) to becoming an instrument (e.g., the warrant of an argument or a tool).

Thus, like Rasmussen and Stephan (2008), I propose documenting local shifts in order to accumulate evidence of global transitions. Further, the local shifts I propose for documenting the emergent models transition can be understood as analogous to the kinds of shifts in argumentation that Rasmussen and Stephan took as evidence for the development of classroom mathematical practices. Therefore, I see this work as a generalization of the principles that underlie their method and hence as a coherent starting point for integrating the constructs of RME with the analytic framework provided by the Emergent Perspective.

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ADAPTING MODEL ANALYSIS FOR THE STUDY OF PROOF SCHEMES

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Model analysis is a quantitative research method used in physics education research; we have adapted this method to study proof schemes in a transition-to-proof course. Model analysis accounts for the fact that students may hold more than one idea or conception at a time, and may use different ideas and concepts in response to different situations. Model analysis is uniquely suited to study students' proof schemes, as students often hold multiple, sometimes conflicting proof schemes, which they may use at different times. Model analysis treats each student's complete set of responses as a data point, rather than treating each individual response as a separate data point. Thus, model analysis can capture information on the self-consistency of a student's responses. We collected data from a transition-to-proof course at the beginning and the end of the Fall 2012 semester. We find that model analysis offers significant insights not offered by traditional analysis.

Key words: Mathematical Proofs, Proof Schemes, Model Analysis, Transition to Proofs

Introduction and Background

The idea of proof is central to mathematics, but the ability to write (or even to read) proofs is notoriously difficult for students to develop. One barrier to student success in proof writing is a poor concept of what makes a mathematical proof. Several researchers have developed taxonomies to describe the various notions of proof held by students; perhaps the most extensive is Harel and Sowder's (1998, 2007) taxonomy of proof schemes. Proof schemes describe the type of argument an individual (or group) finds to be a convincing mathematical proof. This taxonomy has been successful in helping researchers to understand one source of students' difficulty with mathematical proof: when students have inadequate proof schemes, they have difficulty in writing, or even understanding, mathematical proof (Recio & Godino, 2001; Housman & Porter, 2003; Zazkis & Liljedahl, 2004).

While current research has successfully used the taxonomy of proof schemes to understand some of the difficulties that individuals have in writing proofs, Harel and Sowder (2007) have asserted that proof schemes can develop at the level of a group or community; for instance, the community of learners in a given classroom. To our knowledge, no current research has examined proof schemes at the community level.

In this paper we will discuss how we adapted the method of model analysis, a technique pioneered by physics education researchers Bao and Redish (2001), to analyze proof schemes. Model analysis treats each student's complete set of responses as a data point, rather than treating each individual response as a separate data point. Thus, model analysis can capture information on the self-consistency of a student's responses. This allows the data to be analyzed with the understanding that students do not always use the same conceptions in response to every situation. This point is particularly important for analyzing proof schemes, as students often use different proof schemes to understand or write proofs in different contexts.

Our research question is the following:

What insights can model analysis provide into community-level proof schemes that traditional quantitative analysis cannot?

Review of Relevant Literature

We primarily draw from three pieces of research: The first is Harel and Sowder's taxonomy of proof schemes, the second is the findings of of Stylianou, Blanton and Rotou (in press), who have recently completed a large-scale quantitative analysis of the views of proof held by students before taking a transition-to-proof course, and whose questionnaire we have used in our research. Finally, we describe Bao and Redish's method of model analysis, which we use to analyze our data.

Proof Schemes

Harel and Sowder have developed a framework for describing students' overall conception of what justifies a mathematical proof, which they refer to as a proof scheme. The original framework (Harel & Sowder, 1998) is one of several understandings of proof discussed by Balache (2002), and is perhaps the most extensive of such taxonomies. This framework of proof schemes has been used in several studies (Recio & Godino, 2001; Housman & Porter, 2003; Zazkis & Liljedahl, 2004) as a taxonomy for students' conceptions of their justification for proof. Harel and Sowder define proof scheme with the following: "A person's (or community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" (Harel & Sowder, 2007). Harel and Sowder's taxonomy consists of seven major types of proof scheme, organized into three broader categories: external conviction, empirical, and deductive. External conviction proof schemes are possessed by students who are convinced a theorem is true by external forces; empirical proof schemes describe the proof schemes of students who are convinced by evidence, rather than logical reasoning; deductive proof schemes construct and validate theorems by means of logical deductions. An individual's proof scheme may not consistently fit into a single category. Rather, the category of proof scheme can depend on context, and different contexts may activate different categories of proof scheme.

Views of mathematical proof held by early undergraduate students

Stylianou, Blanton, and Rotou (in press) have completed a large-scale analysis of the views of proof held by 535 early undergraduate students, and how those views relate to their attitudes and beliefs towards proof. The part of their study which is most relevant to this paper is a multiple-choice instrument distributed to the 535 participants, which included four mathematical statements and, for each statement, four arguments purporting to prove the statement. The four arguments were designed by the authors to invoke either empirical proof schemes (with numeric or visual presentation) or deductive proof schemes (with narrative or symbolic presentation), with one of each of these four presentations for each statement. For each statement, students were asked to choose the argument that they thought would be the closest to their own approach, the argument they thought was the most rigorous, and the argument they would use to explain the problem to a peer.

Stylianou et al found that the students' choice of the argument closest to their own approach often did not match the types of arguments students actually constructed in interviews. They did, however, find in follow-up interviews that students choice of the argument closest to their own approach often indicated what the student felt was the "best" proof, that they would ideally construct on their own. They also found that over half of the students identified the deductive-symbolic argument as the most rigorous proof for each of the four conjectures, while less than one-fifth chose either empirical argument as the most rigorous. Finally, for all four conjectures, the majority of the students did not choose a deductive argument as the most explanatory.

Model Analysis

We will use model analysis, a quantitative analysis technique developed by Bao and Redish (1999; 2001) for analyzing students' concepts of force in physics education research. According to Bao and Redish (2001), "The method is especially valuable in cases where qualitative research has documented that students enter a class with a small number of strong naive conceptions that conflict with or encourage misinterpretations of the scientific view." Michael Oehrtman (2006) has successfully used model analysis to analyze students' concepts of functions in first-year calculus courses.

The term model, as used by Bao and Redish, refers to a mental model of a particular concept, including ideas, conceptions, and beliefs about the concept. Proof schemes, then, can be thought of as a "model" of the concept of proof. Model analysis will be detailed in the next section, where we describe how it will be adapted for our project.

Bao and Redish describe the method of model analysis as consisting of five steps (2006). These steps, and how they were adapted for our project, are as follows:

> (i) Through systematic research and detailed student interviews, common student models are identified and validated so that these models are reliable for a population of students with a similar background. (Bao & Redish, 2006)

This step describes what Harel and Sowder accomplished when creating their taxonomy of proof schemes (Harel & Sowder, 1998). These categories were used as the "models" described by Bao and Redish.

> (ii) This knowledge is then used in the design of a multiple-choice instrument. The distracters are designed to activate the common student models, and the effectiveness of the questions is validated through research. (Bao & Redish, 2006)

For this step, we use the questionnaire developed by Stylianou and Blanton (in press). The effectiveness of these questions has been validated by their research. The questionnaire present four theorems, and for each theorem, asks students to choose from four different arguments which they believe to be the closest to their own approach, which they consider to be the most rigorous, and which they consider the most explanatory. Stylianou and Blanton categorize each of these responses with a proof scheme (deductive or empirical) and a style (symbolic or narrative for deductive proofs, visual or numeric for empirical proofs) for a total of four types. We have modified the questionnaire slightly by allowing a response of "None of the above" to some questions, but this alteration should not invalidate the effectiveness of the questions.

> (iii) One then characterizes a student's responses with a vector in a linear "model space" representing the (square roots of the) probabilities that the student will apply the different common models. (Bao & Redish, 2006)

The model space described in this step is represented mathematically by a linear vector space, where each common model is represented by an element of an orthonormal basis. That is, each of the categories of proof scheme (empirical and deductive) will be assigned a dimension in the vector space. We also assigned (as do Bao and Redish) a third dimension to a "null" model, considered to be activated when students choose a response of "none of the above." For each student, we created a vector inside this model space that represents the student's responses to the questionnaire. Each entry in the vector is meant to represent the probability with which the student uses the associated category of proof scheme to respond to similar types of questions. Of course, these probabilities can only be approximated by the student's responses to the questionnaire.

> (iv) The individual student model states are used to create a “density matrix," which is then summed over the class. The off-diagonal elements of this matrix retain information about the confusions (probabilities of using different models) of individual students. (Bao & Redish, 2006)

For each model state vector, a density matrix is created by taking the outer product of the model state vector with itself. The diagonal entries of the density matrix are simply the probabilities calculated in step (iii). The off-diagonal entries are non-zero only when a student has responses from more than one category of proof scheme. Thus, when a student is inconsistent, those inconsistencies are preserved by the off-diagonal entries of the matrix.

The off-diagonal entries are considered ‘high’ or ‘low’ relative to the corresponding diagonal entries. Letting a_{12} represent the off-diagonal entry corresponding to model 1 and model 2, and letting a_{11} and a_{22} represent the corresponding diagonal entries, the statistic $\frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}}$ will be referred to as the *mixed model statistic* for these two models. Bao and Redish suggest that when this statistic is greater than 0.5, the level of inconsistency is significant relative to the diagonal entries.

To study a large number of data points, the density matrices will be averaged together: that is, the entries in each position are added together and divided by the total number of data points. Bao and Redish refer to the resulting matrix as the class density matrix.

> (v) The eigenvalues and eigenvectors of the class density matrix give information not only how many students got correct answers, but about the level of confusion in the state of the class's knowledge. (Bao & Redish, 2006)

The class density matrix contains information on the students' responses to the questionnaire. An eigenvalue decomposition allows for trends in the data to be identified. In this way, the eigenvalue decomposition allows for information about the class as a whole to be extracted from the data. That is, the community-level proof scheme held by the class can be identified.

An Illustrative Example

At this point, it will be helpful to demonstrate the method of model analysis through an example. We will also highlight an advantage of model analysis over traditional statistics: by treating each student’s entire set of responses as a single data point, model analysis allows us to differentiate between two possible extremes.

Suppose that 50% of students’ responses to a question are empirical, and 50% are deductive. If each response is considered as a separate data point, there is no way to differentiate between two possible extremes: first, that 50% of the students consistently chose empirical responses, where the other 50% consistently chose deductive; second, that *all* of the students chose empirical responses 50% of the time and deductive 50% of the time. In the first situation, each student is very consistent; in the second, each student is very inconsistent.

When we create model space vectors for each student, in step (iii) described above, we will use the first entry to represent the “none of the above” dimension, the second for “empirical”, and the third for “deductive.” In the first situation, half of the model state vectors will be $[0,1,0]$, the other half will be $[0,0,1]$. The resulting class density matrix, step (iv), will be:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

The diagonal entries represent the probabilities of selecting each model (0, 50% and 50% respectively), while the off-diagonal entries of 0 indicate that there is no inconsistency in students' responses.

The eigenvalues and eigenvectors of this matrix, step (v), are:

$\lambda_1 = 0, \mathbf{v_1} = [1,0,0]$

$\lambda_2 = 0.5, \mathbf{v_2} = [0,1,0]$

$\lambda_3 = 0.5, \mathbf{v_3} = [0,0,1]$

The two equal eigenvalues of 0.5 indicate that there are two distinct groups of students of equal size, one with model space vectors purely in the empirical direction, and one purely in the deductive direction.

By contrast, in the second situation, where all of the students are inconsistent, every student will have a model space vector of $[0,0.7071,0.7071]$. The resulting class density matrix is:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

In this class density matrix, the off diagonal entries are large relative to the diagonal entries, showing a high level of inconsistency.

The eigenvalues and eigenvectors of this matrix are:

$\lambda_1 = 0, \mathbf{v_1} = [1,0,0]$

$\lambda_2 = 0, \mathbf{v_2} = [0,0.7071,-0.7071]$

$\lambda_3 = 1, \mathbf{v_3} = [0,0.7071,0.7071]$

In this situation, there is only one large eigenvalue, indicating that all of the students have the same model space vector of $[0,0.7071,0.7071]$. Squaring these entries reveals that every student selected an empirical response 50% of the time and a deductive response 50% of the time.

Methods

Our study collected data at the beginning (pre-instruction) and at the end (post-instruction) of the Fall 2012 semester in a transition-to-proof class. We used the same multiple-choice instrument developed by Stylianou et al to collect data, with the slight modification of allowing students to choose "none of the above" rather than one of the four arguments. See the Appendix for one representative of the four mathematical statements and possible responses for that statement. A total of 38 students participated in both the pre- and post-instruction surveys. We used model analysis in addition to traditional statistical analysis in order to capture information on students' self-consistency.

Results

We first present a traditional statistical analysis of our data. Table 1 shows the number and percentages of each of the four categories of response:

Response Type	Solution chosen as "closest to your approach"	Solution chosen as "most rigorous"	Solution chosen as "most explanatory"
Pre:Instruction:			
None	1 (0.66%)	0 (0.00%)	3 (1.97%)
Empirical	51 (33.55%)	26 (17.11%)	72 (47.37%)
Deductive	100 (65.79%)	126 (82.89%)	77 (50.66%)
Post Instruction:			
None	1 (0.66%)	3 (1.97%)	1 (0.66%)
Empirical	12 (7.89%)	25 (16.45%)	62 (40.79%)
Deductive	139 (91.45%)	124 (81.58%)	89 (58.55%)

Table 1: Number and percentages of each category of response (pre- and post-instruction) (n=38, 152 total responses for each question)

Our pre-instruction results are similar to those of Stylianou et al: more than 80% of the arguments chosen as the most rigorous were deductive, but fewer than 50% of the responses selected as the most explanatory were deductive. We must take the responses to "closest to your approach" as somewhat unreliable, since Stylianou et al report that these responses do not reflect the actual proofs constructed by students.

Our post-instruction results indicate some improvement. More students selected deductive arguments as both the most rigorous and the most explanatory. The differences between the pre- and post-instruction proportions of students choosing deductive arguments are significant at the $p = 0.01$ level for the question of "closest to your approach", and at the $p = 0.05$ level for "most explanatory." The difference for "most rigorous" is not significant, and in fact, the numbers of deductive responses are almost identical.

The analysis above, however, cannot tell us about students' self-consistency. We now present a model analysis of our data.

First, we present the pre- and post-instruction results for the questions in which students were asked to choose the "most rigorous" argument. Table 2 shows the class density matrices, with eigenvalues and eigenvectors, for the pre and post-instruction responses to these questions.

Pre-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1711 & 0.1990 \\ 0 & 0.1990 & 0.8289 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 1, \mathbf{v_1} = [1,0,0]$ $\lambda_2 = 0.1156, \mathbf{v_2} = [0,0.9632,-0.2687]$ $\lambda_3 = 0.8844, \mathbf{v_3} = [0,0.2687,0.9632]$
Mixed Model Statistic	$\frac{0.1990}{\sqrt{0.1711}\sqrt{0.8289}} = 0.5284$
Post-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0.0197 & 0.0180 & 0.0207 \\ 0.0180 & 0.1645 & 0.0566 \\ 0.0207 & 0.0566 & 0.8158 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 0.0173, \mathbf{v_1} = [0.9933,-0.1146,-0.0176]$ $\lambda_2 = 0.1614, \mathbf{v_2} = [-0.1126,-0.9896,0.0892]$ $\lambda_3 = 0.8213, \mathbf{v_3} = [0.0277,0.0866,0.9959]$
Mixed Model Statistic	$\frac{0.0566}{\sqrt{0.1645}\sqrt{0.8158}} = 0.1545$

Table 2: Results of model analysis for the question of "most rigorous."

The diagonal entries of the class density matrix correspond exactly to those found in Table 1. This is expected, as the diagonal entries represent the percentages of each type of response. The off-diagonal elements represent the levels of inconsistency; higher off-diagonal entries indicate that more students chose responses from both of the corresponding models. For example, the entry of 0.1990 in the pre-instruction class density matrix indicates higher levels of inconsistency between the empirical and deductive models, while the entry of 0.0566 in the post-instruction class density matrix indicates lower levels of inconsistency. Both here and in our subsequent results, we examine the *mixed model statistic* for the empirical and deductive models. In the pre-instruction class density matrix, this statistic is 0.5284, indicating significant, but not extremely high, levels of inconsistency. In the post-instruction class density matrix, this statistic is 0.1545, indicating consistency.

The eigendecompositions of the pre- and post-instruction class density matrices give us information about the model state vectors held by most of the students in the class. Larger eigenvalues correspond to eigenvectors that point in a direction (in the model space) similar to those held by larger subgroups of participants. Bao and Redish suggest that eigenvalues of 0.65 or higher represent strong primary eigenvalues, which in turn correspond to eigenvectors that indicate the model state vectors of a large subgroup of the students. Eigenvalues less than 0.4 represent secondary eigenvalues, whose associated eigenvectors represent the model state vectors of small subgroups.

Both the pre- and post-instruction results indicate strong primary eigenvalues associated with eigenvectors that point mainly in the deductive direction in the model space. The primary eigenvector of the pre-instruction results points slightly more in the empirical direction than that of the post-instruction results.

While the pre-instruction class density matrix shows that there is a `significant' amount of inconsistency between the empirical and deductive dimensions, the eigendecomposition shows that the empirical component of the primary eigenvector is relatively small compared to the deductive component.

It can be illustrative to plot these primary eigenvectors on a two-dimensional plane, in Figure 1 below. In this plot, we can safely ignore the dimension corresponding to 'none of the above,' and plot the empirical dimension on the horizontal axis and the deductive dimension on the vertical axis. Point P represents the primary eigenvector of the pre-instruction results, Point Q represents that of the post-instruction results. Each point is determined by $(\lambda x^2, \lambda y^2)$, where λ represents the primary eigenvalue, while *x* and *y* correspond to the empirical and deductive dimensions of the primary eigenvector, respectively. The *x* and *y* values are squared in order to represent the actual probabilities of selecting each model, rather than the square roots used when calculating the class density matrix. Therefore, the sum $x^2 + y^2 \approx 1$, since the component of the eigenvector representing "none of the above" is small. Multiplying by λ allows us to represent the strength of the eigenvalue as the distance from the origin.

The diagonal from (1,0) to (0,1) represents the maximum eigenvalue of 1, the diagonal from (0.4,0) to (0,0.4) represents the line below which eigenvalues are considered to be 'small'. The two line segments coming from the origin have slopes of 3 and 1/3; these separate the plot into three regions: consistently empirical on the bottom right, consistently deductive on the upper left, and a mixed model state in between. Figure 1 indicates that the primary eigenvector of the pre-instruction results is mainly in the deductive direction, with a relatively strong eigenvalue, but that the post-instruction primary eigenvector is almost completely in the deductive direction.

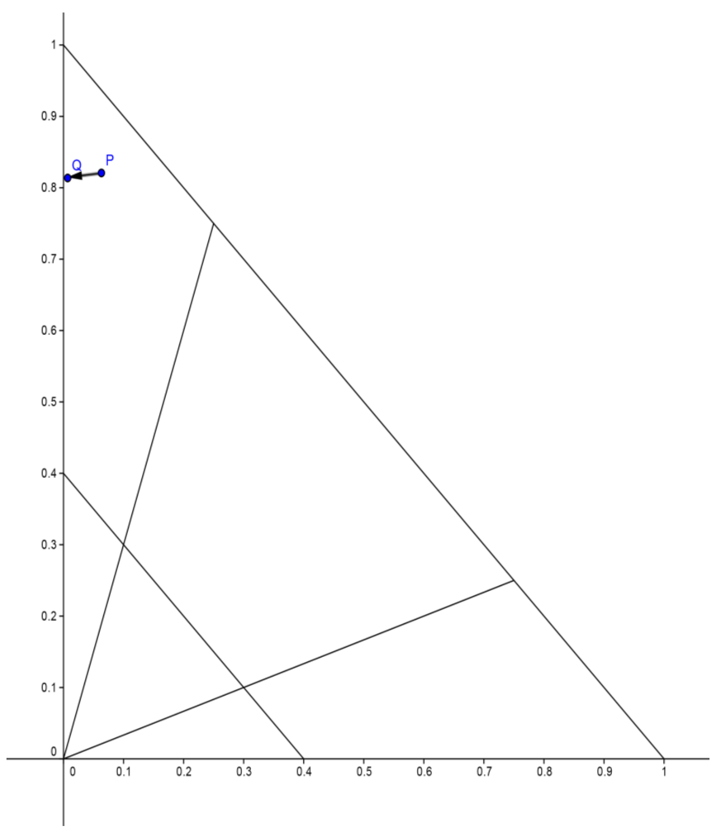

Figure 1: Plot of the primary eigenvalues for the question of "most rigorous."

The pre- and post- instruction results for the questions in which students were asked to choose the "most explanatory" argument are given in Table 3:

Pre-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0.0197 & 0.0246 & 0.0186 \\ 00246 & 0.4737 & 0.2457 \\ 0.0186 & 0.2457 & 0.5066 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 0.0183, \mathbf{v}_1 = [0.9988, -0.0457, -0.0151]$ $\lambda_2 = 0.2440, \mathbf{v}_2 = [0.0231, 0.7292, -0.6839]$ $\lambda_3 = 0.7377, \mathbf{v}_3 = [0.0423, 0.6828, 0.7294]$
Mixed Model Statistic	$\frac{0.2457}{\sqrt{0.4737}\sqrt{0.5066}} = 0.5016$
Post-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0.0066 & 0.0114 & 0.0000 \\ 0.0114 & 0.4079 & 0.2043 \\ 0.0000 & 0.2043 & 0.5855 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 0.0062, \mathbf{v}_1 = [0.9993, -0.035, 0.0122]$ $\lambda_2 = 0.2743, \mathbf{v}_2 = [0.0356, 0.8356, -0.5480]$ $\lambda_3 = 0.7195, \mathbf{v}_3 = [0.0088, 0.5485, 0.8361]$
Mixed Model Statistic	$\frac{0.2043}{\sqrt{0.4079}\sqrt{0.5855}} = 0.4180$

Table 3: Results of model analysis for the question of "most explanatory."

The pre-instruction mixed model statistic of 0.5016 indicates significant inconsistency between the two models. The eigendecomposition of the pre-instruction class density matrix shows a relatively strong primary eigenvalue of 0.7377, with an associated primary eigenvector which indicated a mixed model state, with the empirical component almost as large as the deductive component.

Post instruction, the mixed model statistic of 0.4180 indicates lower levels of inconsistency than the pre-instruction. The eigendecomposition of the post-instruction class density matrix shows a slightly weaker primary eigenvalue of 0.7195. The associated eigenvector, while still mixed, points more in the deductive direction than that of the pre-instruction results. Figure 2 shows the plot of the pre- and post-instruction primary eigenvectors labeled as points P and Q, respectively.

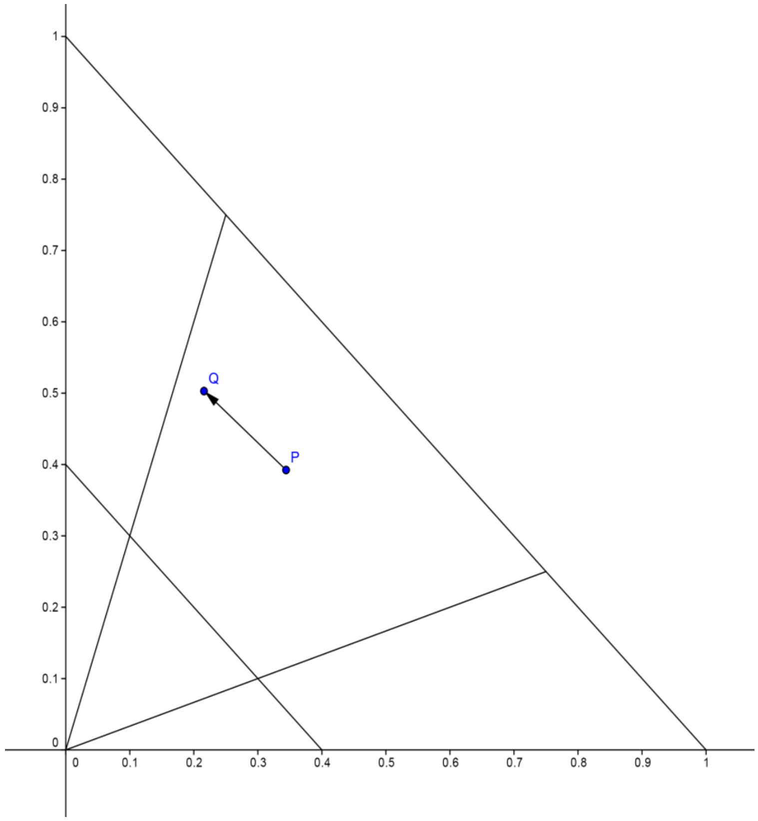

Figure 2: Plot of the primary eigenvalues for the question of "most explanatory."

Finally, the pre- and post- instruction results for the questions in which students were asked to choose the argument closest to their own approach are given in Table 4:

Pre-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0.0066 & 0.0066 & 0.0093 \\ 0.0066 & 0.3355 & 0.1855 \\ 0.0093 & 0.1855 & 0.6579 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 0.0064, \mathbf{v}_1 = [0.9998, -0.0142, -0.0102]$ $\lambda_2 = 0.2510, \mathbf{v}_2 = [0.0087, 0.9098, -0.4150]$ $\lambda_3 = 0.7426, \mathbf{v}_3 = [0.0152, 0.4148, 0.9098]$
Mixed Model Statistic	$\frac{0.1855}{\sqrt{0.3355}\sqrt{0.6579}} = 0.3948$
Post-Instruction:	
Class Density Matrix:	$\begin{bmatrix} 0.0066 & 0.0093 & 0.0066 \\ 0.0093 & 0.0789 & 0.0847 \\ 0.0066 & 0.0847 & 0.9145 \end{bmatrix}$
Eigenvalues and Eigenvectors:	$\lambda_1 = 0.0054, \mathbf{v}_1 = [0.9913, -0.1312, 0.0051]$ $\lambda_2 = 0.0716, \mathbf{v}_2 = [0.01310, 0.9863, -0.1002]$ $\lambda_3 = 0.9230, \mathbf{v}_3 = [0.0082, 0.1000, 0.9950]$
Mixed Model Statistic	$\frac{0.0847}{\sqrt{0.0789}\sqrt{0.9145}} = 0.3153$

Table 4: Results of model analysis for the question of "closet to your approach."

The pre-instruction mixed model statistic of 0.3948 indicates a low level of inconsistency. The eigendecomposition of the pre-instruction class density matrix shows a strong primary eigenvalue of 0.7426 associated with an eigenvector that mainly points in the deductive direction.

The post-instruction mixed model statistic of 0.3153 indicates lower inconsistency than pre-instruction. The eigendecomposition of the post-instruction class density matrix shows a very strong primary eigenvalue of 0.9230, with an eigenvector that points almost entirely in the deductive direction. Figure 3 shows the plot of the pre- and post-instruction primary eigenvectors labeled as points P and Q, respectively.

The question of "closest to your approach" deserves special attention. As Stylianou et al indicate, there are often large discrepancies between the arguments students choose as the closest to their own approach and the proofs they actually construct. However, if these students are indicating (as Stylianou et al suggest) that these are the arguments that they would ideally construct, we take these results to mean that students recognize that "ideal" proofs are deductive.

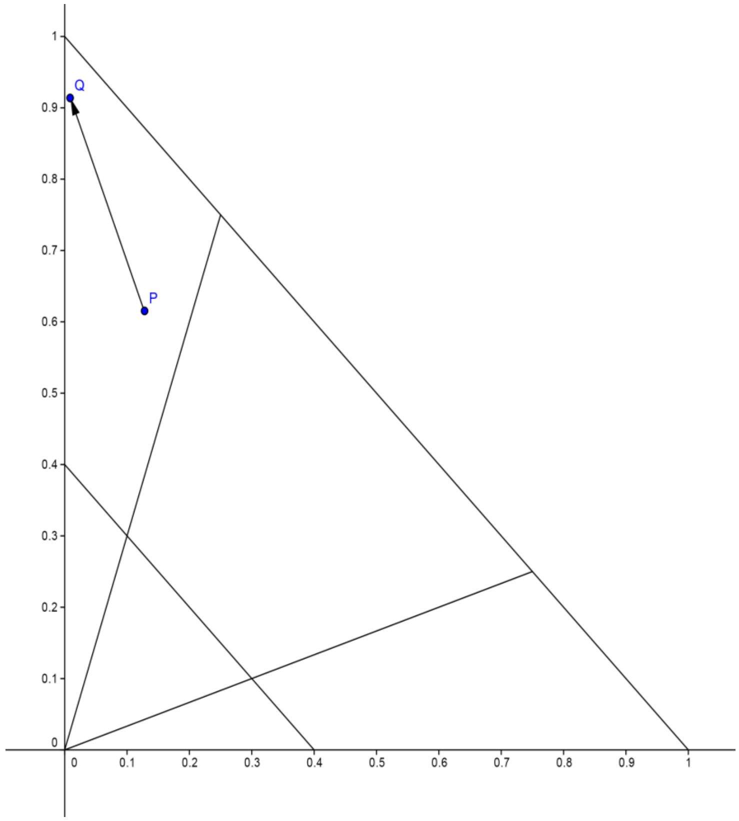

Figure 3: Plot of the primary eigenvalues for the question of "closet to your approach."

Discussion

Our results indicate that for both the questions of "most rigorous" and "closest to your approach," the majority of students' model state vectors are mainly in the deductive direction, even pre-instruction. The primary eigenvector for the results of the "most explanatory" question suggests that the majority of students have a mixed model state, incorporating both empirical and deductive models. For all three questions, the primary eigenvectors move more in the deductive direction post-instruction. We see this as a positive sign, especially for the question of "most explanatory," even though the primary eigenvector for this question remains mixed post-instruction.

The technique of model analysis provides some insights that the basic statistics of Table 1 do not grant, particularly in the cases where the differences between the number of empirical responses and the number of deductive responses are small. The pre-instruction results for the question of "closest to your approach" in Table 1 do not indicate whether roughly two-thirds of the class consistently choose deductive responses, or if the entire class chooses deductive responses two-thirds of the time, or something in between these two extremes. Our model analysis eigendecomposition for this question indicates that the truth is closer to the first extreme: a substantial majority of the class chose deductive responses most of the time for this question. The post-instruction model analysis for this question provides less insight; the basic statistics in Table 1 are so heavily weighted to deductive responses that it is unlikely that there could be any result other than that which was found: most of the students choose deductive responses almost exclusively.

The pre- and post-instruction results in Table 1 for the question of "most rigorous" are nearly identical. The model analysis for these results shows that there is, however, a shift in students' self-consistency: post-instruction, the class density matrix indicates lower levels of inconsistency. The eigendecomposition indicates that the primary eigenvalue is slightly lower than pre-instruction, but the primary eigenvector has a slightly stronger deductive component. This indicates that post-instruction, this group of students shank slightly but their responses became more consistently deductive. At the same time, the secondary eigenvalue indicates that a smaller group of students also became more consistent in choosing empirical responses.

The model analysis for the question of “most explanatory” shows a larger shift in the deductive direction than that for “most rigorous,” though the post-instruction primary eigenvector is still very mixed. The majority of students chose deductive responses with somewhat more consistency post-instruction.

Conclusion

The main claim of this paper is that the technique of model analysis can tell researchers about the self-consistency of student responses to the proof scheme items of Stylianou et al. We claim that this information about self-consistency is useful for gaining a more complete picture of students’ thinking.

Our model analysis of the data collected in this study allows for a more complete picture than the basic statistics presented in Table 1.The results of this study are encouraging: the majority of students entered the transition-to-proof class with a fairly self-consistent deductive model state for the question of “most rigorous,” and ended the course with an even stronger, more self-consistent, deductive model. However, the size of this majority group shrank slightly, as indicated by the slightly smaller primary eigenvalue. The results from the “most explanatory” question are less encouraging; there is not a consensus among the students that deductive models are the most explanatory. Nonetheless, our results do show improvement in the deductive direction post-instruction. For the question of “closest to your approach”, Stylianou et al point out that students do not have the mathematical tools to produce the type of proof that they indicate as the “closest” to the proof they would produce, however, it is promising that the majority of students chose deductive proofs as those they would try to produce, and that the model state vectors of most students became more consistently deductive post-instruction.

It is important to note that our data set is very limited, and that our results are based on a small sample size. However, our data does show that model analysis can differentiate between self-consistent and inconsistent student responses. We believe that model analysis provides a valuable perspective on data collected using Stylianou et al’s instrument. We currently plan to apply this method to a more robust data set collected by Stylianou et al.

We believe that model analysis provides a valuable addition to the quantitative tools used by researchers. Model analysis is particularly well suited to analyze community-level proofs schemes held by a group. Administering this instrument pre- and post-instruction can illuminate whether students in a class are moving in the deductive direction, the extent of that movement, and the extent to which students are self-consistent. The results of such a study can inform different teaching interventions to be implemented in order to help students to better understand and employ deductive proof techniques.

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Appendix

The following is one of the four mathematical statements used on the instrument. Students chose one of the four solutions (or a response of “none of the above”) for each of three questions:

1. Which **one** would be closest to what you would do if you were asked to solve the problem?
2. From the above solutions, choose the **one** that is the most “rigorous” (mathematically correct):
3. From the above solutions, choose the **one** you would use to explain the problem to one of your peers:

Problem: *For any integers a, b, and c, prove that if a divides b with no remainder, then a divides bc with no remainder.*

Solution A:

If one integer (call it the first one) divides another integer (the second one) without a remainder, then the first integer must be a factor of the second one. Thus, no matter what other third integer you multiply the second one by, the first integer will still always be a factor of that product. This means that if the first integer divides the second integer, then the first integer must divide the product of the second and third integers too.

Solution B:

Let a, b, and c be integers and suppose that a divides b with no remainder. Then b can be written as a multiple of a. That is, $b = ak$, where k is an some integer. Thus, $(b)c = (ak)c$. It follows that since a is a factor of bc, then a divides bc.

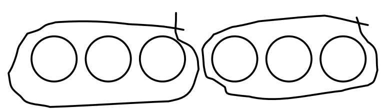
Solution C:

Let $a = 3$, $b = 9$ and $c = 7$. Then 3 divides 9 and 3 divides $9(7)$.
Let $a = 5$, $b = 10$ and $c = 34$. Then 5 divides 10 and 5 divides $10(34)$.
Let $a = 21$, $b = 126$ and $c = 1453$. Then 21 divides 126 and 21 divides $126(1453)$
Let $a = -12$, $b = 96$, and $c = -15$. Then -12 divides 96 and -12 divides $96(-15)$.

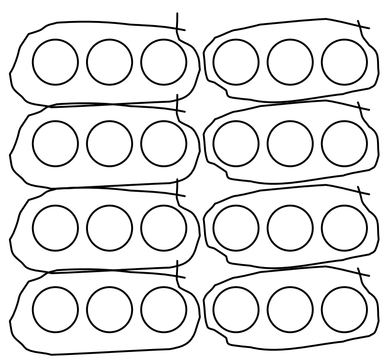
I randomly selected several different types of integers: high and low, positive and negative, prime and composite. Since I randomly selected and tested a variety of types of integers, and it worked in every case, I know that it will work for all integers. Therefore, if a divides b with no remainder, then a divides bc with no remainder.

Solution D:

Say a=3 and b=6
then b can be combined in 2 groups of 3:



Now, say c=4. Then, bc=24, and it
can be combined in 8 groups of 3:



As the picture illustrates, the counters can be divided into groups of 3 with no counter left over. Therefore, a divides bc with no remainder.

PAR FOR THE COURSE: DEVELOPING MATHEMATICAL AUTHORITY

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Perceived mathematical authority plays an important role in how students engage in mathematical interactions, and ultimately how they learn mathematics. This paper elaborates the concept of mathematical authority (Engle, 2011) by introducing two concepts: scope and relationality. This elaborated view is applied to a number of peer-interactions in a specialized peer-assessment context. In this context, self-perceived authority influenced the way feedback was framed (as either questions or assertions).

Key words: Authority, Calculus, Classroom Research, Formative Assessment

Introduction

Individuals who are proficient in mathematics need not rely on an external authority; using the logic of the discipline, mathematicians *know* when they are right. Helping students become mathematicians requires helping them develop such *authority* grounded in mathematical reasoning. In this paper I build on the concept of mathematical authority (Engle, 2011) in the context of undergraduate mathematics education.

According to Engle (2011) authority begins to develop when students are “authorized” to share what they “really” think, and is solidified when students develop into local “authorities,” based on how students are positioned in the social space of the classroom. Implicit is the idea that authority “belongs” to students, existing beyond the confines of a single classroom situation. In what follows, I address the context-dependent nature of authority, and how it may (or may not) be transferred between contexts.

While authority does have some bearing on how one solves problems individually, it becomes most relevant when we think of one’s mathematical interactions with others; mathematical authority is a relational construct. Moreover, as is the case with self-efficacy (Bandura, 1997), it is an individual’s *perceived* mathematical authority that determines how they engage in mathematical interactions, not their *actual* ability to make authoritative statements about mathematics. Thus, it is important to consider authority not as a static attribute, but rather as something that depends on how an individual situates oneself (and is situated by others) in a mathematical interaction (Boaler & Greeno, 2000).

To extend the notion of mathematical authority, I introduce two concepts: scope and relationality. Perceptions of authority operate at a number of levels, increasing in generality (scope): (1) a specific problem, (2) a topic, (3) a mathematical domain, (4) the domain of mathematics, and perhaps (5) academics in general. Generalized authority is more readily transferable than localized authority, but is also less robust. (While an individual with a PhD in analysis has generalized authority that would extend to all of mathematics, a few failures in abstract algebra would be more detrimental to local self-perceived authority than a few failures in a related area of expertise (e.g., Banach spaces), because of the individual’s topic-specific - and more robust - authority in analysis.)

The relational nature of mathematical authority plays out in interactions between multiple individuals (e.g., partner work, small-group work, or a class discussion). Relationally, mathematical authority depends on: (1) one’s self-perception of one’s own authority, and (2) one's perception of another's authority. While scope is related to how authority transfers *between* domains or topics, relationality determines how authority manifests *within* a domain or topic. In general, individuals will act as though they have greater authority in situations where they perceive others to have less authority. For instance, a mathematics student would be much more likely to point out a (perceived) mathematical error to a fellow classmate than to a teacher, because, relatively speaking, she sees herself as having greater *relative* authority compared to the other student than to the teacher. In other words, the student is more likely to "trust" herself when challenged by another student than the teacher, because of the relative perceptions of authority.

There are notable exceptions to this generality though, particularly for individuals with low or high (general) self-perceptions of mathematical authority. For individuals with low perceived authority, they will act as though they lack authority regardless of the perceptions of other individuals. Even if such an individual perceives others to have *even less* authority, because the individual's self-perceived authority is so low, she still will not act as though she has authority. Moreover, individuals often have limited knowledge of the mathematical expertise of others, and thus must make situate themselves based on assumptions about others. Individuals with low levels of self-perceived authority will likely assume themselves to have lesser authority than their peers.

In contrast, individuals with a high-level of self-perceived authority may be willing to challenge the authority of others even if they perceive the other individual to have greater general mathematical authority. For instance, a competent (and confident) student in mathematics may be willing to challenge the authority of someone who is perceived as having greater general authority (e.g., a teacher, or successful competitive mathematician), because she has great enough self-perceived authority that she is willing to trust her own mathematical deductions (at least in this particular topic or problem). One of the reasons that it becomes possible for the student to challenge an individual with greater perceived authority is that the challenge is with respect to a specific problem or topic (i.e. limited in scope), and not the domain of mathematics overall (which is more general). In this way, it is possible for the teacher's mistake in a given problem (localized in context) to remain consistent with her overall perceived (general) authority.

Engle (2011) describes a hypothesized trajectory for developing mathematical authority: (1) learners are authorized to share what they think, (2) are recognized as authors of those ideas, (3) become contributors to the ideas of others, and (4) develop gradually into authorities about something. Because authority is a relational construct, it makes sense that it would develop out of social interactions. As students are recognized as authors of their own work and contributors to the work of others, their authority in that context is increased. However, unless this burgeoning sense of authority is reinforced in other contexts, it is likely that individuals will attribute their authority to the nature of that context, rather than attributing it to themselves, and thus not internalize authority.

In general, authority emerges from developing expertise with a specific set of problems or topics. For instance, one might perceive oneself as an authority on differentiation with the chain rule. If authority remains localized to this specific context, then it is unlikely that the individual will develop generalized mathematical authority. However, if the individual also develops authority in integration, epsilon-delta proofs, etc., then the individual will begin to develop authority in the domain of calculus. Developing authority in other mathematics classes as well will result in further generalization. This emphasizes the need for students to develop authority in a broad scope of activities, if it is to be internalized, generalized, and become transferable.

Having elaborated the concept of authority, I now consider a specialized peer-assessment activity designed to help students internalize mathematical authority.

Method

This paper draws from student interactions during the first six weeks of an ongoing study in introductory college-level calculus ($N = 53$). The larger study is focused on helping students develop skills of explanation and reflection. Students completed daily reflections, consisting of a self-rating (0 to 100%) of perceived understanding of the day's lesson and two other prompts. These self-ratings of understanding were averaged over the first 6 weeks of the course.

Students also engaged in a weekly peer-assessment and reflection (PAR) activity, sequenced in four steps. Students: (1) completed the PAR problem (individually), (2) self-assessed their understanding, (3) traded with a peer and gave peer-feedback during class, and (4) revised and turned in a final solution. The PAR activity helped students develop authority by allowing them to be authors of mathematics (in solving nontrivial problems) and contributing to the work of others through peer-assessment and feedback (cf. Engle, 2011). The PAR activity was framed such that students need not perceive themselves as experts in order to engage (e.g., rather than assessing "right" and "wrong," students provided feedback about what they understood and didn't understand in the solution; c.f. Reinholz, in press). While it is not the focus of this paper, the PAR activity requires students to be mathematically accountable to their assertions, in order to prevent students from developing unbounded (and unwarranted) authority (Engle, 2011).

Results and Analysis

To illustrate the role of perceived authority in peer-interactions, I present three cases of students with varying levels of self-perceived authority. Students' self-assessments of understanding in their daily reflections were used as a proxy for self-perceived authority (i.e. a student who consistently rated a high-level of understanding would be said to have a high-level of perceived authority, while a student who rates high levels of understanding for some topics would be said to have high-levels of authority *locally* for those topics). Although students were chosen based on their self-assessments, at least in this sample, self-assessments corresponded with actual performance on Exam 01. For this short paper, I focus on how scope of authority impacted peer-interactions.

Student 1: Low self-perceptions of authority (average 76.25% self-rated understanding)

Student 1 consistently rated low levels of understanding, which corresponded with the types of feedback he gave. All feedback was phrased as questions, even when it appeared the student was commenting on a perceived error in his peer's solutions:

- PAR01: Why was the second graph parabolic?
- PAR02: How did you know which equations to use for #2?
- PAR03: I'm unsure about #2 because what if the x can be canceled out?
- PAR04: For 2C why does the rate go down at the beginning?
- PAR05: In part (d) what is "it" referring to in your answer?

In PAR03, PAR04, and PAR05, the student gave critical feedback to the peer (in PAR01 and PAR02 the feedback appeared to be genuine questions, not critical feedback); in PAR03 he noticed an issue with the example his peer gave, in PAR04 he noticed that the graph was inaccurate, and in PAR05 the feedback related to the clarity of the explanation. Yet, all feedback was given in the form of questions (e.g., "did you think about X?") rather than as assertions. This feedback style reflects the student's low levels of self-perceived authority. Rather than making assertions that something might be incorrect, the student formulates questions that less directly challenge the peer's authority.

Student 2: High self-perceptions of authority (average 97.5% self-rating)
In contrast, student 2 had high global levels of self-rated understanding. This student's feedback became progressively more authoritative as the semester progressed (i.e. feedback was initially phrased as questions and over time became phrased as assertions):

- PAR01: Why aren't your axes labeled correctly?
- PAR02: Why does a line with a hole work for number 1?
- PAR03: 0/0 is indeterminant; why can't it equal 1 if you do the algebra?
- PAR04: Check where you have corners; corners are not differentiable.
- PAR05: There is a problem with the limit as x approaches a; if you can't find f(a) then the function doesn't exist so it can't be differentiable

While the feedback given by the student during the first 3 weeks seems clearly to be evaluative (e.g., why aren't your axes labeled correctly? is hardly a genuine question), it is not until later in the semester that the student began phrasing the feedback as assertions, as in PAR04 and PAR05. The increasing authority with which this student gives feedback corresponds with the student's continual experiences of success, which reinforced self-perceived authority (i.e. each time the student rated high levels of understanding and received teacher-feedback of success, authority was increased).

Student 3: Mixed self-perceptions of authority (average 92.5% self-rating)
Student 3 also appears to have a high level of self-perceived understanding, but if we look at the student's self-assessments for each of the PAR problems, we see that the perceptions of authority are localized in scope (in contrast to the other students whose self-assessments in their daily reflections were consistent with self-assessments in PAR). Student 3 gave the following peer-feedback:

- PAR01: I believe the x-intercept is wrong, as in the radius. The radius of the wheel will minimize at 1cm, not at 0cm.
- PAR02: Does rounding always work?
- PAR03: Why only graphs? Why no algebra with your written responses?
- PAR04: Explaining your process would really help in #2.
- PAR05: Could the quadratic function be more distinguishable?

Looking at this student's feedback, PAR01 and PAR04 were both stated authoritatively (as assertions rather than questions), but the feedback on other problems was stated as questions. There is no consistent feedback pattern as with the other students. However, when we look at the student’s self-assessments for these particular problems we gain further insight in this seeming inconsistency (the PAR05 self-assessment was left blank):

- PAR01: I didn’t notice any errors, and my assumptions made came from the information given, so they were justified; my answer makes sense in reality.
- PAR02: I am not sure if it was solved correctly. I have an assumption that “L” was the presumed limit, but I am not sure. My whole work was based on this, so if my assumption is wrong all of my work is wrong.
- PAR03: I checked my answers with those I could find in the textbook, and they all seemed to match up. The straightforward approach I used seems to be useful, never assuming that the answer you find is what you initially expect.
- PAR04: I think it was solved correctly. Logically, the graphs I created matched the actual physical behavior of the problem.

The self-assessments for PAR01 and PAR04 indicate the student felt confident in his understanding, which was reflected in his authoritative feedback. In contrast, the self-assessments for PAR02 and PAR03 were less confident (e.g., “I’m not sure if it was solved correctly,” and “I checked my answers with those I could find in the textbook,” which was evidently a subset of the actual answers given). For student 3 the high-level of general self-perceived authority seemed to be trumped by low perceptions of authority in a localized context (such as a single problem).

Discussion and Conclusions

Perceived mathematical authority influences how an individual acts in a situation independently of that individual’s actual understanding. Perceived authority is both relational, dependent on an individual’s relative perceptions of authority between individuals in a situation (relationality), and varies in scope (from local to more general). I presented a number of examples illustrating how the scope of self-perceived authority influences students’ engagement in a specialized peer-assessment context. In particular, students who lacked authority in a given context tended to give less authoritative feedback (phrased as questions rather than assertions), even if their overall self-perception of authority appeared to be high.

Based on the above examples, self-perceived mathematical authority appears to have a profound impact on the types of feedback students give in a peer-assessment context. Students with higher perceived authority were more likely to give feedback in the form of assertions, while students with lower perceived authority tended to leave feedback in the form of questions, which are less likely to result in a challenge to the peer’s authority. While students gave evaluative feedback and suggestions regardless of their perceived authority, the way in which feedback was framed was different. Although it is out of the scope of the present paper, the way in which feedback is framed is likely to influence whether and how other students respond to it. Students who are overconfident may mistakenly give overly authoritative feedback that could mislead other students, while students who are underconfident may lack leave feedback that is ignored because other students perceive it as too hypothetical or uncertain.

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A Microgenetic Study of One Student's Sense Making About the Temporal Order of Delta and Epsilon

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The formal definition of a limit, or the epsilon delta definition is a critical topic in calculus for mathematics majors' development and the first chance for students to engage with formal mathematics. This report is a microgenetic study of one student understanding of the formal definition focusing on a particularly important relationship between epsilon and delta. diSessa's Knowledge in Pieces and Knowledge Analysis provide frameworks to explore in detail the structure of students' prior knowledge and their role in learning the topic. The study documents the progression of the student's claims about the dependence between delta and epsilon and explores relevant knowledge resources.

Keywords: limit, formal definition, students' prior knowledge, microgenetic study, learning

The formal definition of a limit of a function at a point, as given below, also known as the epsilon-delta definition, is an essential topic in mathematics majors' development that is introduced in calculus. We say that the limit of $f(x)$ as x approaches a is L, and write, $\lim_{x\to a} f(x) = L$ if and only if, for every number ε greater than zero, there exists a number δ greater than zero such that for all numbers x where $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. The formal definition provides the technical details for how a limit works and introduces students to the rigor of calculus. Yet research shows that thoughtful efforts at instruction at most leaves students – including intending and continuing mathematics majors – confused or with a procedural understanding about the formal definition (Cottrill et al., 1996; Oehrtman, 2008; Tall & Vinner, 1981).

Although studies have sufficiently documented that the formal definition is a roadblock for most students, little is known about how students actually attempt to make sense of the topic, or about the details of their difficulties. Most studies have not prioritized students' sense making processes and the productive role of their prior knowledge (Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). This may explain why they reported minimal success with their instructional approaches (Davis & Vinner, 1986; Tall & Vinner, 1981). A small subset of the studies have begun exploring more specifically student understanding of the formal definition (Boester, 2008; Knapp and Oehrtman, 2005; Roh, 2009; Swinyard, 2011). They suggest that students' understanding of a crucial relationship between two quantities, epsilon and delta within the formal definition warrants further investigation. Davis and Vinner (1986) call it the *temporal order* between epsilon and delta, that is epsilon first, then delta (p. 295) and found that students often neglect its important role.

This report explores how students make sense of the formal definition of a limit in relation to their intuitive knowledge. Specifically, it investigates the micro changes in one student's understanding of the temporal order of delta and epsilon in the formal definition. Through a fine-grained knowledge analysis of student interviews, this report investigates the range of resources one student navigate through and/or refine as he developed his claim about the temporal order.

Theoretical and Analytical Framework

The Knowledge in Pieces (KiP) theoretical framework (Campbell, 2011; diSessa, 1993; Smith et al., 1993) argues that knowledge can be modeled as a system of diverse elements and complex connections. From this perspective uncovering the fine-grained structure of student knowledge is a major focus of investigation, and simply characterizing student knowledge as misconceptions is viewed as an uninformative endeavor (Smith et al, 1993). Knowledge elements are context-specific; the problem is often inappropriate generalization to another context (Smith et al, 1993). For example, "multiplication always makes a number bigger" is not a misconception that just needs to be removed from students' way of thinking. Although this assertion would be incorrect in the context of multiplying numbers less than 1, when applied in the context of multiplying numbers greater than 1, it would be correct. Paying attention to contexts, KiP considers this kind of intuitive knowledge a potentially productive resource in learning (Smith et al., 1993). This means that instead of focusing on efforts to replace misconceptions, KiP focuses on characterizing the knowledge elements and the mechanisms by which they are incorporated into, refined and/or elaborated to become a new conception (Smith et al., 1993). Documenting the micro changes in learning is one of the foci of investigation (Parnafes & diSessa, in press; Schoenfeld et al., 1993). Similarly, we view students' prior knowledge as potentially productive resources for learning. We also assume that student knowledge is comprised of diverse knowledge elements and organized in complex ways, and thus learning is seen as the process of reorganization and elaboration of students' prior knowledge.

Methods

The data is part of a larger pilot study for my dissertation where I interviewed 7 calculus students about their understanding of the formal definition. This report focuses on a case of a student, AD who used a diverse set of knowledge resources to make sense of the temporal order between delta and epsilon. AD self identified as male, and White-non Hispanic. He was an intended mathematics major, who took first-semester calculus in high school and received a 5 on his AP Calculus AB and BC. The student was selected because he "changed his mind" about the temporal order several times during the interview before arriving at the claim that epsilon came first. These changes provide opportunities for a closer look at the influence of different cued knowledge resources in his thinking and how they might have gotten elaborated or refined.

Analysis

The analysis focuses on the part of the interview where I asked the student to comment specifically about the temporal order of delta and epsilon. I broke down the part into segments based on the change in students' claim (*epsilon first, delta first* or *no order*). At times, depending on the question, the student might have characterized the temporal order in terms of dependence (epsilon depended on delta or vice versa), control (trying to control delta or epsilon), temporal order (epsilon first or delta first) or which one was set first. In ambiguous cases, I follow the position that the cued knowledge resources support. I define *knowledge resources* as relevant prior knowledge that might be used to reason and justify the issue at hand. *Cued knowledge resources* are assertions students bring up as part(s) of a mechanism to justify a currently held position or opinion. We identify cued knowledge based on what students say in the moment. Reasonable interpretations for the statement will be considered and be put through the process of *competitive argumentation* (Schoenfeld, Smith & Arcavi, 1993) using other parts of the transcripts. With each of the cued knowledge resources, particular care will be given to investigate their origin and when it originally came up, and also when possible to the students' commitment to the particular knowledge. The analysis below shows the progression of AD's claim about the temporal order, and the cued knowledge resources involved in his reasoning.

The student, AD initially argued that epsilon depended on delta (delta first) because "delta is giving you an interval for *x*, and then, like, epsilon is evaluating *x* and subtracting the limit" (turns 288-290). By the end he argued that delta depended on epsilon (epsilon first) because "epsilon's [set] first and you break down epsilon… and you find delta" (turns 393-411). Before exploring the different knowledge resources used by AD in making those claims, I report the changes that happened in the span of 17-minute episode between those two claims. The diagram below shows the changes in AD's claim about the temporal order. Each box represents a segment, and the color characterizes the overall nature of the argument about the temporal order. Red is for *delta first*. Yellow is for *no order*. Green is for *epsilon first*.

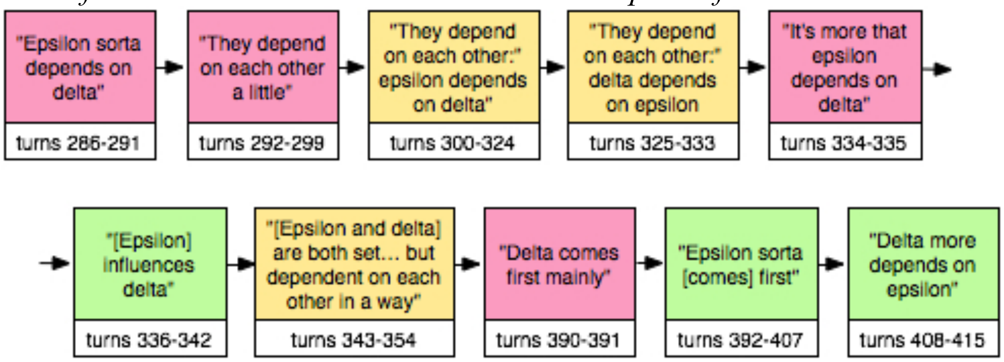

Figure 1. AD's progression of claims about the temporal order of epsilon and delta

The changes might have reflected the instability of AD's claim, but it was not due to lack of resources. Analysis shows that at each segment different knowledge resources were cued, they interacted with existing ones, and at times became conflicting. Below, I explore some of these resources in the first segment to give an idea of this process.

AD started by saying that "epsilon sort of depended on delta" in the first segment. AD argued that epsilon sort of depended on delta because delta gave an interval for *x* and epsilon evaluated the *x* and subtracted the limit from it. I claim that he cued particular views about what epsilon was and what delta *did* as well as the dependence relationship between *x* and $f(x)$.

286 AD Um delta, no, epsilon sorta depends on delta.
287 AA Epsilon depends on delta…
288 AD Because, um delta is giving you an interval for *x*,
289 AA Uh-hm.
290 AD And then like epsilon is evaluating x and subtracting the limit,
291 AA Uh-hm.

Part of AD's argument focused on what delta *did* and epsilon *was*. He said that delta gave an interval for *x* and epsilon was evaluating *x* and subtracting the limit. Earlier part of the interview indicated that AD had a more nuanced view of both delta and epsilon. About delta, AD stated that "[d]elta is another small number such that the interval, it makes the interval, sm-small, but big enough so you can actually, it's not just a point but it's uh, that you get numbers that are close to the limit" (turn 192). More than just giving an interval for *x,* delta created a particular size of interval (small but big enough), and served a particular purpose (to get numbers close to the limit). But it seems that the only aspect of delta that was cued in this turn was the fact that *delta constrained the x values*. For epsilon, his statement on 290 seems to suggest that he was saying $|f(x) - L| = \varepsilon$. Earlier accounts, however, showed that he was aware that $|f(x) - L|$ had to be less than epsilon. He stated, "Epsilon's just a number and you're using it to make sure that $f(x)$ minus L, the absolute value is just less than some certain nu- number, and it must be greater than zero, so you call it epsilon" (turns 158-160). He later also said that epsilon made sure that $f(x) - L$ was small (turns 188-190). But here, AD only cued the fact that epsilon evaluated x and subtracted the limit. Why did AD only cue certain aspects of delta and epsilon? I argue that he might have done so to cue another knowledge resource: $f(x)$ depends on x.

Focusing on the fact that delta gave him an interval for x and epsilon evaluated that x, could help establish the dependence between delta and epsilon through the dependence between x and $f(x)$. In fact, AD argued exactly this earlier, when he was explaining the meaning of the *if-then statement*, $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. He said, "… this one [*points at* $|f(x) - L| < \varepsilon$] is *dependent* on this one because in this one [*points at* $0 < |x - a| < \delta$] you're choosing the x, this [*points at* $|f(x) - L| < \varepsilon$] is evaluating the function at x" (turn 238, emphasis added). AD attended to the if-then statement to conclude the dependence, but he focused on the *evaluation* of x in order to argue for the dependence. This also suggests that AD might have also used his interpretation of the if-then statement to arrive at his conclusion. But earlier accounts showed that AD had different interpretations for the if-then statement. He treated the two inequalities in the if-then statement as two conditions to satisfy instead of an implication (turns 126-128, 142, 224, and 240). This suggests that AD used the if-then statement to infer the temporal order of delta and epsilon, but in a very particular way. He used the if-then statement to infer the purpose for delta and epsilon. That is delta sets an interval for x and epsilon evaluates such x and subtracts the limit.

In sum, in the first segment, AD cued the following knowledge resources: 1) delta constrains the interval for x; 2) epsilon is involved in evaluating the x and subtracting the limit; 3) his interpretation of the if-then statement, through which he inferred the meaning of delta and epsilon; and 4) the dependence between x and $f(x)$. The fourth knowledge resource is in fact one of the most common knowledge resources found in most students I interviewed. In the presentation, I will explore how some of these resources get *reused* and *refined* in later segments, as well as other resources that were cued during the 17-minute episode.

Conclusion and Implications

This case shows how one student made sense of the temporal order of delta and epsilon. AD's progression with the claim suggests the range of resources AD cued in the episode, and how they might have been competing resources. The analysis showed that the student cued particular aspects of his knowledge to make claims about the temporal order. It also shows that students can interpret parts of the statement of the definition in particular ways. While I was able to document the resources cued in this episode, the question remains, why do certain resources get reused and refined while others were abandoned. That is the next step in the analysis. The question for the reader would be, is the analysis demonstrated in this report convincing and informative? In what ways, can I make it more rigorous or convincing?

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Students' Knowledge Resources About the Temporal Order of Delta and Epsilon

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The formal definition of a limit, or the epsilon delta definition is a critical topic in calculus for mathematics majors' development and the first chance for students to engage with formal mathematics. Research has documented that the formal definition is a roadblock for most students but has de-emphasized the productive role of their prior knowledge and sense making processes. This study investigates the range of knowledge resources included in calculus students' prior knowledge about the relationship between delta and epsilon within the definition. diSessa's Knowledge in Pieces provides a framework to explore in detail the structure of students' prior knowledge and their role in learning the topic.

Keywords: limit, formal definition, students' prior knowledge, fine-grained analysis

The formal definition of a limit of a function at a point, as given below, also known as the epsilon-delta definition, is an essential topic in mathematics majors' development that is introduced in calculus. We say that the limit of $f(x)$ as x approaches a is L, and write. $\lim_{x\to a} f(x) = L$ if and only if, for every number ε greater than zero, there exists a number δ greater than zero such that for all numbers x where $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. The formal definition provides the technical details for how a limit works and introduces students to the rigor of calculus. Yet research shows that thoughtful efforts at instruction at most leaves students – including intending and continuing mathematics majors – confused or with a procedural understanding about the formal definition (Cottrill et al., 1996; Oehrtman, 2008; Tall & Vinner, 1981).

Although studies have sufficiently documented that the formal definition is a roadblock for most students, little is known about how students actually attempt to make sense of the topic, or about the details of their difficulties. Most studies have not prioritized students' sense making processes and the productive role of their prior knowledge (Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). This may explain why they reported minimal success with their instructional approaches (Davis & Vinner, 1986; Tall & Vinner, 1981). Thus, understanding the difficulty in the teaching and learning of the formal definition warrants a closer look – with a focus on student cognition and with attention to students' prior knowledge. It also calls for a theoretical and analytical framework that focuses on understanding the nature and role of students' intuitive knowledge in the process of learning.

A small subset of the studies have begun exploring more specifically student understanding of the formal definition (Boester, 2008; Knapp and Oehrtman, 2005; Roh, 2009; Swinyard, 2011). They suggest that students' understanding of a crucial relationship between two quantities, epsilon and delta within the formal definition warrants further investigation. Davis and Vinner (1986) call it the *temporal order* between epsilon and delta, that is epsilon first, then delta (p. 295) and found that students often neglect its important role. Swinyard (2011) found that the relationship between the two quantities is one of the most challenging aspects of the formal definition for students. Knapp and Oehrtman (2005) and Roh (2009) document this difficulty for advanced calculus students. This difficulty is also prevalent among the majority of calculus students who struggled with the formal definition in Boester (2008). While studies have shown the existence and prevalence of this difficulty, little is known about why this relationship is difficult for students.

This report is a preliminary analysis of a pilot dissertation data. The dissertation explores how students make sense of the formal definition of a limit in relation to their intuitive knowledge. Specifically, it investigates the micro changes in student understanding of the temporal order of epsilon and delta within the formal definition of a limit. Through a fine-grained analysis of student interviews, this preliminary report focuses on one of the questions that will be explored in the dissertation. What claims do students make about the relationship between delta and epsilon, and what is the range and nature of the resources they use to make these claims?

Theoretical Framework

The Knowledge in Pieces (KiP) theoretical framework (Campbell, 2011; diSessa, 1993; Smith et al., 1993) argues that knowledge can be modeled as a system of diverse elements and complex connections. From this perspective uncovering the fine-grained structure of student knowledge is a major focus of investigation, and simply characterizing student knowledge as misconceptions is viewed as an uninformative endeavor (Smith et al, 1993). Knowledge elements are context-specific; the problem is often inappropriate generalization to another context (Smith et al, 1993). For example, "multiplication always makes a number bigger" is not a misconception that just needs to be removed from students' way of thinking. Although this assertion would be incorrect in the context of multiplying numbers less than 1, when applied in the context of multiplying numbers greater than 1, it would be correct. Paying attention to contexts, KiP considers this kind of intuitive knowledge a potentially productive resource in learning (Smith et al., 1993). This means that instead of focusing on efforts to replace misconceptions, KiP focuses on characterizing the knowledge elements and the mechanisms by which they are incorporated into, refined and/or elaborated to become a new conception (Smith et al., 1993). Similarly, we view students' prior knowledge as potentially productive resources for learning. We also assume that student knowledge is comprised of diverse knowledge elements and organized in complex ways, and thus learning is seen as the process of reorganization and elaboration of students' prior knowledge.

Methods

The data for this report comes from the pilot data for one of the author's dissertation. We interviewed seven calculus students using a protocol developed for the dissertation. Each of these students has received some form of instruction on the formal definition. So we anticipate some knowledge about the definition to be a part of their prior knowledge. The protocol was designed to elicit student understanding of the formal definition, but more specifically their understanding of the relationship between delta and epsilon. To explore the stability of students' knowledge across different contexts, we asked students about the temporal order of the two variables in three different contexts: dependence, control, and their *temporal order* (see the table below). Each individual interview lasted about 2 to 3 hours. These interviews were videotaped following recommendations in Derry et al. (2010).

Analysis

The first part of the analysis places students in categories based on their claim about the temporal order of epsilon and delta. There will be three categories: *epsilon first*, *delta first*, and *no order*. For a student to be classified into the category *epsilon first*, s/he would respond in the following way to the four questions. S/he would say that: 1) delta depends on epsilon; 2) one is trying to control x using delta, based on a given epsilon; 3) epsilon comes first and then delta; 4) the four variables are ordered in such a way where epsilon comes first then delta. For a student to be classified into the category *delta first*, s/he would respond in the following way to the five questions. S/he would say that: 1) epsilon depends on delta; 2) one is trying to control $f(x)$ using epsilon, based on delta; 3) delta comes first and then epsilon; 4) the four variables are ordered in such a way where delta comes first then epsilon. For a student to be classified as *no order*, there needs to be variance in responses across the different questions. In this study, we found few inconsistencies between the four different ways of asking the question.

The second part explores the range and nature of knowledge resources. We define *knowledge resources* as relevant prior knowledge that might be used to reason and justify the issue at hand. *Cued knowledge resources* are assertions students bring up as part(s) of a mechanism to justify a currently held position or opinion. We identify cued knowledge based on what students say in the moment. The analysis focuses on discussions around the four questions about the temporal order of delta. Reasonable interpretations for the statement will be considered and be put through the process of *competitive argumentation* (Schoenfeld, Smith & Arcavi, 1993) using other parts of the transcripts. With each of the cued knowledge resources, particular care will be given to investigate their origin and when it originally came up. Until there is consistent evidence of stance taken by a student, it would be impossible to make claims about the stability or how committed the student might be to the specific claim they made.

Results

Relationship Between the Epsilon and Delta

Five of seven students interviewed concluded that delta came first, 2 students concluded that epsilon came first and no student fell into the no order category. The table below shows the claims students made about the temporal order between epsilon and delta across the different contexts. We determine the student's final categorization by what the student said last about the relationship between epsilon and delta.

Student Initial	Question 15: Which depends on which?	Question 17: Which variable are you trying to control?	Question 18: Which one comes first?	Question 19: Order of [the four] variables.	Final categorization
DC	δ depends on ε	[Skipped]	N/A	N/A	Epsilon first
DL	ε depends on δ	δ you can control, ε you're trying to control.	N/A	N/A	Delta first
JJ	ε depends on δ	Control δ and [trying to] control ε.	N/A	N/A	Delta first
AD	δ depends on ε	[Skipped]	ε is first, you break down the epsilon to find delta.	Student decided not to try.	Epsilon first
DR	ε depends on δ	Trying to control δ so that you can get a smaller ε.	You get delta first then you get ε as your result.	$x, f(x), a, L, \delta, \varepsilon$	Delta first

SR	ε depends on δ	Trying to control x and $f(x)$, but not sure.	Calculate delta first, then used to calculate ε.	x, δ , $f(x)$, ε	Delta first
OB	ε depends on δ	Try to control x and δ to find ε.	Delta comes first, ε second.	δ, x, a, $f(x)$, ε	Delta first

Range and Diversity of Knowledge Resources

One very common knowledge resource that emerged in the pilot study was the output $f(x)$ was dependent on the input x. Students often associated this knowledge resource with that that argues that epsilon is a quantity related to $f(x)$ and delta is a quantity related to x. So $f(x)$ depends on x meant that epsilon must depend on delta, and so delta was first. Five of seven students used this fact to justify that epsilon depended on delta. DC cued these knowledge resources below.

> "Um [inaudible] well given that the, um, delta does generally or does seem to refer to the x value or the range of x values, the domain of x values that you want to be paying attention to, generally I think of functions, um, since a function is a relationship between dependent and independent variables, I tend to think of x as being you know as they are the, uh, independent variables. And so the y as being the ones that are altered by the x. So that's how you plug in numbers for functions, that's how you utilize functions in most cases. So it makes more sense to me to think that as epsilon being dependent on delta, where I'm assuming that delta is referring to x and epsilon is referring to y values" (turns 137-145).

DC reasoned that delta referred to a range of x-values and thus epsilon referred to a range of y-values, and since $f(x)$ or y depended on x, then it 'makes more sense to him' that epsilon depended on delta. So in this case we would argue that DC used the following knowledge resources to conclude that delta was first: 1) the dependence between x and $f(x)$; 2) delta refers to x values; and 3) epsilon refers to y values. Observe the similarity between what DC said with what DR said in her interview. "Um, see cus I was looking at it like the x or the $f(x)$ or the yeah, the $f(x)$ depends on the x and that's how I was like saying that epsilon depends on delta because epsilon like is related to the $f(x)$ or whatever" (turn 578). DR also relied on the dependence between x and $f(x)$, and she, too saw epsilon as "related to the $f(x)$." From the pilot studies other resources emerged from the data and we expect more to emerge as we interview more students. This analyses show the diverse knowledge resources students used to make sense of this relationship, and that most of these resources supported the assertion that epsilon depended on delta. No wonder students have a hard time with this relationship!

Conclusion and Broader Implications

This report shows that most students argue that within the formal definition, delta comes first. Students draw upon a range of resources, many of which support that claim the delta comes first. The remaining questions are the following. Once we recognize the range of resources that students use to make conclusions about the relationship between delta and epsilon, how do we begin to help students navigate through them? More specifically, how can we better assist students to refine or elaborate on the productive knowledge resources to make appropriate conclusions about the temporal order? A better understanding of the nature of these resources can facilitate the design of instruction that can help students bridge and reorganize these resources for a better understanding of the formal definition.

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COMPUTATIONAL THINKING IN LINEAR ALGEBRA

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In this work, we examine students' ways of thinking when presented with a novel linear algebra problem. We have hypothesized that in order to succeed in linear algebra, students must employ and coordinate three modes of thinking, which we call computational, abstract, and geometric. This study examines the solution strategies that undergraduate honors linear algebra students employ to solve the problem, the variety of productive and reflective ways in which the computational mode of thinking is used, and the ways in which they coordinate the computational mode of thinking with other modes.

Key words: Justification, Linear Algebra, Problem Solving, Procedural Understanding

Purpose and Background

The field of linear algebra has recently attracted much attention in the literature; many studies have examined students' difficulties in linear algebra (e.g., Carlson, 1993; Sierpinska, 2000). By contrast, this study examines students' abilities. We focus on the ways in which students are able to think productively, in order to provide a model of successful thinking in linear algebra. In particular, we examine the productive ways students use computational thinking to reason through a novel problem about basis.

We have adopted a three-fold taxonomy of ways of thinking in linear algebra. The ways of thinking we have identified, which we refer to as *abstract*, *computational*, and *geometric* thinking, are roughly parallel to those of Sierpinska (2000) and Hillel (2000). *Abstract thinking* is indicated by working with vectors as formal objects characterized by the vector space axioms, or by statements of definitions and theorems in coordinate-free language. *Computational thinking* is typically indicated by explicit reference to particular algorithms, such as row reduction or the Gram-Schmidt process, and representation of vectors in terms of their components. It includes not only carrying out a computation, but also choosing the appropriate computation to solve a particular problem and understanding what the result of that computation means in context. Finally, *geometric thinking* is indicated by the use of language such as line, plane, ray, angle, length, or intersection, and geometric knowledge such as the Pythagorean Theorem. Usually this language is used in the context of $\mathbf{R}^2$ or $\mathbf{R}^3$, or by analogy with one of these spaces.

We argue that in order to be successful in linear algebra, students must come to be able to use and coordinate each of these three ways of thinking; we follow Hillel (2000) in attributing many student difficulties to their trouble switching between, or relating, these three languages. These ways of thinking are not sharply demarcated; powerful tools for reasoning can be found at their overlaps. This study was designed to address the following research questions: What strategies do students use to solve a novel problem (detailed below)? What are the uses, affordances, and constraints of each mode of thinking? In what ways do students coordinate these modes of thinking?

Design, Settings, and Methods

Eight first-year undergraduate honors linear algebra students from a large public university in the southwestern United States volunteered to participate in individual clinical interviews, conducted near the end of the course. The interview centered on the "Michelle problem":

> Michelle would like to create a basis for $\mathbf{R}^4$. She has already listed two vectors **v** and **w** that she would like to include in her basis, and wants to add more vectors to her list until she obtains a basis. What instructions would you give her on how to accomplish this?

Based on a prior study (Wawro, Sweeney, & Rabin, 2010), this problem was expected to provide rich opportunities for each mode of thinking. The problem was initially presented abstractly, as above, and students were encouraged to describe a general method by which the problem could be solved. After this, specific vectors were given, and the students were asked to try using their method on these given vectors. We asked a number of follow-up questions to probe students' intuition and solution procedures, whether students could formulate their solution procedure in an algorithmic way, and whether they could justify their procedure.

The interviews were videotaped and transcribed, and students' written work produced during the interviews was retained. These recordings, transcripts, and written documents formed the corpus of data analyzed in this study. Using grounded theory (Strauss & Corbin, 1994), we coded students' utterances as instances of abstract, computational, or geometric thinking, referring to written work for confirmatory evidence, and documented the ways in which students used each way of thinking.

As we began analyzing the data, we noted a preponderance of computational thinking. Since we saw a much more balanced use of modes of thinking in the pilot study, this result was unexpected. We were thus led to ask, in addition to the initial research questions, why students used computational thinking so much, and in what productive ways they used it.

Results

One result from our data is the surprising variety of ways in which students were able to productively use computational thinking. Computational thinking is often maligned in the literature and in educators' opinions. The common perception is that computational thinking is exclusively procedural, that students simply wish to feed numbers unreflectively into an algorithm, and that they commonly make significant errors while doing so. This analysis highlights, by contrast, the *productive* and *reflective* ways students can use computational thinking in linear algebra. For example, using computational thinking, students were able to generate strategies for choosing additional vectors, produce proofs of their methods, and get "un-stuck" when they encountered roadblocks.

Generating strategies

Computational thinking can inspire the creation of strategies for choosing vectors. Evan (all names are pseudonyms), for instance, initially proposed a strategy of pure guess and check. The interviewers asked him if there was a strategy he could use to make informed guesses:

> *Evan*: Well, I'd first, I'd reduce the first, these two vectors [i.e., **v** and **w**] and let – so I'll just make a 4x2 matrix. And to reduce – If I can have two pivotal columns, so like, for these two, because they are linearly independent, so sure I can have two pivotal columns. And I can just pick another two that has different pivotal columns [*sic*: he means rows], like in the third column or fourth column, to get one.

When asked for a way to make better guesses, he suggested row-reducing **v** and **w** to see where the pivots were, then choosing standard basis vectors to provide the missing pivots. Thus, by using computational reasoning to think about the test his vectors must pass, he was able to engineer vectors that are guaranteed to pass it. Although he does not formulate his reasoning as a proof or justification, it essentially serves as such.

Computational proofs

We had anticipated that the justification questions would prompt abstract thinking. However, we found that several students were able to produce fully valid proofs using computational thinking alone. Bob, for example, produced a proof by analyzing his algorithm. His strategy was similar to Evan's; he row-reduced the two given vectors, then used standard basis vectors to supply the missing pivots. He argued abstractly that the new vectors will be independent, and then justified this claim computationally (transcript edited for length and clarity):

> *Bob*: Well, by the definition of linear independence, their matrix has to row-reduce to the identity. All the columns will be pivotal. So, by using these facts about linear independence and pivotal columns, this procedure is a way to find two more columns that will be pivotal columns independent of the other ones already found. With the two vectors she already has, she has two pivotal columns here, and they both represent pivotal ones. It's just a way to find the other two columns that won't form the same pivotal row as another one, so that they'll all be independent.

This technique of proof by analysis of an algorithm is a particularly valuable way of constructing formal justifications. Many textbook proofs in linear algebra proceed in a similar fashion. The fact that students are capable of producing such justifications is perhaps an argument for teachers to highlight this method when discussing proof techniques.

Roadblocks and resolutions

Students often encounter difficulties and "get stuck" when solving a problem. We initially conjectured that one way of getting around such roadblocks would be for students to transition to another way of thinking. However, we found that in many situations where students encountered difficulties relating to their computational thinking, they were able to resolve the difficulty by continuing to think computationally.

Greg is an example of a student who encountered such a roadblock. His solution method was to guess a vector to potentially include in the basis, then show that it would work by showing "that it's not a linear combination of these two [i.e., **v** and **w**]." He augmented the two given vectors with a vector he guessed and proceeded to row-reduce the resulting matrix. He hit a roadblock when the third row of his matrix reduced to [0 0 | -15], saying, "I'm actually kinda confused about what this tells me. Did I make a mistake?"

It appeared that Greg is used to rows of the form [0 0 | 1] signaling that something bad has happened. However, he was able to step back and reason (computationally) about this result, coming to the conclusion that it was the appropriate one:

> *Greg*: All right. Well actually, if I continue row-reducing this, then I would get a 1 here... then that would make it unsolvable. So then I suppose ... Yeah. Okay. And then, that would mean that this can't be a linear combination of these two. So then, it's not in the span.
>
> *Int*: And is that good or bad, for purposes of this problem?
>
> *Greg*: That's good. This could be an additional vector for a basis.

The fact that the system he had constructed was "unsolvable" made him second-guess himself for a moment, but by reflecting on the framing of the algorithm, he was able to realize that this is exactly what he had wanted. By *framing* we mean the context in which the algorithm is applied; its goals and the meaning of its inputs and outputs. Reasoning about the framing of an algorithm in this way is a productive overlap of the computational and abstract modes of thinking.

Our talk will explicate a number of other findings, including an elaboration of students' productive uses of the other two ways of thinking. Additionally, we will discuss at greater length the ways in which students coordinated and transitioned between multiple ways of thinking.

Conclusion

Our data contribute to the reconceptualization of procedural knowledge as a useful and productive mathematical resource (see, e.g., Star, 2005). Additionally, they show that computational thinking is more varied, flexible, and sophisticated than the common perception. We have presented evidence of students producing sophisticated strategies for choosing vectors, justifying their approaches, and resolving problems they encounter through the use of computational thinking. This evidence of student ability provides direct recommendations for pedagogical practice in linear algebra.

This study contributes to the literature by documenting student abilities in linear algebra. While it is true that these are advanced students, and thus our results may not generalize to the broader population of undergraduate linear algebra students, it is still useful to know what students are capable of. Although they are honors students, they are still freshmen, so it is reasonable to expect that their abilities are within the range of potential development of other students. Our study provides evidence that the fog need not always roll in (Carlson, 1993).

Discussion Questions

1. In what other ways might computational thinking be useful in linear algebra? in other areas of undergraduate mathematics?
2. How distinct are the three modes of thinking? Should they be thought of as separate but coordinated, or as shading into one another at their boundaries?
3. What other overlaps between the three modes of thinking might be expected?

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GUIDED REFLECTIONS ON MATHEMATICAL TASKS: FOSTERING MKT IN COLLEGE GEOMETRY

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Abstract: *This study is a part of ongoing research on development of Mathematical Knowledge for Teaching (MKT) in mathematical content courses. Reflective practice represents a central theme in teacher education. The purpose of this reported study was to understand the role of guided reflections on mathematical tasks in a college geometry course. We were also interested in understanding how guided reflections on mathematical tasks would effect teachers' development of MKT. Our research data consist of participants' reflections, teaching scenarios, and pre-post test results. In this study we developed a workable framework for data analysis. Audience discussion will address questions related to the proposed analysis framework and development of MKT in college mathematics courses.*

Key Words: [College Geometry, Mathematical Knowledge for Teaching, Reflections, Framework, Pedagogical Content Knowledge]

Theoretical Background

Recently there has been high interest in the knowledge of secondary mathematics teachers (Chamberlin, 2009; Proulx, 2008; Hough, O'Rode, Terman & Weissglass, 2007). In the following study, we look at the role of reflections in mathematics teachers' knowledge development. Various studies documented a connection between reflection on mathematical experiences and an increase in mathematical knowledge (Burk & Littleton, 1995; Chamberlin, 2009; Bjuland, 2004; Wheatley, 1992). Specifically, our study focuses on the effect of reflections on Mathematical Knowledge for Teaching (MKT). Much work has been done on understanding MKT in elementary mathematics (Ball, 1991; Ball, Hill, & Bass, 2005, Hill & Ball, 2004). Our study is situated in college geometry.

In 2004, Hill, Ball, and Schilling developed a framework for characterizing teachers' MKT consisting of two areas, Pedagogical Content Knowledge (PCK) and Subject Matter Knowledge (SMK). Our interest in this study is to observe secondary mathematics teachers' growth in MKT by assessing their change in PCK and SMK and the quality of their reflections on mathematical tasks. In this study, we attempt to answer the following research questions:

1) In what way do reflections on mathematical tasks affect the growth of MKT in secondary mathematics teachers?
2) Do the type and quality of the reflections have an effect on what kind of growth (SMK or PCK) the teachers experience?

Data Sources

The preliminary report is based on early analysis of participants' reflections, teaching scenarios, and pre-post test results. During the summer of 2011, 18 students enrolled in a *Teaching Geometry from Problem Solving to Proving* course at a small northeastern liberal arts college. The content of this course included a research-based curriculum, which was designed as a part of an MSP Grant. This task-based course (see Appendix A for an example of a task used in the course) was designed around problem-solving episodes, where the students would engage in investigating mathematical concepts through a reflective process; the objective was to increase the knowledge of mathematics while providing frequent opportunities to think about how they would teach the mathematical concepts being investigated. The course focused on Euclidean Geometry, including some fundamental concepts such as congruence, similarity, construction, area, etc. To measure the change in MKT, we administered a pre-test before the course, and a post-test (exactly the same content) after the course. As a part of formative course assessment, students solved and designed a variety of mathematical problems, along with reflections on their learning experiences. The data collected included participants' reflections, teaching scenarios, and pre-post test results. Typical course activities were multifaceted, targeting several domains of teachers' knowledge, in a coherent and interconnected manner.

Research Methodology

Designing the Framework: Reflections

The research methodology for this project was developed on Chamberlain's original work (2009). Originally, we assessed the participants' reflections in four categories: identification of the purpose of the task, recognition of cognitive difficulties that their students might have when trying to complete the task, situation of the task in their teaching by acknowledging where in the curriculum the task would fit or what the appropriate grade level for which the task should be utilized, and finally identification of the pedagogical strategies that could be used to teach the task. However, we decided to hone in on the first category, identification of purpose, because we noticed that there were two different types of purpose being recognized by the participants: mathematical and instructional.

Thus, we defined each type of purpose, differentiating from strong to weak within these purposes and continued our research through this scope (See Table 1 in Appendix A). A weak mathematical purpose is one that generalizes the steps taken to solve the problems or states what mathematical concept is being addressed in the given task. A strong mathematical purpose recognizes connections to mathematical contexts not directly used in the task. A weak instructional purpose included a general comment on the strategy used to complete the task. A strong instructional purpose provided connections made to teaching outside of the specific task and took the students into consideration. Since mathematical and instructional ideas are not completely separate, there were some participants that identified both types of purposes. Their responses were placed into both classifications and assessed according to the strength within those categories. Examples of these comments are included in the results portion of this paper and are useful in clarifying the meaning of each category. After finalizing the rubric, we read each of the reflections and scored them according to the rubric.

Designing the Framework: Pre- and Post-Tests

At the stage of assessing the change of subject matter knowledge and pedagogical content knowledge of each of the teachers, we decided to analyze the results of the pre- and post-tests.

Looking at particular subsets of all the questions answered on the test, we were able to distinguish between those that tested the SMK and those that tested the PCK. We used questions from the Graduate Record Examinations as a means to measure the SMK because these items were specifically designed “to indicate knowledge of the subject matter.” (www.ets.org) Only geometry questions were selected from various GRE test sources. A total of nine multiple-choice GRE questions were examined (see Appendix B for examples of the mathematical questions). The results of the pre- and post-tests were compared to assess for the change in knowledge. Four categories were formed to differentiate the mathematical growth: No Growth/Decay for scores that did not change or went down; Moderate Growth for scores that grew between 1% and 15%; Significant Growth for scores that grew between 16% and 35%; and Exceptional Growth for scores that grew 36% or more.

We also selected several questions from the National Assessment of Educational Progress tests to assess each teacher’s growth of PCK. The NAEP items were specifically designed to “present information on strengths and weaknesses in [secondary school] students’ knowledge of mathematics and their ability to apply that knowledge in problem-solving situations.” (www.nagb.org) We decided that the participants’ performance on these particular questions was a sufficient tool to assess the teachers’ PCK; looking at their ability to solve [secondary students oriented] problems would give insight into participants’ ability to teach the concepts these problems incorporate. We selected two multiple-choice questions and two written response questions to assess the PCK (see Appendix B for examples of the instructional questions). The four categories, presented above were applied to differentiate the levels of growth: No Growth/Decay for scores that did not change or went down; Moderate Growth for scores that grew between 1% and 15%; Significant Growth for scores that grew between 16% and 24%; and Exceptional Growth for scores that grew 25% or more.

Preliminary Results

Our hypotheses: those teachers that identify strong mathematical purposes would grow mathematically; those that identify weak mathematical purposes would not grow mathematically; those that identify strong instructional purposes will grow instructionally; and those that identify weak instructional purposes would not grow instructionally. Though the preliminary results indicate that data collected from 8 teachers support the hypothesis and data collected from 10 teachers refute it, the developed framework allowed us to successfully investigate the interconnected nature of MKT. Preliminary findings suggest reflections on mathematical tasks positively affected the growth of MKT in 15 of the 18 participants; specifically, 14 participants showed an increase in PCK, 8 participants showed an increase in SMK, and 7 participants showed an increase in both areas.

Questions

What types of reflections would spark substantial growth in MKT in a mathematics content course? What are other meaningful ways of foster growth of MKT in college mathematics courses? What theoretical perspectives would provide a better lens to observe and analyze the phenomenon of developing MKT in mathematics courses through reflections?

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Appendix A:

An example of a mathematical task used in the course (Adopted from *Developing Thinking in Geometry* by Johnston-Wilder & Mason, 2005):

Task 1: Draw a triangle. Draw a square outward on each side of this triangle. Join the outer corner of each square to the nearest outer corner of the next square to form three additional 'flanking' triangles, each one between two adjacent squares. Which is the largest of the triangles? Does it depend on your starting triangle?
Task 2: Draw a triangle and label the vertices with the co-ordinates (0, 0), (1, 0), and (x, y). Draw a square outward on each side of the triangle. Join each square to its neighbor by joining the nearest vertices of each of the squares. Find the areas of all four triangles.
Task 3: Draw a triangle, draw a square outward on each side, and join the vertices of the square to get three more 'flanking' triangles. Show that each of the new triangles has the same area as the original triangle.

Reflection question: *What was the relationship between these three tasks? Could this assignment be used in your classroom? If not, why?*

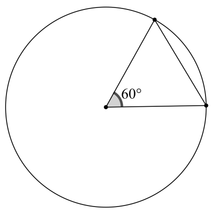
Table 1: Final rubric for the purpose used to analyze the teachers' reflections

	0 (Undefined)	1 (Weak)	2	3 (Strong)
Mathematical	No purpose provided	Solution to the task or stated the mathematical concept being addressed	Provided more detail about the concept or outlined the steps required to complete the task	Explained a solution that included the connections to the context not directly used in the task
Instructional		General comment made on the strategy used to complete the task	Provided some insight on the strategy needed to teach and/or complete the task	Recognized the students learning process and how the task used pedagogical strategies and the reason for using such approach

Appendix B:

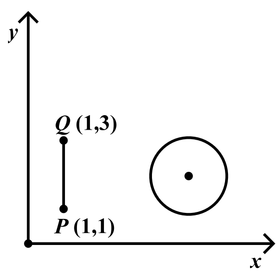
Samples of mathematical questions from the pre- and post-tests:

Select one of the following four answer choices:



a) The circumference of a circle is greater than 12
b) The circumference of a circle is less than 12
c) The circumference of a circle is equal to 12
d) The relationship cannot be determined from the information given

The figure shows line segment PQ and a circle with radius 1 and center (5, 2) in the *xy*-plane.

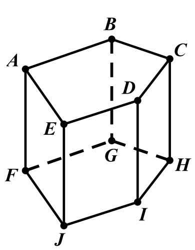


How many of the following values could be the distance between a point on line segment PQ and a point on the circle?

{2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0}

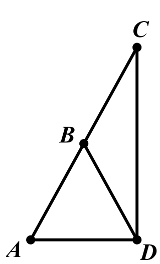
a) Only one value
b) Only two values
c) Only five values
d) All of the values
e) None of the values

Samples of instructional questions from the pre- and post-tests:



In the figure above, points A, E, and H are on a plane that intersects a right prism. What is the intersection of the plane with the right prism?

a) A line
b) A triangle
c) A quadrilateral
d) A pentagon
e) A hexagon



Write the proof in the space provided.

Given: B is the midpoint of AC and AB=BD

Prove: Angle CDA is right

CROSSING COMMUNITY BOUNDARIES: COLLABORATION BETWEEN MATHEMATICIANS AND MATHEMATICS EDUCATORS

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Effective mathematics teachers are able to make connections between mathematical content and pedagogy in their professional practice. One of the most readily prescribed approaches for facilitating teachers' ability to make such connections is through the development of collaborations between mathematicians and mathematics educators in venues related to teacher professional development. Most prior research related to collaborative endeavors between these two groups has focused on the products, rather than the process, of collaboration. In this preliminary research report, I present the results of an interpretative phenomenological case study that investigated the team-teaching experiences of a mathematician and a mathematics educator within the context of an undergraduate mathematics teacher preparation program. I present extracts from interviews that highlight the instructors' perceptions related to crossing the boundaries of their professional communities of practice, and engage participants in discussion about relevant "boundary crossing" in their own institutional contexts.

Key words: Teacher Education, Team-Teaching, Communities of Practice, Phenomenology

Introduction.

Research shows the importance of teachers' ability to make connections between mathematics content and mathematics pedagogy as a means of increasing student achievement (Ball & Bass, 2000; CBMS, 2001, 2012; Hill, Rowan, & Ball, 2005). One of the most readily prescribed approaches for facilitating teachers' ability to make such connections is through the development of collaborations between mathematicians and mathematics educators in venues related to teacher professional development (Bass, 2005; CBMS, 2001, 2012; Cheng, 2006; Ferrini-Mundy & Findell, 2001; McCallum, 2003; Millman et al., 2009; Wu, 2006). In its seminal publication, *The Mathematical Education of Teachers,* the Conference Board of Mathematical Sciences (CBMS, 2001) recommended that "the mathematical education of teachers should be seen as a partnership between mathematics faculty and mathematics education faculty," while at the same time acknowledging that "the reality today is that there is considerable distrust between mathematics faculty and mathematics education faculty both within institutions and through public exchange" (p. 9).

Over a decade later, with its release of *The Mathematical Education of Teachers* (part II), CBMS (2012) suggested that considerable strides have been made in relation to collaborative endeavors between mathematicians and mathematics educators. Examples of the products of such collaborations are prevalent throughout mathematics education literature (e.g., Eaton & Carbone, 2008; Koirala, Davis, & Johnson, 2008). However, the researchers who have reported on these collaborations have focused primarily on the products of such collaborative efforts, with little attention to the process of collaboration. In order to build on and learn from collaborative efforts between members of the mathematics and mathematics education communities, it is imperative to investigate the nature and process of existing collaborations.

Research Questions.

In this preliminary research report, I present a selection of results from an interpretative phenomenological case study (see Bleiler, 2012) investigating a team-teaching collaboration between a mathematician (Dejan) and a mathematics educator (Angela) within the context of a university undergraduate teacher preparation program for prospective secondary mathematics teachers (PSMTs). Dejan and Angela worked together to plan, teach, and assess a mathematics content course (Geometry, Fall 2010) and a mathematics methods course (Teaching Senior High School Mathematics, Spring 2011) for PSMTs. The overarching research question guiding this inquiry is the following: In what ways do a mathematician (Dejan) and a mathematics educator (Angela) make sense of their experiences engaging in a team-teaching collaboration within a mathematics content course and a mathematics methods course for PSMTs?

To provide insight into the overarching question above, I used the following sub-questions as a guide to understand particular elements of Dejan and Angela's perceived experiences during their collaboration: (1) In what ways do Dejan and Angela make sense of their similarities or differences in relation to their perceptions of teaching and learning?, (2) In what ways do Dejan and Angela make sense of their roles within the team teaching collaboration?, (3) What do Dejan and Angela perceive as the affordances, if any, of their experiences in the team-teaching collaboration?, and (4) What do Dejan and Angela perceive as the constraints, if any, of their experiences in the team-teaching collaboration?

Methodology.

To answer these questions, I employed interpretative phenomenological analysis (IPA) (Smith, Flowers, & Larkin, 2009) as my methodological framework. In comparison to more traditional approaches to phenomenology (e.g., Husserl, 1970; van Manen, 1990), in which phenomenological research is conceptualized as "the study of the lifeworld—the world as we immediately experience it pre-reflectively rather than as we conceptualize, categorize, or reflect on it" (van Manen, 1990, p. 9), interpretative phenomenological analysis is focused on the lived experiences of individuals within a particular context as those experiences are reflected on and interpreted by the individuals themselves (Smith, Flowers, & Larkin, 2009). Because the main question guiding this research is focused on the ways Dejan and Angela make sense of their experiences while team-teaching, IPA is particularly useful as a methodological framework.

Interviews with each of the instructors served as the primary source of data for this study, as interviews were the most appropriate means of gaining an understanding of instructors' perceived experiences. Individual (one-on-one) interviews were conducted at the beginning and end of each of the two semesters of team-teaching (4 individual interviews per instructor). In addition, two group interviews (researcher and two instructors together) were conducted at the end of each semester of team-teaching. Other data sources (e.g., video-recordings of class sessions, audio-recordings of instructor planning sessions, researcher observation notes) were collected to triangulate the findings and to inform the contextual description of the case.

Theoretical Perspective.

I utilized situated learning theory (Lave and Wenger, 1991; Wenger, 1998) as an interpretive lens to describe and explain Dejan and Angela's meaning-making throughout their collaboration. In particular, I followed Wenger's (1998) perspective that learning and meaning are situated within communities of practice. I aimed to provide insight into the ways Dejan's and Angela's identities and understandings as members of the mathematics and mathematics education communities, respectively, were instrumental in the ways they made sense of their team-teaching experiences. Therefore, I analyzed the interview transcripts from the perspective that the instructors' words were representative of their personal understanding of "being" in their respective communities (Hemmi, 2006) as opposed to directly representative of the mathematics and mathematics education communities of practice.

Results.

The themes that emerged from this interpretative phenomenological analysis illustrate (a) how crossing community boundaries led to Dejan and Angela's increased awareness of their practice, (b) the roles of coach and student taken on by Angela and Dejan throughout the collaboration in an effort to increase Dejan's awareness of the needs of PSMTs, and (c) the influence of mutuality as a driving force in the instructors' collaborative experiences.

Using the situated learning perspective as an interpretive lens proved useful to describe and gain greater understanding of the instructors' perceived experiences. Some of the major insights derived from this analysis were (a) the importance of the dual processes of participation and reification to facilitate learning and meaning between instructors, (b) the ways in which a lack of shared history can hinder communication between collaborators, (c) the influence of a community's "regime of mutual accountability" on collaborators' decision making and interactions, and (d) the value and complexities of brokering and crossing boundaries.

Conference Presentation and Future Directions.

During this preliminary report session, I will provide a brief overview of the study described above and then engage participants in a reading of several extracts from Dejan's and Angela's interviews that highlight the instructors' perceived experiences of collaboration, focusing specifically on the instructors' perceptions related to crossing boundaries of their professional communities of practice. I will seek feedback from participants related to their own experiences "crossing boundaries" (Wenger, 1998) between the mathematics and mathematics education communities by posing the following questions: Have you been involved in collaborative efforts that "cross boundaries" between the mathematics and mathematics education communities? Does your professional identity lie more so in one of these communities than the other? How does your identity and understanding as a member of the mathematics (mathematics education) community influence your perspective when collaborating with members of other communities of practice? How does your identity and understanding as a member of the mathematics (mathematics education) community influence your perspective when teaching/preparing undergraduates and/or prospective teachers?

I expect the discussion that transpires during this session will provoke members of the audience to think deeply about their own educational experiences and community identities, their assumptions about the teaching and learning of mathematics that stem from those experiences and identities, and how those assumptions might help or hinder progress in the education of prospective teachers at their own institutions. As suggested by Barritt (1986), "By heightening awareness and creating dialogue, it is hoped research can lead to better understanding of the way things appear to someone else and through that insight lead to improvements in practice" (p. 20).

The ultimate goal of this research program is to build a significant library of cases from which a cross-case inductive analysis can be conducted. From such an analysis, I plan to collaborate with other researchers to build a practice-based theoretical framework to guide the implementation of successful collaborations between mathematicians and mathematics educators across different contexts related to mathematics teacher professional development. The discussion that evolves from this proposed conference session has the potential to inform future research with respect to possible considerations for the development of case investigations of collaborative endeavors in different contexts.

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DIFFICULTIES IN USING VARIABLES – A TERTIARY TRANSITION STUDY

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This article describes the results obtained from a diagnostic instrument to establish the difficulties in understanding and using variables that engineering students have at the moment of their entrance to a public Mexican university that does not examine the candidates prior to admittance. This work is part of a study that analyses the possible impact that failing to understand the uses of variables may have in understanding systems of linear equations.

Key words: Uses of Variables, Simultaneous Linear Equations, 3 Uses of Variables Model, Tertiary Transition.

This study has been held in a Mexican public university that does not examine the applicants to be admitted; it only requests a High School Certificate. If the number of applicants is greater than the number of places available, the decision is taken by means of a lottery in front of a notary public. Furthermore, the students that this university receives are in most of the cases people that have interrupted their studies, from high school to university, and that come from the least favoured zones in Mexico City, where it is located.

All these preliminaries already justify the wish to know how our students understand and use variables, particularly since they are to become engineers and do not have a strong mathematical background. But our main interest focused in identifying specific difficulties in using the variables, to apply a didactic treatment based on the 3 Uses of Variables Model (3UV Model; Ursini & Trigueros, 1997, 1999, 2001) and later analyse how a rich/poor conception of variable interferes in achieving a correct mental construction of the solution to simultaneous linear equations, from the linear algebra perspective. Some researchers had suggested that not understanding the many uses of variables correctly can contribute to difficulties in understanding linear algebra (Dorier, Robert, Robinet & Rogalski, 2000), in particular the solution to a system of linear equations (Trigueros, Oktaç & Manzanero, 2007).

In this article we will focus our attention in the first part of our study, which consisted in designing a diagnostic instrument and to process the results using the 3UV Model as a conceptual framework. We show the results we obtained and the conclusions to which they led us.

Theoretical Framework and Research Methods

From the research literature we realized that we had to propose an instrument that would make it possible to identify students' common difficulties while working with variables, linear equations in one unknown and linear equations in two unknowns (all of them necessary prerequisites to the study of systems of linear equations). Some of the elements we took into consideration in our diagnostic instruments are: different factors that make literal symbols hard to understand (Wagner, 1983), arithmetic difficulties interfering with the correct solution of equations (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996) and accepting and finding multiple solutions to a linear equation in two variables (Panizza, Sadovski & Sessa, 1999). Since our main objective was to know how students performed with respect to the different uses of variables and how flexibly they could adapt to changes in the use of variables along one same problem, we chose the 3UV Model as our theoretical framework.

The 3UV Model is a theoretical framework proposed by Ursini and Trigueros (1997, 1999, 2001) as "a basis to analyse students' responses to algebraic problems, to compare students' performance at different school levels in terms of their difficulties with this concept, and to develop activities to teach the concept of variable " (Trigueros & Ursini, 2008, p.4-337). The 3UV model takes into consideration the three most frequently present uses of variable in elementary algebra: specific unknown, general number and variables in functional relationship. Its authors emphasized aspects corresponding to different levels of abstraction at which each one of the uses of variable can be handled. These aspects are described in the following paragraphs:

According to the 3UV Model, the understanding of variable as unknown requires to: recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem (U1); interpret the symbols that appear in equation, as representing specific values (U2); substitute to the variable the value or values that make the equation a true statement (U3); determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations (U4); symbolize the unknown quantities identified in a specific situation and use them to pose equations (U5).

The understanding of variable as a general number, according to the 3UV Model, implies to be able to: recognize patterns, perceive rules and methods in sequences and in families of problems (G1); interpret a symbol as representing a general, indeterminate entity that can assume any value (G2); deduce general rules and general methods in sequences and families of problems (G3); manipulate (simplify, develop) the symbolic variable (G4); symbolize general statements, rules or methods (G5).

As the 3UV Model considers it, the understanding of variables in functional relationships (related variables) implies to be able to: recognize the correspondence between related variables independently of the representation used (F1); determine the values of the dependent variable given the value of the independent one (F2); determine the values of the independent variable given the value of the dependent one (F3); recognize the joint variation of the variables involved in a relation independently of the representation used (F4); determine the interval of variation of one variable given the interval of variation of the other one (F5); symbolize a functional relationship based on the analysis of the data of a problem (F6) (Trigueros and Ursini, 2008).

As Trigueros and Jacobs (2008, p.105) recall it, "according to Trigueros and Ursini (1999, 2001, 2003) a well-developed understanding of algebra necessitates the ability to differentiate among the three uses of variable and to flexibly integrate their uses during the solution of any problem".

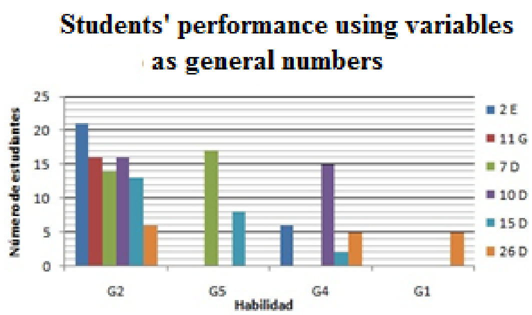
Using both, the information coming from research literature and the abilities described in the 3UV Model, the instrument questions were designed to give us insight in whether or not the students presented the difficulties described in the research literature, how students related to the different uses of variables and how flexibly they could identify a change of use in the variable in some situations. The result was a 28-question-intrument that was applied to 25 students in two 75-minute-sessions. After the first session, the students gave back the instrument together with the answers, so that at the beginning of the second session, they would continue from the point where they left the questionnaire in the previous session. The instrument was applied in the first two sessions of a first-semester-course in algebra and analytic geometry (AAG).

Preliminary results – results of the Diagnostic Instrument

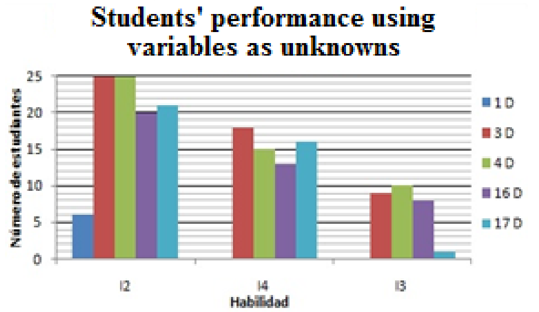
In this section we describe the results for each use of the variables and present a graph showing the general results we obtained of how students performed for each exercise of the instrument that was related to the abilities considered in the respective use of the variables being described. We decided to show general results rather than specific performance for specific exercises, to show a broader view of how rich/poor the variable conception of our students is when they enter university and to show the various aspects that would have to be taken into consideration to project a potential didactical treatment to help them enrich their conception of variable, thinking that they need to use variables fluently to solve and understand the concept of solution to simultaneous linear equations.

For all the graphs that will be presented, we show in the horizontal axis the abilities of the 3UV Model for the respective use of the variables. In the vertical axis, we show the number of students that performed correctly for the respective exercise-ability, which is represented by a bar. Each bar is labelled by the number of exercise in the diagnostic instrument, followed by a capital letter indicating which type of exercise it was. E holds for “Give an Example” tasks, G holds for “Twist” questions, D holds for Performance questions and R holds for Reflection questions, all these categories according to Zazkis and Hazzan (1999).

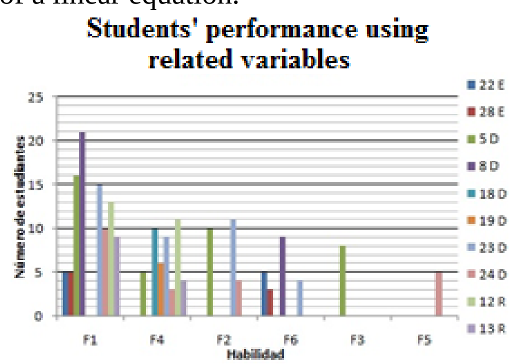
Results for the use of variables as general number: our results show that students perform better for G2, than for the rest of the abilities, but that the complexity of the exercises has a direct effect: the higher the complexity of the exercise, the lower the performance. Using exercises of different complexity results in a stronger change in performance for G4 and G5. For instance, manipulating a sequence of sums and differences that involve the variable does not represent a big challenge, but manipulating a perfect square trinomial, that requires a substitution, to rewrite it as a square binomial, already represents quite a difficult task; manipulating a sequence of operations involving variables as denominators, turned out to be an extremely difficult task. Symbolizing an open expression that involves a variable added to a number, is a relatively simple task; but symbolizing the result of a product of variables or numbers and variables is not that simple a task. In the case of G1, a regular high-school-substitution to rewrite an expression turned out to be very challenging for most of the students, who avoided it completely.



Results for the use of variables as unknowns: when presented with exercises that involve the use of variables as unknowns, students seem to be most at ease with U2, but we found that when the context is not that familiar to them, it is not clear for them when a variable is really an unknown: they tend to treat the variables in a two-variable linear equation as unknowns, not noticing that the variables are related by the equation. This is similar to what has been reported by Malisani and Spagnolo (2009) and Panizza, Sadovski and Sessa (1999). In general, we found that students consider a literal to be an unknown if it appears in an expression with an “=” sign, and that in the absence of it, they sometimes add a “= 0” to be able to manipulate the expression. Regarding U4, we found that students tend not to use algebraic procedures if it is not that difficult to solve the equation by arithmetic operations and that only in the case of equations presenting the variable on both sides of the equation they used algebraic operations from the beginning in the solution of the problem. Linchevsky and Herscovics (1996) had already detected this problem. We also found that students are not used to substitute for the variable the value or values that make the equation a true statement (U3) if they are solving the equation algebraically, as if solving it would mean to find a value through the algebraic procedure and not to find a value that satisfies it, which has also been pointed out before by Sfard and Linchevsky (1994). Another detected problem is that students tend to forget what the purpose of the problem is, and they rarely turn their attention back to the questions to check whether finding a solution was enough for solving the problem posed or if they would still need to do something else; this had been reported by Trigueros and Jacobs (2008).

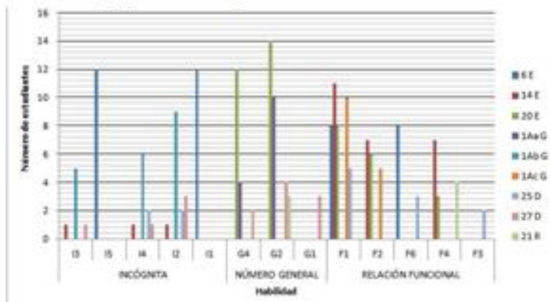


Results for the use of related variables: our main result when it comes to analysing students' performance for related variables is that with the exception of some cases in which the context is more familiar to the students, they do not know how to act when confronted with equations in related variables. For those familiar situations, they only perform relatively well for F1. They even present difficulties in determining the values of the dependent variable given the value of the independent one (F2) and vice versa (F3), if a specific value is not given for one of the variables explicitly, which coincides with what Panizza, Sadovski and Sessa had reported (1999). Not even in the case of related variables of the form "$\boldsymbol{y} = \boldsymbol{k}\,\boldsymbol{x}$", with $\boldsymbol{k}$ a specific explicit constant given in the problem statement, students managed to symbolize the relationship between the variables, showing how weak their F6 ability is. Recognizing the joint variation of the variables (F4) was shown not to be that difficult only in the case of very simple familiar cases (either because the problem context was a familiar situation to them or the structure was familiar to them) and when they had to reflect about someone else's response instead of answering directly. Determining the interval of variation of one variable given the interval of variation of the other (F5) was not easy even in the case of a graphic representation of a linear equation.



Results for transitions in the uses of variables: if students have to adapt to a change in the use of a variable along the solution of the same problem, they have really strong difficulties. Of all the problems that involved a change of variable for the solution, only one student was able to solve completely most of them (and when not, it was due to arithmetic or algebraic operations). Students in general would interrupt their solution process after wandering for a while trying different things without a clear structure of what they planned to do, until they would just give up without summing-up, or they would stick to one trial and follow it until they somehow would end up with a response that they never verified to be a solution.

Students' performance solving problems that imply a change is the use of variables



Concluding Remarks

We conclude from the analysed data that the students taking part in this study still have a long way to develop a rich conception of variable, and that, as Trigueros and Jacobs (2008) pointed out, “students need help in developing a rich conception”. It is necessary to help students in advancing their conceptual understanding of variables, considering the unfavourable conditions under which the students of this university have to work and not “to expect that as students encounter algebraic expressions, word problems, and problem-solving exercises, they will construct (all by themselves!) a robust, flexible and coherent conception of variable as a mathematical entity” (Trigueros and Jacobs, 2008, p.110). For the purposes of our research, now that we know in which aspects our students have a weak conception of variable, we have to first test how they perform in solving and interpreting systems of linear equations and, second, by means of analyzing the respective results with the help of the 3UV Model, make a selection of the elements that have to be strengthened to understand a system of linear equations and propose a didactic treatment accordingly. This work is in process and will be reported in future papers.

Questions to the audience

a) Considering the weak background in mathematics that our students have and that they find it very difficult to do homework due to their jobs schedules, what didactical considerations could help them in making the best use of the class time to develop the necessary abilities to enrich their concept of variable and perform better in their engineering courses?
b) What kind of short projects could be developed along a semester-course to foster the acquisition of deeper understanding of the concepts introduced in the Algebra and Analytic Geometry and Linear Algebra courses?

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SYSTEMATIC INTUITIVE ERRORS ON A PROVE-OR-DISPROVE MONOTONICITY TASK

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Despite the importance of intuition and analysis in proof tasks, students have various difficulties with both types of reasoning. Such difficulties may be attributed to insufficient intuition, logical reasoning skills, or concept images. However, dual-process theory asserts that intuition can form inaccurate or incomplete representations of tasks based on systematic errors before analysis can respond. Thus, students' difficulties may be attributed to systematic intuitive errors rather than inadequate intuitive or analytical reasoning. In this study, I conducted task-based interviews with four undergraduate and one graduate mathematics major in which they completed prove-or-disprove tasks. In this paper, I discuss the systematic intuitive errors committed by these students on a monotonicity task. These errors led all five students to believe incorrectly that the statement in the task was true. Furthermore, each student engaged in correct mathematical reasoning guided by their incorrect intuitive representations.

Key words: Reasoning and proof, Intuition, Dual-process theory, Task-based interviews

Intuition and analysis are fundamental components of mathematics that play key roles in producing proofs and counterexamples (Fischbein, 1987; Tall, 1991; Wilder, 1967). However, undergraduate students have a multitude of difficulties with both types of thinking that inhibit effective reasoning on proof tasks. Intuitive difficulties include (a) lack of intuition (Moore, 1994) and (b) narrow intuitions based on examples and visualizations (Moore, 1994; Raman, 2003). Analytical difficulties include (a) limited logical reasoning skills (Harel & Sowder, 2007; Selden & Selden, 1987) and (b) incomplete or inaccurate concept images (Tall & Vinner, 1981). Furthermore, students have difficulties connecting their intuitive understandings to analytical arguments. Students' intuition may not lead directly to a proof or counterexample, or students may not recognize the relationship between their intuition and a proof or counterexample (Raman, 2003). This may result in students' inability to understand formal mathematical statements or begin a proof (Moore, 1994).

Dual-process theories of reasoning assert that intuition and analysis correspond to distinct types of cognitive processing, each with specified characteristics and roles (Evans, 2006, 2008, 2010; Kahneman, 2002). Although dual-process theories result from cognitive psychology research, Leron and Hazzan (2006, 2009) have suggested using them to analyze recurring errors in mathematical tasks. Dual-process theories suggest that certain systematic errors that recur across tasks and participants can be attributed to flawed intuitive reasoning that steals the show before analytical reasoning even takes the stage.

Preliminary results from a study in which students decided whether to prove or disprove a mathematical assertion and constructed corresponding proofs or counterexamples indicate that students' errors may be attributed to systematic intuitive errors. Furthermore, students' analytical reasoning was incorrect only because it was based on these errors.

Theoretical Framework

Dual-process theory asserts that reasoning uses two distinct types of cognitive processes – intuitive and analytical (Evans, 2008). Although these processes collaborate, intuition often dominates and influences analysis in unproductive ways (Evans, 2010).

Intuition is often quick, automatic, requires little cognitive effort, and is developed through experience (Evans, 2008; Fischbein, 1987; Wilder, 1967). The automatic operation of intuition frequently provides a default response to a task (Evans, 2010; Wilder, 1967) that takes into account prior knowledge and beliefs, task features, and the current goal of the reasoning to create a representation of the task (Evans, 2006). However, the representation may be distorted or deficient due to systematic accessibility errors of intuitive reasoning (Evans, 2010; Kahneman, 2002). *Accessibility* is the ease with which certain knowledge is evoked or certain task features are perceived (Kahneman, 2002). Two key accessibility errors involve (a) attribute substitution, and (b) knowledge and task feature relevance.

Attribute substitution errors occur when a more easily accessible attribute is substituted in a task for a less easily accessible attribute (Kahneman, 2002). Participants often intuitively notice similarities between the current task and previously encountered tasks, substitute accessible attributes for less accessible ones based on these similarities, and unknowingly change the given task to a similar more accessible task (Kahneman & Frederick, 2002).

Relevance errors occur when knowledge and task features are deemed irrelevant because they are not readily accessible (Evans, 2008, 2010; Kahneman, 2002). When forming intuitive task representations, (a) less accessible relevant knowledge is often not applied to the task (Weber, 2001), (b) less accessible relevant task features are often overlooked, and (c) more accessible irrelevant task features are often overemphasized (Evans, 2008).

Analysis is frequently slow, deliberate, requires much cognitive effort, and is developed through reflective and logical thinking (Evans, 2010). Analytical reasoning is often brought into action in response to an intuitive representation of a task (Evans, 2010; Kahneman, 2002). However, the power of intuition may result in analytical reasoning being (a) bound to a biased or incomplete intuitive representation or (b) invoked solely to justify an intuitive representation, thus failing to consider alternative representations of a task (Thompson, 2009). Thus, analytical reasoning may not be able to overcome faulty intuitive reasoning.

Method of Inquiry

The participants in this study were four undergraduate mathematics majors, called Ann, Brian, Chris, and Dave, in transition-to-proof mathematics courses at two private liberal arts colleges in Ohio and West Virginia and one graduate student in mathematics, called Ben, at a public university in Ohio. I conducted individual, semistructured, task-based interviews with each participant (Goldin, 2000). Each interview was audio-recorded and transcribed. During the interview, participants worked on three prove-or-disprove tasks, including the following:

> *Monotonicity task*: Definitions: A function f: $\mathbb{R} \to \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$). Similarly, a function f: $\mathbb{R} \to \mathbb{R}$ is said to be **decreasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) > f(x_2)$). Prove or disprove: If f: $\mathbb{R} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ are decreasing on an interval I, then the composite function $f \circ g$ is increasing on I.

I instructed the students to think aloud during the tasks and to clarify or expand on their thinking as necessary. Upon completion of the tasks, I asked the students about difficulties they had with the tasks and general strategies they used for prove-or-disprove tasks.

Analysis included the following: (a) categorizing students' reasoning as intuitive or analytical, (b) identifying errors in students' reasoning, (c) classifying errors as intuitive or analytical, (d) categorizing intuitive errors as relevance or attribute substitution errors, (e) determining the impact of intuitive errors on analytical reasoning.

Preliminary Results

Each student believed that the false statement in the monotonicity task was true and attempted to prove it. Ann and Brian each committed an attribute substitution error, and Chris, Dave, and Ben each committed a relevance error.

Ann and Brian made attribute substitution errors that prohibited them from making significant progress on the task. Ann substituted the similar concept of *negative times negative equals positive* for the task concept of *decreasing composed with decreasing equals increasing*. Brian substituted the incorrect concept *odd times odd equals even* in place of *decreasing composed with decreasing equals increasing*. As an example of attribute substitution, consider Ann's intuitive response to this task:

> Well my first thought is just simply that if the two functions f and g are both decreasing, then at that point, then both of the slopes would have to be negative, or something in there would have to be negative, which, and then I go to the simple [idea] that a negative times a negative is a positive. That would be increasing.

Ann changed the given task, replacing it with a similar and more accessible task. She then illustrated her idea with an example in which she composed two negative functions resulting in a positive function. Thus, her analytical reasoning supported her intuition. However, she was unable to begin a proof of the task statement.

Chris, Dave, and Ben made relevance errors that led them to construct false proofs for the task. They each ignored the interval restriction in the task, responding as if the task was to prove or disprove the following: If $f\colon \mathbb{R} \rightarrow \mathbb{R}$ and $g\colon \mathbb{R} \rightarrow \mathbb{R}$ are decreasing, then the composite function $f \circ g$ is increasing. The proofs that Chris, Dave, and Ben constructed were correct proofs for this similar, more accessible task. Each student produced essentially the same false proof of the task, for which Ben's proof is representative:

> Suppose $x_1 < x_2$. Then g decreasing implies $g(x_1) > g(x_2)$. Now apply f to $g(x_1)$ and $g(x_2)$. $g(x_2) < g(x_1) \rightarrow f(g(x_2)) > f(g(x_1))$. Started with $x_1 < x_2$, conclude $f(g(x_1)) < f(g(x_2))$. Therefore $f \circ g$ is increasing.

Each student focused on the basic idea that a decreasing function composed with a decreasing function would result in an increasing function. There was no spoken or written consideration of the interval restriction by any of these three students.

Discussion

Each student demonstrated some intuition on the monotonicity task and created a deficient intuitive representation of the task based on a systematic intuitive error. This led each student incorrectly to judge the statement to be true. The students' subsequent work was based on this ill-formed intuitive representation, but their analytical reasoning was mathematically correct. Ann and Brian constructed examples to help them support their intuitive representations and correctly interpreted the information in the examples, but were unable to move beyond examples to more general representations of the task. Chris, Dave, and Ben constructed correct proofs to a similar, yet different, mathematical assertion.

These students demonstrated intuition, valid mathematical reasoning, and use of relevant mathematical knowledge on this task. Furthermore, these students connected their intuition to their analytical work on the task. So why did they still all think this false statement was true and back up their claim with legitimate mathematical work? These students were victims of the quick, automatic processing of their intuition which developed an inaccurate or incomplete representation of the task for their analytical reasoning to process. Due to the power of these intuitive errors, the students employed analysis to support them rather than correct them. Thus, the students' intuition had already warped the task before their analytical reasoning had the chance to respond.

Questions for the Audience

Does the dual-process theory perspective have useful practical implications? Is the monotonicity task a "trick" question? Would a larger scale comparison study with undergraduate and graduate students be a worthwhile step to further this work?

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COOPERATIVE LEARNING AND TRAVERSING THE CONTINUUM OF PROOF EXPERTISE: PRELIMINARY RESULTS

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This paper describes preliminary results of a study aimed at examining the effects of working in cooperative groups on acquisition and development of proof skills. Particular attention will be paid to the varying tendencies of students to switch proof methods (direct, induction, contradiction, etc) based on their level of proof expertise. Namely, as students progress from novice to expert provers, they tend to change proof methods more frequently until they reach the final stages of development (Hart 1994).

Key words: Transition to Proof, Proof Writing Expertise, Proof Methods

Introduction

Although proof is essential to studying mathematics, much research in the past two decades shows that students struggle with constructing and validating proofs (Almeida, 2000; Harel & Sowder, 1998; Levine & Shanfelder, 2000, Moore, 1994; Selden & Selden, 2003a, 2003b; Weber, 2001; Weber, 2003). Several innovative course structures have been introduced for so-called bridge courses (Almeida, 2003; Bakó, 2002; Grassl & Mingus, 2004), but little dedicated research has been done on the effectiveness of such courses. However some common themes have emerged about the necessity for and efficacy of active learning strategies, and there is a general trend away from lecture and toward more student-centered models models. In particular, this can be seen within the Modified Moore Method community (McLoughlin, 2010).

Cooperative learning (CL) is one such model. "CL may be defined as a structured, systematic instructional strategy in which small groups work together to produce a common product" (Cooper, 1990). There are five specific features that, when combined, distinguish CL from other active and collaborative learning strategies: positive interdependence, individual accountability, student interaction, attention to social skills, and teacher as facilitator. While the efficacy of CL has been researched (Johnson & Johnson, 1991), the majority of this research has been undertaken with precollegiate populations.

Studies done on CL and active learning in the context of physics instruction (Deslauriers, et al, 2011; Heller & Hollabaugh, 1992; Heller, et al., 1992) give hope that CL could be effective in helping students acquire and develop their proof skills. This paper looks at some of the preliminary results of a study exploring the relationship between CL and proof-skill development. Specifically, the study was designed to examine how working in a CL seminar environment affected 1) students' attitudes about proof, 2) students' ability to construct proofs, and 3) students' abilities to validate student-generated arguments. The second of these will be addressed in this paper.

Hart (1994) compared expert and novice proof writers through use of a proof test. He categorized 29 undergraduate math majors by their level of proving expertise using three specific tasks from the test, and rated the students in one of four levels: Level 0: pre-understanding, Level 1: syntactic understanding, Level 2: concrete semantic understanding, Level 3: abstract semantic understanding (p. 56). He then examined the students' individual proof production processes noting similarities that arose among students at the same level of understanding. The mathematical context of Hart's study, abstract algebra, differed vastly from that of the study addressed in this paper, but some generally applicable findings were reported.

He noted that it is important not to try to get novice provers to perform like expert provers all at once; there is a continuum of expertise that must be traversed, even though the progression across it is often not smooth. In particular, he noted that expert provers switch proof methods less often than the most novice provers, but that the tendency to change plans increased between levels at all but the final step (pp. 59-60). This study examined whether a cooperative learning environment would enable students to become more expert provers and whether this would be marked by a similar change in tendency to change plans.

Methods

The subjects for this study were seven "seminar" students (five male, two female) and three "comparison" students (two male, one female). All students were undergraduates at a large, public university with a declared major or minor in mathematics.

All subjects took pre- and post- assessments of their proof construction skills via three proof prompts in basic number theory. The assessments of the seminar students were conducted in the presence of the researcher, and those subjects were asked to think aloud as they attempted to construct the proofs. The assessments of the comparison students were conducted in a group setting with each subject working independently and silently. At least 11 weeks passed between pre- and post- assessments for all subjects. In this paper, I will focus on the subjects' proof construction performance of all but one student. This particular student, Zach, did not make much effort on post-assessment, instead spending much of the interview complaining about the research methods. As a result, he performed more poorly on the post-assessment than he had on the pre-assessment (see Table 2).

The three prompts listed below were presented as true theorems dealing with elementary number theory concepts accessible to all of the subjects regardless of prior background and testing varying proof skills the researcher believes occur across content areas (see Table 1).

Assessment Item	Hypothetical Skill(s) Tested
1. Prove: If m^2 is odd, then m is odd.	• Use of indirect proof methods. • Avoidance of a more accessible converse argument.
2. Prove: If n is a natural number, then n^3-n is divisible by 6.	• Ability to identify pertinent subclaims and construct subarguments (divisibility by 2 and 3).
3. A *triangular number* is defined as a natural number that can be written as the sum of consecutive integers, starting with 1. Prove: A number, n, is triangular if and only if $8n+1$ is a perfect square. (You may use the fact that $1+2+...+k=\frac{k(k+1)}{2}$.)	• Use of the specifics of a definition to form a basis for a proof. • Ability to identify the logical implications of "if and only if" statements. • Use of previously established results (to prove $8n+1$ a perfect square implies that n is triangular, the result of item one needs to be applied).

Table 1. Assessment Items

Between assessments, the seminar students met with the researcher for eight, 90-minute sessions during which they worked on problem sets in cooperative groups. The cooperative groups were consistent throughout the study and were formed to be heterogeneous based on gender and on skill level as demonstrated on the pre-assessment. The members of each group spent a few minutes at the beginning of each session getting to know each other and 5-10 minutes at the end of each session doing group processing exercises. Both of these exercises facilitated the development of the social skills necessary for effective cooperative work, and the rotating roles (manager, explainer, skeptic, presenter) the students assumed each session assured their personal accountability and positive interdependence. After a brief introduction each session, the students worked with each other and the researcher functioned solely as a facilitator, encouraging the student-to-student interactions.

The problem sets dealt with function concepts, primarily injectivity and surjectivity, and the seminar group did not work with number theoretical concepts. This was done so that any changes from pre- to post-assessment would reflect changes in the subjects' proving skills independent of mathematical context.

The video recordings of the seminar students' assessments and of all seminar sessions were transcribed, and the transcriptions of the assessments were coded for instances in which subjects changed proof methods (direct, contradiction, contrapositive, induction) or switched to a different proof but returned later. All written proof attempts were also analyzed for correctness (0 – no progress or completely flawed, 1 – minimal progress or progress with substantial flaws, 2 – some progress with some flaws, 3 – substantial progress but incomplete or with minor flaws, 4 – correct proof). Specific errors appearing in the proofs were also coded according the list of common errors and misconceptions by Selden and Selden (2003b).

Preliminary Findings

Six of the seminar students showed dramatic improvement from pre-assessments to post-assessment. Those six were all able to reprove the results proved on the pre-assessment, though sometimes in a different manner, and all six were able to prove additional items. Despite the fact that their performances on the pre-assessment did not differ greatly from those of the seminar students, the three comparison students, all of whom were enrolled in at least one proof-based course, showed no noticeable improvement on the post-assessment.

Most of the six seminar students under consideration changed proof methods more frequently on the post-assessment than they had on the pre-assessment. The students who had the greatest change were those who had the weakest performances on the pre-assessment. There was only one student, Bill, who changed proof methods less frequently on the post-assessment than on the pre assessment, and he was one of the strongest students on the pre-assessment (see Table 2). These results mirror Hart's (1994) findings that as students progress from novice to expert provers, they are more likely to change plans mid-proof, except at the final stage of development when that tendency decreases. This progression was even shown on individuals' performances on specific tasks (see Table 3) illustrating the "rather unstable, irregular, developmental process" (Hart, 1994, p. 61).

Student	Pre-Assessment		Post-Assessment	
	Total Score	Number of Switches	Total Score	Number of Switches
Omar	0	0	4	1
Ursula	0	3	8	6
Ingrid	1	0	8	6
Ivan	4	3	6	3
Zach*	4	4	2	0
Nathan	6	1	8	2
Bill	6	5	9	4

Table 2. Seminar Student Performance on Assessments

The considered students who performed the best on the pre-assessment, Ivan, Nathan, and Bill, all had items on the post-assessment for which they changed plans less but performed as well or better. However, the lowest-performing students on the pre-assessment, Omar, Ursula, and Ingrid, all changed plans at least as many times on every item on the post-assessment as they had on the pre-assessment (see Table 3 for examples).

BILL	Description of Performance	Score	Number of Switches
Item 1 - pre	Produced a valid proof by contradiction.	4	3
Item 1 - post	Produced a valid proof by contrapositive	4	1
Item 2 - pre	Produced a proof that n^3-n is even, and recognized he was missing that 3| n^3-n.	2	3
Item 2 - post	Produced a proof that n^3-n is even, and recognized he was missing that 3| n^3-n.	2	1
Item 3 - pre	Manipulated the equation $8n+1=x^2$, but the manipulations were unproductive.	0	0
Item 3 - post	Produced a proof of both directions, but was missing the justification that 8n+1 is necessarily odd, so if it is a perfect square, then it is the square of an odd number.	3	2
URSULA			
Item 1 - pre	Produced an empirical contradiction argument with use of a single example, m=2.	0	1
Item 1 - post	Produced a valid proof by contrapositive.	4	2
Item 2 - pre	Did not identify subgoals, attempted a proof by induction, but could not get to conclusion even though the work was error-free.	0	1
Item 2 - post	Identified subgoals, attempted a proof by contradiction again and successfully proved that 3| n^3-n. Had the work to get 2| n^3-n, but did not recognize that k^2-k is necessarily even.	2	3
Item 3 - pre	Correctly stated givens and goals, attempted to use $1+2+...+k=\frac{k(k+1)}{2}$, but set up $8(\frac{k(k+1)}{2})+1=triangular$. Made no progress from there.	0	1
Item 3 - post	Proved that n triangular implies 8n+1 is a perfect square. Made no progress on reverse direction.	2	1

Table 3. Student Performance on Individual Items

Based on Hart's (1994), these data show that not only were the seminar students able to improve their performances between assessments, but that they matured along the spectrum of novice to expert provers. Additionally, the stark difference seen between the improvement of the students in the seminar group and the lack of improvement of those in the comparison group indicates that the cooperative seminar was key to the seminar students' improvement. This study suggests the need for further study along these lines in randomized comparative trials of the effects of cooperative learning on students' development of proof skills. Also, while proof researchers often talk about proving skills as if they were not context-dependent, this study shows strong support that some proving abilities are truly context-independent, but this needs to be studied in much more detail.

Questions

1. What other indicators that the research subjects progressed from more novice to more expert provers might I look for?
2. The tasks Hart used to define expertise levels were very different from my tasks. Is it reasonable to draw similarities and parallels from his work given that I can't apply the levels to my own students directly?
3. What proof skills do you think may be context-independent? Do you think there are any?

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PROVIDING OPPORTUNITIES FOR COLLEGE-LEVEL CALCULUS STUDENTS TO ENGAGE IN THEORETICAL THINKING

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Previous research has reported an absence of a theoretical thinking component in college-level Calculus courses. While valid arguments can be made for or against the necessity of incorporating such a component, our belief is that students who wish to engage in theoretical thinking should be given the chance to do so. Our goal is to determine whether instructors can provide their students with opportunities to engage in theoretical thinking, despite constraints they often face such as time, course content, and assessment material. This report presents a preliminary analysis of a study in which we presented students in a Calculus class with tasks intended to provoke theoretical thinking. Using Sierpinska et al.'s (2002) model for theoretical thinking we show that students who engaged in these optional tasks where in fact engaging in theoretical thinking. We conclude that, despite the institutional constraints, incorporating such a component is indeed feasible.

Keywords: Theoretical thinking, Calculus, Quizzes, Institutional constraints

This paper presents preliminary results of an ongoing research. The study was conducted in a prerequisite Calculus class (college-level) offered at a university in Montreal. These are multi-section courses taught by different instructors but designed by a single course examiner. The course examiner writes the outline of the course, the weekly assignments, and the midterm and final exams. In addition to specifying the topics to be taught each week, the outline includes a list of "recommended" exercises from a common-assigned textbook which are indicative of the types of problems that will appear on the weekly assignments and midterm and final exams. The problems on these assessments can be described as "routine" problems in the sense of (Lithner, 2003; and Selden et al., 1999). Therefore, instructors of these courses face several constraints: a fixed outline (i.e., they cannot change it), with a fixed order for delivering the content, a pre-chosen set of exercises, assessments that they cannot modify; plus constraints associated to classroom time and class-size. Our goal is to explore whether in a course and setting such as the one described here, and despite the mentioned constraints imposed on the instructor, students can be provided with opportunities to engage in theoretical thinking. To answer this question, we propose and investigate the effectiveness of one means to engage students in theoretical thinking, described further below.

The motivation behind this research is the reported absence of a theoretical thinking component in Calculus courses; in fact, previous research describes Calculus students' predominant behaviors that are not indicative of theoretical thinking (e.g., Boesen et al., 2010; Hardy, 2010; Lithner, 2000; Selden et al., 1999), reporting a link between students' behaviors and the tasks set before them. Moreover, the development of mathematical reasoning skills, personal sense-making, and convictions of mathematical concepts are seen to arise from interactions that take place in the classroom (e.g. Yackel, 2004; Hanna, 1991); this study addresses these results particularly.

The study was conducted over a thirteen-week semester and the tool that was used to foster theoretical thinking was *weekly quizzes*. The quizzes consisted of one or two questions that were designed in a way such that meaningfully answering a question would require students to think theoretically. To determine whether the quizzes were successful in engaging students in theoretical thinking, the model developed by Sierpinska et al. (2002) was used after modifying and adapting it for this study.

Theoretical perspective

With a concern for the individual and a desire to characterize theoretical thinking, Sierpinska et al. (2002) were inspired by Vygotsky's work; namely his distinction between scientific and everyday (or 'spontaneous') concepts (1987). In particular, Vygotsky characterized scientific concepts as formed in the mind on the basis of concretization from general statements, as opposed to spontaneous concepts that are formed on the basis of generalization and verbalization from concrete experience. Furthermore, Vygotsky argued that theoretical thinking does not develop spontaneously in children as one last "stage" of their cognitive development, but requires special nurturing. Based on these assumptions Sierpinska et al. (2002) constructed a model of theoretical thinking, shown below, in which the main postulated categories of theoretical thinking are "reflective", "systemic", and "analytic" thinking. The authors presented features of theoretical thinking that were relevant to their study in a Linear Algebra class and operationalized the model with theoretical behaviors which, when displayed by a subject, are indicative of the occurrence of theoretical thinking.

Category of TT **• Feature of TT**	**Description**
TT1 Reflective	Theoretical thinking is thinking for the sake of thinking.
TT2 Systemic	Theoretical thinking is thinking about systems of concepts, where the meaning of a concept is established based on its relations with other concepts and not with things or events.
• TT21 Definitional	The meanings of concepts are stabilized by means of definitions.
• TT22 Proving	Theoretical thinking is concerned with the internal coherence of conceptual systems.
• TT23 Hypothetical	Theoretical thinking is aware of the conditional character of its statements; it seeks to uncover implicit assumptions and study all logically conceivable cases.
TT3 Analytic	Theoretical thinking has an analytical approach to signs
• TT31 Linguistic sensitivity	
o TT311 Sensitivity to formal symbolic notations	
o TT312 Sensitivity to specialized terminology	
• TT32 Meta-linguistic sensitivity	
o TT321 Symbolic distance between sign and object	
o TT322 Sensitivity to the structure and logic of mathematical language	

Table 1 – Sieprinska et al.'s (2002) model for theoretical thinking

The learning of Calculus and of Linear Algebra likely prompts different aspects of theoretical thinking due to their distinct natures, and in analyzing the model we noticed that we needed to customize the features and consider different theoretical behaviors since we were assessing theoretical thinking in Calculus; the operationalization of the model is discussed below. Clearly, this is not an exhaustive operationalization of the model since the listed theoretical behaviors are pertinent to the questions chosen for the quizzes of this particular study.

Methodology

The study was conducted in an integral Calculus class stretched over one term (thirteen weeks) with an average of thirty five students attending the two classes (1h15 each) per week. The instrument used for the study was a set of quizzes each consisting of a question related to material previously covered in the course. Once a week, students were given a quiz and fifteen minutes of the class time to respond to the question. The time allocated to these quizzes was done in a way so that the course outline was completed as required. It was explained that taking the quiz was optional and that students would be rewarded with 'bonus marks' on their course grade for a complete response or an incomplete response containing valid arguments. The quiz questions were of a conceptual nature, aimed at prompting a type of behavior that we characterize as a display of theoretical thinking. There was usually not a single solution path that had to be followed, but students were asked to "justify" their answers and generally be as expressive as possible. We show three of the quiz questions in the table below.

Is it true that $\int_a^b f(x)dx + \int_c^d g(x)dx = \int_c^d g(x)dx + \int_a^b f(x)dx$ where a, b, c, and d are real numbers?	Explain, in your own words, why this theorem is true: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$	If $g(x)$ is continuous for all real numbers and $\int_1^{\infty} g(x)dx$ is convergent, is $\int_{10}^{\infty} g(x)dx$ is also convergent?

Table 2 – Examples of quiz questions (from left to right: questions 2, 7, and 11)

Once corrected by the instructor, quizzes were returned to the students with a grade and written suggestions for improving the quality of their responses when they were inadequate. Students were not provided with an answer to the questions as we believed that this could inhibit their own creativity and possibly encourage them to mimic the instructor's answers or style.

Responses were analyzed based on our model: In the operationalization of the model we identified a total of thirteen theoretical behaviors (TB). To justify that a TB was indicated by a response, we created an *a priori* list of phrases ('features of discourse') describing possible elements of discourse in the response to each question; we considered these indicative of theoretical thinking and interpreted the performance of these actions as a display of a particular type of TB. Features of discourse that display the same TB were grouped together (as shown in Table 3). Due to space constraints, we display our model (Table 4) with only samples of TBs that correspond to the features of theoretical thinking.

Feature of TT	TB	**Features of discourse** (of possible answers to different questions)
TT1 Reflective	$TB1_2$ Generalizing a solution	• Writing a single *general* statement for positive and negative series by considering the absolute value of terms • Indicating that integral[a,∞) is convergent for *all* a≥1 • Indicating that the integral of g is convergent over *any* subinterval of [1, ∞) • Remarking that the addition of *any* non-zero constant to the integrand would result in a diverging integral

Table 3 – Sample of how our model is operationalized with features of discourse and TBs.

Feature of TT	Samples of corresponding TB
TT1 Reflective	$TB1_2$ Generalizing a solution
TT2 Systemic • TT21 Definitional • TT22 Proving • TT23 Hypothetical • TT24 Contextual	 $TB21_1$ Referring to definitions when deciding upon meaning $TB22_1$ Engaging in a proving or reasoning activity $TB23_1$ Being aware of the conditional character of a mathematical statement $TB24_1$ Modeling a problem
TT3 Analytic	$TB3_2$ Being sensitive to logical connectives, particularly to implication and its negation

Table 4 – Our model of theoretical thinking, including sample theoretical behaviors

Results and analysis

Two types of analysis were carried out so far; a *question* analysis determining which TBs (and thus types of TT) each question invited, and a *class* analysis: whether (and how) the class engaged in theoretical thinking in responding to each question. At a later stage an analysis of each student's engagement in theoretical thinking across the quizzes will be carried out.

Table 5 indicates how many times each TB was invited by each question (if at all) and overall, as well as how many times each category of theoretical thinking was invited. These are indicated by "count QX", "count TB", and "count TT" respectively (due to space constraints, details for questions 2 and 7 only are shown). For instance, our analysis showed that responding to question 7 could involve representing a situation graphically. This was identified with $TB24_1$ "Modeling a problem" which corresponds to "Contextual thinking"; a feature of Systemic thinking. $TB24_1$ thus appears once in question 7:

	REFLECTIVE			SYSTEMIC								ANALYTIC	
	$TB1_1$	$TB1_2$	$TB1_3$	$TB21_1$	$TB22_1$	$TB22_2$	$TB23_1$	$TB23_2$	$TB24_1$	$TB24_2$	$TB24_3$	$TB3_1$	$TB3_2$
Count Q2	1		2				1			3			
Count Q7	1			1	1	1			1	1			1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Count TB	12	4	4	4	9	2	3	3	9	11	3	1	1
Count TT	20			44								2	

Table 5 – Number of times TBs (and TT) are invited in questions 2 and 7, and by questions overall

Tables 6a and 6b indicate the class's engagement in TT in questions 2 and 7 respectively; where "class TB count" indicates the number of students who displayed the corresponding TB, and "count TT" the total number of times the corresponding type of theoretical thinking was displayed in student responses.

Student count = 42	REFLECTIVE			SYSTEMIC				ANALYTIC
TB invited by question	$TB1_{11}$	$TB1_{3(a)}$	$TB1_{3(b)}$	$TB23_1$	$TB24_{2(a)}$	$TB24_{2(b)}$	$TB24_{2(c)}$	n/a
Class TB count	3	0	11	7	0	21	1	
Count TT	14			29				

Table 6a – Number of students displaying TB, number of times TT was displayed – Question 2

Student count = 31	REFLECTIVE	SYSTEMIC					ANALYTIC
TB invited by question	$TB1_{13}$	$TB21_1$	$TB22_1$	$TB22_2$	$TB24_1$	$TB24_2$	$TB3_2$
Class TB count	8	8	4	3	1	9	0
Count TT	8	25					0

Table 6b – Number of students displaying TB, number of times TT was displayed – Question 7

These preliminary results indicate that the questions succeeded in engaging students in theoretical thinking with a highest occurrence of *systemic* thinking (which was indeed the type of thinking that was most invited by the questions). Table 6a indicates that at least 21 out of 42 students were engaged in (systemic) thinking in question 2. Likewise, Table 6b indicates that at least 9 out of 27 students were engaged in (systemic) thinking in question 7. The results also show that some TBs were more popular than others, and that in general some questions were more effective at provoking theoretical thinking in students than others. This is perhaps an indication that particular features of a question make it more (or less) engaging, which could be a call for further investigation.

Implications for teaching

College-level Calculus instructors are often compelled to certain teaching constraints- we do not deny this; rather we take on a different perspective: We have shown that this reality need not stand in the way of incorporating what we believe to be an essential part of a Calculus course; a theoretical thinking component. Our study shows that by simply posing additional non-routine tasks chosen in a way to promote theoretical thinking, creating a space in which students can actively engage in theoretical thinking (should they wish to do so) is indeed possible, despite these constraints.

Questions for the audience

1- What could be a different type of analysis, yielding additional (or different) results?
2- How can we effectively measure a student's *progress* in engaging in theoretical thinking?
3- Could a different choice of a model of TT change/ enrich the results of this study? If so, how?

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' STATISTICAL PREPARATION: INTERPRETING THE NEWS

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Undergraduate mathematics programs must prepare teachers for the challenges of teaching statistical thinking as advocated in standards documents and statistics education literature. This preliminary report presents initial results from a study of pre-service secondary mathematics teachers at the end of their undergraduate educations. Although nearly all had completed a required upper-division statistics course, most were challenged by two tasks which required a critical analysis of the use of statistics in newspaper articles. Some patterns emerged in the incorrect answers, including a tendency to focus on potential sampling issues which were not relevant to the tasks. The session will explore the nature and sources of these difficulties with statistical thinking and statistical communication and it will explore the implications for undergraduate mathematics and statistics teacher preparation.

Key words: Statistical Thinking, Distributional Thinking, Teacher Preparation

Statistical literacy must be a goal of K-12 education (Franklin et al., 2007); it is essential to informed citizenry, to decision-making, and to economic empowerment (Utts, 2003). The achievement of this goal is dependent upon the undergraduate preparation of mathematics teachers who are proficient in statistical thinking and can foster that ability in their students. That is, K-12 teachers must understand and be able to communicate "the need for data, the importance of data production, the omnipresence of variability, and the quantification and explanation of variability" (Aliaga et al., 2005, p. 14). Indeed, recent K-12 standards/guideline documents, namely, the American Statistical Association's GAISE standards (Franklin et al., 2007) and the widely-adopted Common Core State Standards (CCSS) for Mathematics (CCSSI, 2010), have highlighted the importance and need for students to develop statistical literacy and engage in statistical thinking. Yet, many in- and pre-service teachers are products of a system where the learning of data and chance at the K-12 level has been underemphasized (Shaughnessy, 2006). Against this backdrop of increasing statistical demands on K-12 teachers, this preliminary report explores the statistical thinking of pre-service teachers of secondary mathematics (PSMTs) at the end of their undergraduate mathematics programs.

In particular, two statistical thinking questions were asked of 23 senior-level students enrolled in a capstone mathematics course for undergraduate mathematics majors intending to be secondary mathematics teachers. Twenty-two of the students had completed, as a pre-requisite for the course, an upper division semester-long calculus-based statistics course offered in a department of mathematics & statistics. Each of the two questions required students to comment on whether statistics reported in a newspaper article supported a claim from the article. Initial analysis indicates that (1) many of these PSMTs may be uncomfortable with statistical thinking and (2) there are some emergent patterns in the students' incorrect answers.

Perspective

In 2012, the Conference Board of the Mathematical Sciences (CBMS) released a draft of recommendations for teacher preparation. They called for statistical preparation with a focus "on data collection, analysis, and interpretation needed to teach the statistics outlined in the CCSS"

(p. 8). This type of preparation may not be the status quo; Rossman, Chance, Medina, and Obispo (2006) pointed out that many mathematics teachers “do not have ample opportunities to develop their own statistical skills and understanding of statistical concepts before teaching them to students” (p. 332). They connect this issue to the structure of teacher preparation programs in which PSMTs receive little instruction in “communication skills and statistical judgment” (p. 332). There have been few studies of teachers’ statistical knowledge for teaching. Groth (2007) proposed a hypothetical framework for this knowledge and lamented that “there is a daunting amount of work to accomplish in building programs that are effective in helping teachers develop knowledge for teaching statistics” (p. 433).

Preliminary Results

The data are comprised of student work on two questions from a capstone mathematics course for PSMTs which was taught by the author of this report. Question 1 was a homework problem; there is no information about the level to which students coordinated on this task. Question 2 was assigned on a take home exam; coordination was prohibited on the exam.
Both questions required students to examine newspaper quotes. Of the semester-long course, three weeks were devoted to discussion of statistics, though no in-class activities required students to interpret news reports.

Question 1. USA Today published an article called *Is `failure to launch' really a failure?* (Jayson, 2006). Here are two sentences from it:

> i. “High housing costs are only part of the reason young adults are staying home in greater numbers than ever before.”
>
> ii. “Since 1970, the percentage of people ages 18 to 34 who live at home with their family increased 48%, from 12.5 million to 18.6 million, the Census Bureau says.”

(A) Does the second sentence support the statement that “young adults are staying home in greater numbers than ever before”? (Assume “young adult” means “people ages 18 to 34”.) (B) Is there anything misleading? (C) What questions would you like answered in order to further clarify the provided statistics?

There were two main issues with the quotes in Question 1 which students were to identify: (1) The article incorrectly stated that the percentage (not the number) of people living at home increased, and (2) The article was misleading in its failure to acknowledge that, from 1970 to 2006, the population of the United States also increased. In fact, it increased by approximately 48% (US Census Bureau, 2012). Among the 20 responses, only one made note of Issue (1). In order to examine student recognition of Issue (2), responses were initially coded based upon whether an issue of proportionality was acknowledged. Twelve responses clearly addressed proportionality. However, three of these twelve did not mention it in connection with the primary reason why the second sentence did not support the quoted statement; instead they brought up the possibility of a population increase as one of their questions in part C.

Question 2. Here's a quote from the New York Times (Future, 2009).

> “Immigrant children lagged in mastering standard academic English, the passport to college and to brighter futures. Whereas native-born children's language skills follow a bell curve, immigrants' children were crowded in the lower ranks: More than three-quarters of the sample scored below the 85th percentile in English proficiency.”

Does the statement that “more than three-quarters of the sample scored below the 85th percentile in English proficiency” support the statement that “Immigrant children lagged in mastering standard academic English”? Explain why or why not.

The intention of Question 2 was for students to engage in some form of distributional thinking. Only nine of the 23 students supplied an explanation which acknowledged that a sub-population with only 75% scoring below the 85^{th} percentile would likely be outperforming the general population. Of the 14 other students, seven did demonstrate distributional thinking to some extent. Often, however, this thinking was unproductive and inaccurate.

Some Implications and Questions for Audience

In general, the formal upper division courses typically required of mathematics majors do little to prepare them for their careers as mathematics or statistics teachers (CBMS, 2012; Monk, 1994). Rossman et al. (2006) and Groth (2007) raised concerns about the lack of appropriate statistical education for pre-service teachers. Preliminary analysis indicates that there are consequences to that lack of instruction. All but one student had completed the required upper division statistics course, yet, on Question 1, only 12 out of 20 students addressed issues related to proportional reasoning. On Question 2, only nine out of 23 successfully demonstrated distributional thinking. Indeed, many of the PSMTs, at the end of their undergraduate mathematics education, struggled with statistical thinking. Furthermore, there are some notable characteristics of the incorrect answers provided in this study. Many incorrect answers were valid observations which did not relate to the question. This may be an instance of what Kahneman and Frederick (2002) referred to as attribute substitution, often characterized by situations “in which a difficult question is answered by substituting an answer to an easier one” (p. 50). In particular, many students focused on potential methodological issues which are typically not discussed in news media, but may be discussed in statistics courses. That is, the students may have been more comfortable with discussing potential sampling issues (about which they had no information) than with a context-based critical evaluation of the use statistics in a non-academic source.

These preliminary results, accompanied by examples of student work, will be discussed in the proposed session. The following questions will guide discussion after an initial presentation:

1. What do these preliminary results and the student work suggest about these students’ statistical preparation for teaching? What can be said, in general, about the undergraduate preparation of secondary level teachers of statistics?
2. With the adoption of the Common Core State Standards, this is a time of widespread K-12 curricular change in statistics. How can undergraduate teacher preparation programs help teachers meet the demands associated with these changes? What statistics education research is needed to support this? How can the challenges of institutional change be confronted?

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EMERGENT MODELING AND RIEMANN SUM

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This research focuses on mental challenges that students face and how they resolve these challenges while transitioning from intuitive reasoning to constructing a more formal mathematical structure of Riemann sum while modeling "real life" contexts. A pair of Calculus I students who had just received instruction on definite integral defined using Riemann sums and illustrated as area under the curve participated in multiple interview sessions. They were given contextual problems related to Riemann sums but were not informed of this relationship. Our intent was to observe students' transitioning from model of to model for reasoning while modeling these problem situations. Results indicate that students conceived of five major conceptions during their first task and their reasoning from the first task that became a model for reasoning about their next task. In this paper we detail those conceptions and their reasoning that became model for reasoning on the second task.

Keywords: Emergent modeling, Riemann sum, Quantitative reasoning, Definite integral

Introduction and Research Questions

Riemann sums provide a foundation upon which one can understand why definite integrals model various situations. Previous research has detailed mental challenges that students face while reasoning about accumulation contexts, and has stressed how students could perform routine procedures for definite integral without being able to explain their reasoning (e.g., Artigue, 1991, Hall, 2010, Orton, 1983; Sealey, 2006). Research that has detailed how students might shift from more intuitive understanding to a more formal understanding has focused on roles of quantitative reasoning (Sealey, 2006; Thompson, 1994) and how that reasoning can support a more conceptually accessible formation of the Fundamental Theorem of Calculus (Thompson & Silverman, 2008). Other research has detailed the importance of conceiving of appropriate structural elements of the Riemann sum within context in order to complete approximation tasks (Sealey & Oehrtman, 2008). But when students come to understand Riemann sums as a model of a particular situation, how does their reasoning about that model influence their reasoning in constructing Riemann sum models of subsequent situations? This research attempts to answer the following questions. (1) What challenges do students face and how do they resolve those challenges as they constitute Riemann sum as a model of a contextual approximation problem? (2) How do students utilize their prior reasoning from their constitution of their Riemann sum model as a model for their reasoning about subsequent problems?

Theoretical Perspective and Methods

Rooted on the theory of Realistic Mathematics Education (Freudenthal, 1973), emergent modeling is an instructional design heuristic where modeling is viewed as an active organizing process where models co-evolve as students reorganize their intuitive reasoning and construct more formal mathematical reasoning (Gravemeijer 2002; Heuvel-Panhuizen, 2003). Models are viewed as more than representations but as holistic organizing activities including a solution strategy. *Model of* is the starting phase of emergent modeling where learners consider a model to be context-specific and employ informal solution strategies. Model for is the latter phase of emergent modeling where learners shift from thinking about the problem situation of the model to reasoning about it mathematically. "The model changes character, it becomes an entity of its own, and as such it can function as a *model for* more formal mathematical reasoning" (emphasis in original, Gravemeijer, 2002, p.2). Quantitative reasoning provides a means of modeling where students conceive of quantities, construct relationships between quantities, and meaningfully operate on those quantities that can support the construction of further quantities as one reasons with and about the problem situation (Larson, 2010; Thompson, 2011). When a conceived quantity is specifically attached to an attribute of a problem situation, any representing of this quantity would indicate model of reasoning, but as one reasons about this quantity within a quantitative structure without referring to a problem situation, that reasoning emerges as a model for their reasoning about the mathematics.

Ten interview sessions (50-148 minutes) were conducted with two volunteer Calculus I students, Sam and Chris (pseudonyms), who had been introduced to the definite integral through Riemann sums illustrated as area (Stewart, 2008). Students were given three approximation tasks related to Riemann sums, out of which two emphasized finding under and overestimates to total distance traveled based off of a table containing velocities and a velocity function, respectively (Figure 1). The third task was related to pressure on a dam, but this paper will focus only on the first two tasks since analysis of the third task is ongoing. For these two tasks, additional subtasks included drawing pictures of the actual situation, finding and illustrating error bounds, and graphing. Sessions were videotaped to analyze how students modeled their problem situations. Models were identified based on students' reasoning as exemplified by their representations and verbal utterances. When students directly related their reasoning to the problem situation, this was viewed as model of reasoning. Prior patterns of reasoning and representing when applied to a current problem situation were viewed as indicators of potential model for reasoning.

Task 1: The table below shows the velocity of a car travelling from Conway to Little Rock. In this activity you will approximate the distance travelled by the car during the first 10 seconds of the car entering the southbound I-40 ramp.

T(s)	0	2	4	6	8	10
V(ft/s)	0	21	34	44	51	56

Task 2: NASA's Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After t hours of traveling, its speed in miles per hour is given by the function $v(t) = sin\sqrt{9-t^2}$. In this activity you will approximate the distance travelled by the Lunar Rover in the first two hours.

Figure 1. First two teaching experiment tasks.

Results

The results reported here will focus on student's emerging model of Task 1 (Table 1) and reasoning about Task 1 that reappeared in Task 2 to suggest model for reasoning.

Initially, Sam and Chris realized that the varying velocities and the finite amount of data caused problems with easily completing Task 1. Reasoning from the provided table, their first conception of a distance/rate/time relationship (DRT 1[1]) was modeled as a picture containing snapshots of a car equally distanced between every two seconds (Figure 1, Picture a). After the

[1] Descriptions for DRT 1, DRT 2, DRT 3, Total 1, and Total 2 can be found in Table 1.

Table 1.
Distinct Conceptions During Task 1.

Conception	Description of reasoning
DRT 1: Distance changes as time changes	Omitted explicit detail to amounts of change in velocity. Pictorially represented as a vehicle with constant amounts of changes in distance per two second intervals.
DRT 2: Distance is change in velocity × change in time	Initially supported by their reasoning about amounts of change in distance vary because of changing velocities. Pictorially represented as a vehicle with decreasing amounts of changes in distance per two second intervals which became a model for distance as $d = \Delta V \cdot \Delta t$.
DRT 3: Distance is constant velocity × change in time	Initially only conceived for a vehicle traveling at constant velocities. After adjusting their picture to model a vehicle with increasing amounts of changes in distance and after "supposing" their vehicle as traveling at constant velocities was this conception applied to their context. Formulaically represented as $d = V_p \Delta t$.
Total 1: Total distance approximated by adding up distances are underestimates or inconclusive.	Adding up amounts of change in distances approximates total distance. Coordinated with DRT 2 and then DRT 3. With DRT 3 it was initially represented as $\sum_{p=0}^{5} V_P \Delta t$. For Sam, this sum was an underestimate because the sum would increase towards the exact total distance traveled as more data points were added. For Chris, this sum was inconclusive because the data table did not reveal what happened between data points.
Total 2: Total distance approximated by adding up using max. and min. velocities.	Coordinated with DRT 3. They conceived of maximum and minimum velocities over a time-interval as approximations to varying velocity over that interval. Underestimates and overestimates were represented by $\sum_{p=0}^{4} V_P \Delta t$ and $\sum_{p=1}^{5} V_P \Delta t$, respectively.

Note. DRT = Distance, Rate, and Time relationship.

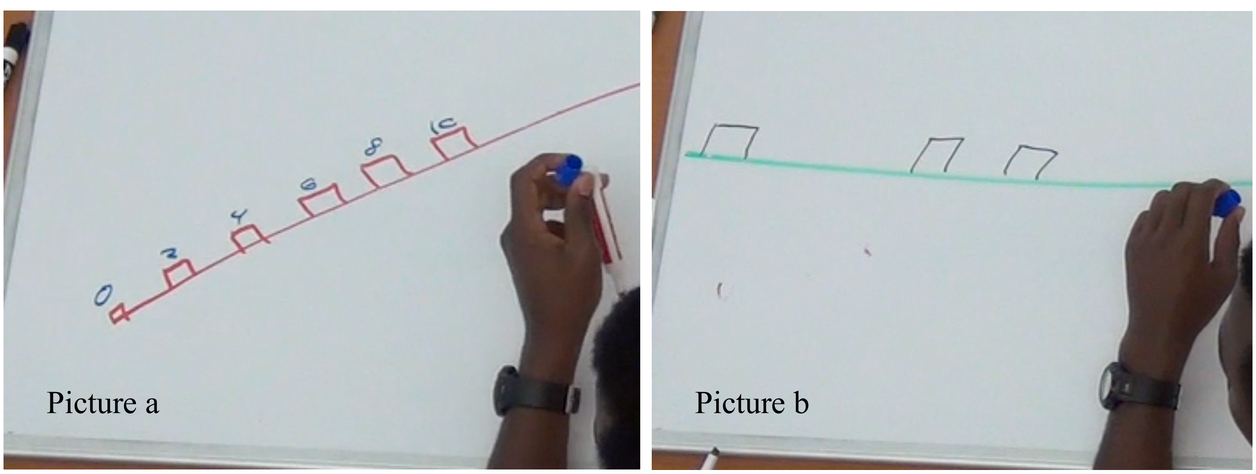

Figure 2. Two pictures showing locations of the car every two seconds.

facilitator prompted them to be "picky" with their picture, they attended to varying amounts of change in distance between snapshots, and represented this conception pictorially with increasing changes in distance between every 2-second snapshot (Figure 1, Picture b) and formulaically as "$d = \Delta V \cdot \Delta t$" (DRT 2). After prompted to think about a "real life" situation of a car merging onto an interstate, they adjusted their picture to indicate increasing distances between snapshots. By this moment they had indicated that adding up individual distances would provide approximations for total distance (Total 1). At first, DRT 3 appeared in response to an additional facilitator question concerning another situation in which a car traveled at 70 mph for two hours and 80 mph for one more hour. Though they concluded that the car in the other situation traveled 220 miles, DRT 2 persisted in their reasoning about the car with varying velocities in Task 1. Once they realized that their formula $d = \Delta V \cdot \Delta t$ for the other situation yielded a conflicting answer when applied to the additional question did they rethink DRT 2. Attempting to calculate error bound, they grappled with finding both under and overestimates for total distance. After three hours since starting this task, once they had conceived of the roles of maximum and minimum velocities as approximations for varying velocities over 2-second intervals, they coordinated DRT 3 with total distance and were able to find both under and overestimates for total distance (Total 2). Later Sam compared getting an exact distance to a perfect video, “We have an infinite number of snapshots, […] a solid image of what- We have a video, a perfect video where there is no frames or anything like that, an ideal video.” They finished with representations seen in Figure 3.

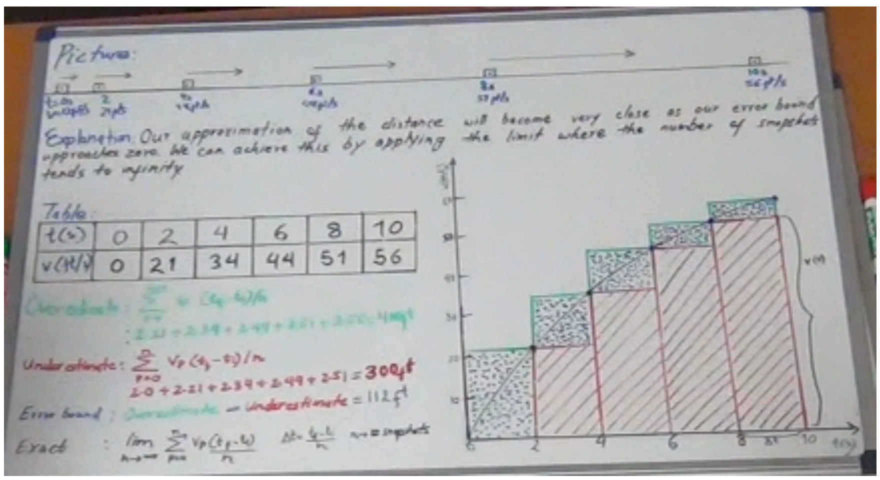

Figure 3. Chris and Sam's multiple representations of Task 1

Immediately after being given Task 2, Sam asked Chris, “Don’t you think the picture looks the same like last time?” He then drew a picture with snapshots of the rover at every half hour interval. To understand the changing velocity of the rover, they constructed a table of values similar to the given table in Task 1. Once their table was constructed, they explicitly quantified amounts of change and coordinated that with labels added to their picture. After being asked, “Where is distance?” they noted distances between snapshots on their picture and proclaimed that finding total distance was, “the same as what we did last time.” Subsequently, they represented total distance as $\sum V_{(t)} \Delta t$. Although imprecise, this representation captures the multiplicative structure between particular velocities and amounts of change in time within a summation. In the process of constructing a numerical approximation to total distance travelled, they employed their prior reasoning concerning maximum and minimum velocity over an interval and coordinated their DRT 3 and Total 2 to calculate under and overestimates, eventually represented as summations with appropriate adjustments to the starting values of the index. From approximations to exact distance, Sam stated, “We go from having pictures, to a flip book, to a video, to like one true continuous string where there is no frame rate.” Chris generalizes, “all of these summarize how you can make an error smaller by increasing the number of snapshots...as we increase the number of snapshots we tending to get the exact displacement so, that's what both of them summarize [pointing to both Task 1 and 2]."

Discussion and Questions

We observe that the challenges presented by Task 1 were not easily overcome by Sam and Chris but their engagement of these challenges supported them in forming patterns of reasoning for more effectively modeling Task 2. For instance, Sam and Chris had to construct appropriate ways for reasoning about a relationship between distance, rate, and time for a car of increasing velocity. They had to conceive of pertinent roles for minimum and maximum velocities for under and overestimates, and relate those to a notion of summing up distances to obtain Riemann sums for under and overestimates. For instance, Sam and Chris' conceptions of minimum and maximum velocities within a model of calculating under and overestimates for total distance during Task 1 was first represented after three hours of work. In contrast, they readily represented these estimates for total distance for Task 2 within thirty-two minutes. How were they able to progress so rapidly during Task 2? Their picture, graph, table, and formulaic expressions from Task 1 served as reference points for them to make connections between their two tasks as they conceived of, represented, and related relevant quantities. For example, before they firmly committed to using their reasoning from Task 1 applied to Task 2, their pictures and tables supported their conceiving of varying velocities, amounts of change in time, amounts of change in distance, and in relating these quantities while building connections across the tasks. As these connections became more apparent, the students progressed in constructing appropriate Riemann sum approximations. We note that it was not merely the end results of Task 1, but elements of their reasoning that went behind creating those end results, including a solution strategy, which served as a model for their subsequent reasoning during Task 2.

We acknowledge that our work with one pair of students does not necessarily generalize to others. Furthermore, the model for reasoning being reported may be more general reasoning that is still tied to Riemann sum approximation problems involving relationships between distance, rate, and time. We also note that since the students were exposed to Riemann sums, they were not reinventing Riemann sum symbolizations but were conceiving of a multiplicative structure within contexts and constructing relationships between this structure and some existing Riemann sum structure. Our questions are: How can we design tasks to better capture students' modeling activities and their transition from model of to model for in the context of definite integral and Riemann sum within a research context? For students who have not been exposed to Riemann sums, how can we modify our tasks to generate an intellectual need for these sums and subsequently support these students in constructing a Riemann sum? How might activities be effectively scaffolded to support the model of / model for transition in a classroom?

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INVESTIGATING STUDENT UNDERSTANDING OF EIGENTHEORY IN QUANTUM MECHANICS

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An initial investigation into students' understanding of Eigentheory using semi-structured interviews was conducted with students at the end of a first-semester course in quantum mechanics. Many physics faculty would expect students to have mastery of basic matrix multiplication after a course in Linear Algebra, and especially so after fairly extensive use of matrices in quantum mechanics in the context of Ising model spin problems. Using a previously published interview protocol by Henderson et al, student reasoning patterns were investigated to probe to what extent there reasoning patterns were similar to those identified among Linear Algebra students. Reasoning patterns appeared quite consistent with previous work; that is, students used superficial algebraic cancellation, and demonstrated difficulty interpreting their result even when they arrived at a correct solution. The interview protocol was modified slightly to probe whether or not students felt the tasks they were engaging in were mathematical or physics-related. Additional questions were added at the end of the protocol about how these concepts were used in their quantum mechanics course. Students were somewhat successful relating them to Hamiltonians and energy eigenvalues, but couldn't articulate the type of physical situations where they might be useful.

Key words: Linear Algebra, Physics, Quantum Mechanics, Interdisciplinary, Interviews

Over the past decades, a great deal of research has taken place within both Mathematics Departments and Physics Departments on the learning and teaching of these respective fields. There is a small, but growing field of researchers that are interested in tapping the knowledge base of both the communities of Physics Education Research (PER) and Research in Undergraduate Mathematics Education (RUME). This work was inspired by a paper at the 2010 RUME Conference, which shed light on students' reasoning on concepts of Linear Algebra (Henderson et al. 2010). The study focuses on students' understanding on matrix multiplication and geometric interpretations of these operations.

These principles are mathematical concepts that physics students use throughout their upper-division courses and into graduate studies in courses such as Quantum Mechanics (QM), Particle Physics, Electricity and Magnetism, and Mechanics. Curricula in undergraduate QM courses in particular require students to apply these mathematical concepts to identify "energy" eigenstates and eigenvalues for a particular system that is defined by a Hamiltonian operator. After completing instruction in a QM class, instructors expect students to be well versed in using these concepts to solve and interpret physical situations.

There are only a handful of investigations on students' use of mathematics at the upper-division in physics. (Pepper et al. 2012, Bucy, Thompson et al. 2006, Bucy et al. 2007) The initial phase of this study is to investigate how QM students respond to this interview protocol and get a sense of their reasoning patterns. All three students had previously taken Linear Algebra, and displayed a range of thinking about eigenvalues, eigenfunctions and operators. All three students were very proficient at "doing the math" in as much as determining the equations they need to solve to determine the eigenfunction, but were uneven in their description of what the results meant.

Framework

The theoretical framework of this work is identifying student difficulties by Heron (2004). This framework has an explicit goal of identifying things that students know well as well as ideas with which they struggle. Its design is meant as the first step in a broader research agenda of developing curricular materials that explicitly build on correct student conceptions and target student difficulties though instructional interventions. It assumes that student responses (be they written, spoken and otherwise) are representative of their thinking about the question at hand. If student thinking is unclear to the interviewer, clarifying questions are asked until the interviewer feels they have been given the best explanation the student is likely to give.

Methods

The interview protocol consists of several questions about interpreting matrix multiplication, and also doing a matrix multiplication (see Fig. 1 and 2). Students' demonstration of their mathematical reasoning was uneven throughout the interviews. One student in particular, demonstrated previously reported difficulties by treating the first problem in Figure 1 as a simple algebra problem, answering that A was equal to 2.

Figure 1. The first prompt from the interview protocol by Henderson et al.

Consider a 2x2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. *How do you think about* $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?

How do you think about what the equals sign means when you see it written in the context of this equation?

Student response: *"When I see the equals sign, I guess it tells me that everything on the left half is the same as everything on the right. Since I also see that the matrice [sic] on the left is equal to the matrice [sic] on the right. It lets me know that the only difference between the left half and the right half on the left is an A and right half is a 2. So I assume the A has to be 2. When I think about the equals sign and that's what it tells me."*

When describing what the equals sign means, the student uses what Henderson et al. called superficial algebraic cancelation. When given the values for the 2x2 matrix (in Fig. 2), he recognizes there is an issue with his previous work. Instead of wrestling with this inconsistency further, the student immediately begins plugging away.

Student response: *"Now I think about this expression… here [pointing to previous] we have a matrix and a constant and I said 'hmm, makes me think A was equals the constant'. Now we have matrix and a constant, and hmm, it doesn't quite sound equal. I know what I can do, I'll multiple this through and see what it looks like."*

The student continues until he has expressions for the values of x and y that make the expression true. There is an explicit prompt later in the protocol to do this, but the student does it all spontaneously.

Figure 2. Prompt from the interview protocol.

Suppose $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. *Now how do you think about* $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?

The student is asked by the interviewer if any of these questions seem more like physics questions or more like math questions, and the student identifies the first question (Fig. 1) as being more like math and the second question (Fig. 2) as being "more like doing physics". This differentiation of knowledge on the part of the student is interesting because it implies that domains of mathematics knowledge and physics knowledge may be different for students. This also implies that students may be framing problems as being of a particular domain that may influence their ability to solve problems (Tuminaro and Redish 2007). The concept of framing ties into the theoretical framework of resources (Hammer and Elby 2003), that is common in physics education research. Resources have a great deal of overlap with diSessa's knowledge-in-pieces model of student thinking.

The general idea of epistemic framing within a resources framework means that if students view a problem to be from a particular domain, they will attempt to access bits (often called resources) of knowledge and procedure that have previously been helpful for answering questions within this domain. Students may have difficulty accessing knowledge they possess from a different domain (e.g., physics) when they frame a particular problem to not include that domain but rather something else (e.g., mathematics). In the above scenario, the student identifies a strategy that they feel can work for them ("multiple this through and see what it looks like"), and later identifies this as "doing physics" when in fact there is no physics context whatsoever.

After the episode where the student demonstrates some understanding about what can be done when giving a matrix operator, the student attempts to reconcile the previous answer.

Student response: *"All of a sudden I think I would've figured out A… if I plug in 10, and -10 I get a new expression for A, but now that's dead in the road. I don't think it'll give me anything new."*

It seems that there is a disconnect between what the student initial identifies as correct and what their work in the second in the interview and the work from the first part of the interview. The student makes no further progress in understanding where the error from the first question lies.

Future Work

Future work will attempt to further probe the boundary of students understanding of mathematics and physics concepts and how students attempt to use physic and mathematics ideas to answer questions in the complimentary domain. Given the piece-wise nature of student responses and the small sample-size of students in upper-division of physics, it seems that a more appropriate theoretical framework would be more suitable for making sense of student thinking. As this work progresses alternative frameworks will be considered in an attempt to cast additional light on students thinking.

Questions

This work is in a very preliminary stage. It wasn't entirely clear that using a protocol from on Linear Algebra concepts would fly at all among physics students. A lot of questions remain for myself as a researcher about what I can do and what I should be doing with this work. I would like to use these first steps as a springboard for a refined protocol that draws out their mathematical knowledge, but also has them tackling explicit physics problems that use these mathematical tools. Could this be done in a single interview? Should these questions be administered over the course of several interviews? Perhaps it would be better to have them done in a group of students to capture more authentic dialogue? Might it be better to capture actual classroom dialogue or ask students to record their own discussions while doing homework or other assigned tasks outside of class?

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EFFECTS OF COLLABORATIVE REVISION ON BELIEFS ABOUT PROOF FUNCTION AND VALIDATION SKILLS

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Although there is much research showing that proof serves more than just a verification function in mathematics, there is little research documenting which functions of proof undergraduate students understand. Additionally, research suggests that students have difficulty in determining the validity of a given proof. This study examines the effects of a teaching intervention called collaborative revision on student beliefs regarding proof and on student proof validation skills. Student assessment data was collected and interviews were conducted with students in the treatment course and in a comparison course. At the end of this study, we will produce a categorization of the proof functions that students appreciate, as well as a determination of the value of the teaching intervention on students' abilities to correctly classify proofs as valid or invalid.

Key words: proof validation, transition to proof, collaborative learning, function of proof

Introduction

This study aims to investigate the effects on beliefs about proof and argument validation skills of students in an introduction to proof course by employing a process called collaborative revision in the classroom. *Collaborative revision* refers to the process in which students present a proof they have written to their classmates and the other students are encouraged to make comments and point out inconsistencies in order to ensure that the proof is valid. Based on feedback from classmates, the student then revises the proof and presents it again, repeating the process until the proof is valid and includes all the relevant details. In this context, this project aims to answer the following research questions:

- What are students' beliefs about the function of proof in mathematics and how do these beliefs change during a course using a collaborative revision teaching intervention?
- How does collaborative revision affect students' proof validation skills?

Background & Related Literature

The role of proof as useful for verification of a statement has been well documented (e.g. Harel, 2007; Mason et al., 1982). However, researchers have categorized many other functions of proof in mathematical practice. Hanna (1990) and deVilliers (1990) suggest that in some cases proofs can be *explanatory* and provide insight as to why a certain statement is true, especially when a conjecture was arrived at by empirical results. DeVilliers (1990, 2002) also argues for four more functions of proof in mathematics. The first is that of the use of proof as a *means of discovery* in that proof can sometimes lead to new results in a field. The second is that proofs serve an important *communicative* role, since they are the main way mathematical knowledge is transmitted. Third, proof has the function of being an *intellectual challenge* in that the completion of a proof can be very satisfactory for many mathematicians. Finally, proof can expose logical relationships between statements and can serve as a tool for *axiomatizing* results in a mathematical system.

Furthermore, Weber (2002) claims that proof can serve as the *justification for a definition* in mathematics or, in the context of teaching and learning, proof can *illustrate techniques* in advanced mathematics. Yackel and Cobb (1996) note that another role of proof can be to *provide autonomy* to students by allowing them to create their own mathematical knowledge.

Indeed, Rav (1999) proposes (and Hanna & Barbeau (2009) agree) that proofs are of the utmost importance in mathematics, since they, instead of theorems, are the main vehicles in which mathematical knowledge is contained and transferred.

If proofs are the bearers of mathematical knowledge, the ability to determine if a given proof is valid is an important skill. Selden & Selden (2003) highlight that this skill is invaluable for not only future mathematics educators because they will someday have to evaluate student proofs for assessment purposes, but also for future mathematicians because they will have to examine others' proofs to learn about new mathematics that is being produced. Additionally, proof validation is intricately linked to proof construction so students in transition to proof courses need both skills (Selden & Selden, 2003).

How then should these important concepts be taught to undergraduate students? There is much research to support the hypothesis that collaborative learning can greatly enhance student learning of appropriate mathematical proofs. Yackel & Cobb (1996) note that participation in a community of learners can be a vital part of students' success in mathematics and learning can be regarded as a relationship between communal classroom processes and individual activity. Additionally, in a study, Strickland & Rand (2012) allowed students to submit multiple revisions of proofs in response to teacher feedback and measured the effects on student learning. The teacher comments given were minimal, often just circling a confusing or incorrect passage of the proof, and students were allowed as many revisions as needed. Although the data set was small, on average, students in the revision group did better on the final exam. Thus, collaborative revision is a way to explore the benefits of combining these proven techniques and this study examines the effects on students in an introduction to proof course using a collaborative revision teaching intervention in the classroom.

Methodology

Participants were drawn from two lecture-based transition to proof courses at a large Midwestern university and from a course designed to supplement these courses, where the teaching intervention was enacted. A comparison group was desired to determine the effects of the teaching experiment when compared to a lecture-based course, but the researcher could only get access to the supplementary course, thus we only have a comparison course, not a proper control.

The 43 participants were given a pre and post assessment designed to evaluate how students think about the function of proof and if students can identify valid proofs. The first question on the assessment, modeled on a study by Healy & Hoyles (2000), is an open response question asking for general thoughts on the purpose of proof in mathematics, phrased as "What do you believe is the purpose of proof in mathematics?" The purpose of this question is to determine which functions of proof students recognize and how important they believe proof to be in mathematics. The same question was asked on both the pre and post assessment to gauge how students' beliefs about proofs change during the course. Additional questions on the assessment concerned the role of proof and statements were given about the myriad functions of proof from the literature, as discussed above. Students were asked to rate their agreement with each statement on a five point Likert scale, from strongly agree to strongly disagree. The purpose of these questions was determine if students consider other functions of proof besides verification and if students believe that some functions of proof are more important than others.

Another part of the assessment required the students to examine four correct and incorrect 'proofs' of a given statement and determine whether each was a valid or invalid proof. The proofs presented to the students are adapted for this study from the proofs given to high school students in Healy and Hoyles (2000) to be appropriate for undergraduates and did not require outside concepts or theorems that students may not remember from a previous course.

Students were first asked to determine if each proof is valid or invalid. The terms valid and invalid are intentionally not defined in order to see if students gain more of an understanding of what a valid proof entails throughout the course of the semester. There were also two other questions asking students, on a three-point Likert scale, how well they feel they understand the proof and how certain they are about their classification.

Student interviews were also conducted with six students (three in the treatment course and three in the comparison course) to allow students to elaborate on their classifications of proofs as valid or invalid and assess if these skills improved in students in the treatment course as compared with students in the other course. Students from the treatment and comparison courses were interviewed individually twice during the course of the semester, once shortly after the pre-assessment and again shortly after the post assessment. These were semi-structured interviews and questions asked to students, according to Zazkis and Hazzan (1998), are performance questions, unexpected "why" questions and reflection questions, where students were to explain their thought processes during the proof validation task on the assessment.

Data Analysis Plan

A quantitative analysis of the questions about proof function evaluated on a Likert scale will be performed to determine which functions students consider most important. Statistical analysis will be completed to determine any significant difference between students pre and post assessment scores, as well as between the treatment and comparison courses. A qualitative analysis of students' responses to the question "What do you believe is the purpose of proof in mathematics?" will be performed using the functions of proof identified in mathematics education literature outlined above as starting categories and adding other categories as needed.

The framework for data analysis of the interviews will be developed iteratively. I will begin with the framework of Harel & Sowder's (1998) proof schemes categorization and data from the interviews will be used to inform the framework and in turn the framework will help classify students into a proof scheme category. Preliminary findings support using this framework as the students seemed to clearly fall into one of the proof scheme categories.

Questions for the Audience

- How can collaborative revision be implemented so that it is balanced with lecture in a transition to proof course?
- What are other implications that you see from this work?
- This work is part of a larger study. What other data do you think would be helpful to collect to fully analyze the collaborative revision process?

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INITIAL UNDERGRADUATE STUDENT UNDERSTANDING OF STATISTICAL SYMBOLS

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In this study we use the tradition of semiotics to motivate an exploration of the knowledge of, and facility with, the symbol system of statistics that students bring to university. We collected a sample of incoming mathematics majors in their first semester of study, prior to taking any statistics coursework, and engaged each in a task-based interview using a think-aloud protocol with questions designed to assess their fluency with basic concepts and symbols of statistics. Our findings include that students find symbols arbitrary and difficult to associate with the concepts. Second, that generally, no matter the amount of statistics that students took in high school, including Advanced Placement courses, they have relatively little recall of topics. Most can calculate the mean, median and mode, but they generally remember little beyond that. Finally, students have difficulty connecting practices or procedures to meaning.

Keywords: semiotics, symbols, statistics, baseline data, recall of K-12 schooling

Research Questions

There have been investigations of students' understanding of measures of center (Mayen, Diaz, Batanero, 2009; Watier, Lamontagne, & Chartier, 2011), variation (Peters, 2011; Watson, 2009; Zieffler & Garfield, 2009), and students' preconceptions of terms related to statistics (Kaplan, Fisher, & Rogness, 2009); however, relatively little work has focused on students' use of symbols (Kim, Fukawa-Connelly, Cook, 2012; Mayen, Diaz, Batanero, 2009). In this study, we attempt to explore students' understanding of the symbolic representation system in statistics from their secondary curriculum to establish the baseline of student knowledge of statistical symbols upon arrival at university. That is, we investigate:

1. What fluency with statistical symbols do undergraduate mathematics majors have upon arrival at university?
2. How does that vary based on prior course-taking?
3. How do students reason about statistical concepts, such as the normal distribution and sampling, upon their arrival at university?
4. How does that vary based on prior course-taking?

Initial results suggest that students, even those that had significant experience with statistics pre-college, struggle to recall the relationship between symbols and definitions.

Literature Review

Understanding symbolic representations of ideas is especially critical for mathematics students. Hewitt (1999, 2001a, 2001b) distinguished between "arbitrary" and "necessary" elements of the mathematical system. The aspects of a concept used by a community of practice are labeled "arbitrary," meaning they can only be learned through instruction and memorization. Those which can be learned through exploration and practice are labeled "necessary." Hewitt found that for students to become proficient at communicating with established members of the community, they need to memorize the arbitrary elements and associate them correctly with appropriate understandings of the necessary elements. Hewitt noted that names, symbols and other aspects of a representation system are culturally agreed-upon conventions. For those who already understand them, the conventions seem sensible, but "names and labels can feel arbitrary for students, in the sense that there does not appear to be any reason why something has to be called that particular name. Indeed, there is no reason why something has to be given a particular name" (1999, p. 3).

The study of symbolic representations and the linkages between symbolic representations and the concepts themselves is at the heart of *semiotics*. Eco (1976) used the term "semiotic function" to describe the linkage between a text and its components and between the components. The semiotic function relates the antecedent (that which is being signified) and the consequent sign (which symbolizes the antecedent) (Noth, 1995). In the statistical community and the representation system in use within the community, there is a complex web of semiotic functions and shared concepts that "take into account the essentially relational nature of mathematics and generalize the notion of representation," and furthermore, "the role of representation is not totally undertaken by language (oral, written, gestures, …)" (Font, Godino, & D'Amore, 2007, p. 4). Throughout this paper, we recognize the inherent arbitrary nature of much of the symbolic system of statistics and draw on the notion of semiotic function as a means of linking a particular representation with the relevant concept.

This study is in line with the tradition of onto-semiotic research in mathematics education (Font, Godino, & D'Amore, 2007). It is situated in the context of statistics education, and designed to explore undergraduate students' understanding of the symbolic system of statistics at the start of university math major. It will attempt to describe the collection of semiotic functions the students had, focusing on those involving symbolic representations for the concept of the sampling distribution. It will also give a preliminary explanation of why the students constructed this particular set of semiotic functions by describing their understandings of the general representational system.

Methods

Data for this study was drawn from 7 participants in a mid-sized public university. All of the participants were first-semester students with a declared major in the Department of Mathematics and Statistics. We engaged each student in a task-based interview, during which the student completed a 14-item survey while using a think-aloud protocol. We created the survey by drawing on items used in a previous study on student understanding of statistics (Kim, Fukawa-Connelly, & Cook, 2012), adapting items from the *Assessment Resource Tools for Improving Statistical Thinking*, and developing our own tasks. Our goals were: to determine whether students could correctly recognize the symbols, to evaluate their understanding of what the symbols represented, and to evaluate their understanding of the related concepts (e.g., ideas related to the normal distribution).

All interviews were video-recorded and transcribed. We analyzed the transcribed interview data in combination with the students' written work using grounded theory (Strauss & Corbin, 1994). First, we made narrative comments indicating where students appeared to make claims about a particular statistical symbol or concept, and identified themes. Then, we collected the themes to create a set of initial codes and descriptions and came to agreement on the coding. Each author then read and coded all of the transcripts. We are currently developing profiles of the students and their overall proficiency with symbols. In coding, we described what students knew and believed about symbols and concepts, as well as areas where they recognized that they lacked appropriate knowledge or held misconceptions.

Results

Students find symbols difficult to distinguish between and associate with related concepts

While previous research examined students' proficiency at the end of an undergraduate class, this study examined what students recalled from their high school years. For example, after Cam calculated a mean, he indicated its label was μ, then changed it to $\bar{x}$, then decided he did not know how to determine what it was. This finding is similar to that which was reported in (Kim, Fukawa-Connelly, & Cook, 2012). When compared with students who were currently enrolled in an undergraduate statistics class (Kim, Fukawa-Connelly & Cook, 2012), while the students in this study had slightly less proficiency with symbols, the difference was not great. The most common difference was that the students in this study had no ability to distinguish, and only limited ability to recall, the difference between symbols for populations and samples.

Students do not remember much at all

We interviewed students with a range of experience of statistics in high school, from no specific courses on statistics to two years of coursework. While the students with more years of experience typically knew that they had learned more about statistics and recognized more symbols as relating to statistics than those without fewer courses, they generally had no greater understanding of the concepts.

For example, Kenny had taken an AP Statistics course (but not the exam). When asked about symbols, he recognized many of them, but when asked to distinguish between a single individual being within one standard deviation and a sample of four being the same distance from the mean, Kenny claimed that they were equivalent situations. He also described a sample of four individuals with a mean as being "four people, each with the same score." Both of these conceptions are problematic; while one could be understood as a lack of sense of scale, the other is a fundamental misunderstanding of the mean as a measure of center. On the other hand, students with less experience in statistics exhibited more advanced reasoning (as articulated by Cook & Fukawa-Connelly, 2012).

Students have trouble connecting practice to conceptual reasoning

Students, even those who had taken pre-college statistics courses, showed an ability to calculate mean and median but had trouble understanding that these are tools used to determine the center of data. For example, when we asked Jeremy if mean and median describe similar things, he said they do not, even though he said they both generally described the middle of data. When we pointed out that he used the word "middle" to describe both a mean and a median, Jeremy acknowledged that he had, but even then reiterated that mean and median were not similar. At the conclusion of the interview, Jeremy said that he had learned how to do things in class, but never learned the concepts.

Jeremy also showed that he was able to apply a rule that he learned, but unable to conceptually expand on the rule. When asked to determine if a particular data point should be considered unusual in the context of normally distributed data, Jeremy referenced the "empirical rule." He said that because the point was within a standard deviation, it was not unusual. However, when asked the same question about the average of 4 random data points from the distribution, he said that the answer would not change because it was the same data.

Students generally have an ability to both calculate and give a verbal description of measures of center

All of the students, regardless of the amount of statistics that they have taken, can correctly determine the mean and median of a set of data. If they recall what the mode is, they are similarly able to determine it. When asked, they have each been able to give a verbal description. Similarly, while students are not typically able to remember the formula for, or how to calculate, the standard deviation, they are able to state that it is a "measure of variation." Jeremy described the standard deviation as a measure of "how concentrated they [the data] are around a certain value." James attempted to recall the formula for standard deviation and recalled that it "had a summation of *x*'s and a summation of *x-squares*" but was not sure of the exact formula. Considering that students typically, at most, calculate the standard deviation of a data set once or twice by hand, it is unsurprising that they retain no knowledge of how to do so.

Discussion

Students must know the symbols used by statisticians and associate them with their accepted statistical meanings. Students must also be able to distinguish statistics from parameters and to view statistics as variables when they are embedded in a certain context. The results show that, even though students have a variety of exposures to statistics prior to arrival at the university, their recall and facility with the concepts of statistics is still minimal. Generally, students have appropriate procedural knowledge for calculation of measures of center, but appear to have limited ability to describe the differences in uses of those measures. Although we need further research on this subject, our results suggest that it is best to treat students who have taken AP Statistics or other advanced coursework as having knowledge and beliefs similar to those who have *not* taken specific course on statistics. The major difference between these two groups seems to be that those who have never taken a statistics course are less confident in their abilities and are less likely to have misconceptions about concepts. We believe that our results have implications for both K-12 and collegiate instruction, suggesting that instruction should focus on developing conceptual understanding that can stay with the students over the long-term, rather than calculating procedures.

Questions for Discussion

1) What was surprising to you? Why?
2) Which of the findings appear to be the most interesting, from a research perspective? Why?
3) What other themes should be looked at, in addition to initial proficiency, in the data we have collected?

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THE ROLE OF TIME IN A RELATED RATE SCENARIO

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Students who graduate with an engineering or science degree using applied mathematics are expected to synthesize concepts from calculus to solve problems. First semester calculus students attempting to understand the derivative as a rate of change encounter difficulties. Specifically, the challenges arise while making the decision to apply an average rate of change or an instantaneous rate of change (Zandieh, 2000) to the problem. This paper discusses how students view the derivative in an applied mathematical setting and investigates how the concept of time and other related quantities contribute to the development of a solution.

Key words: Calculus, rates of change, ladder problem, constant, time

Understanding the usage of the derivative and its related quantities is an essential component to applied mathematics, science, and engineering. This paper discusses ways in which students approach a solution to a standard calculus related rate problem. Generally in these problems, time is the independent variable. There are many quantities that involve time where time is explicitly or implicitly stated. The following literature describes student reasoning of the relationship of time to this calculus problem.

The ladder problem that Monk describes is directly related to applications of the derivative. Monk (1992) discusses Across-Time questions. *Across-Time* questions "ask the student to describe patterns of change in the value of a function that result from a pattern of change in the values of the input variables" (Monk, 1992, p. 176). Difficulties understanding an Across-Time view of functions arise from the students' grasp of relevant concepts. Once students were given a physical model, students are able to obtain correct answers.

Carlson (2002) defines "covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, 2002, p. 354). Sophisticated covariational reasoning ability is important for representing functions graphically and understanding calculus (Thompson, 1994; Zandieh, 2000). Carlson (2002) presented a ladder problem which was a modification of the ladder problem reported by Monk (1992). Students were asked to represent a dynamic situation of a resting vertical ladder being pulled away at the bottom at a constant rate. In particular, they were asked to describe the speed of the top of the ladder as it slides down the wall. The student who had the correct response "performed a physical enactment of the situation, using a pencil and book on a table" (Carlson, 2002, p. 371).

Keene (2007) defines "dynamic reasoning as developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems." From data collected, a characterization of time as an explicit quantity was made. Time was used to reason both quantitatively and qualitatively.

Zandieh and Knapp (2006) discussed the role of metonymy in mathematical understanding of the derivative. When an interviewee was asked what the derivative was, it was referred to as a rate at which something increases. The student's vague description of rate of change made it difficult to determine if this change denoted the limit of the difference or if it referred to a qualitative rate of change.

The aforementioned literature is relevant to using time in developing solutions to applied mathematics problems. My analysis indicates an absence of time being emphasized as a contributing factor to the solution of the standard calculus ladder problem.

Methods

Four students from a second semester calculus for engineers class were interviewed in spring 2012, of which two are discussed in this paper. The interviews were semi-structured and the students were interviewed one at a time. The interviews lasted between twenty-five and forty minutes. The students were asked to use a Livescribe smartpen and notebook to record their thinking processes.

The students were given a 14-inch model which was to represent a 14-foot movable ladder. The wall and floor that supported this mobile ladder was the wall of a room and the top of the table was covered with butcher paper. They used this model ladder to answer the following question.

There are several terms that were observed in the data and coded. These terms were counted once per turn. Terms classified as rate included velocity, speed, and the appropriate usage of units. Time is explicitly stated with usage of units describing speed or rate. An average rate is defined as a change in distance per change in time. Velocity also adds a directional component to speed. Terms describing the change in distances were referred to as the change in the height of the ladder, drop, change in y, and increment.

Results

In this section, descriptions of what two of the four students discussed during the interviews are presented. George and Ted did not initially know how to handle the interview question and each asked the interviewer to clarify the interview question. George does not explicitly state time as he reasons through the question; however, he speaks of constant rate.

George

Before discussing any mathematical quantities that could be related to the problem, he instinctually answered the question. Clarifying questions followed his initial answer.

George: Does it speed up or slow down? I would say that it speeds up. [points to the top of the ladder and shows change in height by spreading his fingers out down along the wall from the top of the ladder]. Like, at the top of the ladder. Well, are we comparing as it [still handling the top of the ladder] relates to the bottom of the ladder? Or are we comparing it to the overall speed of the top of the ladder as it descends?

Notice as he initially discusses the solution to the problem, he discusses several quantities. The rates of the top and bottom of the ladder are mentioned right after he describes with his fingers the vertical drop of the ladder as compared to the bottom of the ladder. The phrase "relates to the bottom of the ladder" could refer to either the rate or the constant incremental distance that the ladder moves along the table.

He draws a picture illustrating the static situation of the ladder leaning up against a vertical wall. He discusses concepts of the derivative as a rate of change until he was suggested by the interviewer to create a table of measurements as the ladder was being pulled away from the base of the wall.

George: Ok, so as we're going down, it actually is accelerating. The increments are becoming greater and greater. So the answer to the question would be that it speeds up.

Interviewer: As evidenced by?

George: The measurements of it.

Once the model was used, words and phrases describing rate were replaced with words describing difference in heights.

Ted

In contrast to George, Ted does explicitly state time. After being asked the interview question from Figure 1, the student started to write in the notebook relating velocity, distance, and time. As he restates that the ladder is moving at a constant slow rate, he creates a graph and table in the notebook. He wanted to quantify the constant rate.

The interviewer inquired about Ted's thinking.

Ted: So I could just say that it's being pulled away a centimeter, once a centimeter per second. Cause velocity is distance over time. So if it's being pulled out in seconds. So the bottom of the ladder is being pulled out 1 centimeter per second.

[…]

Ted: So, I want to measure the time. I need a watch to measure the time but that might not be accurate.

Twelve minutes into the interview, he pulled the base of the ladder away from the wall in two inch increments and made a mark corresponding to the ladder's height on the vertical wall. He noted next to each tick mark on the wall the corresponding horizontal distance.

Ted: Ah HAH! So the top of the ladder slides down. It speeds up! When you pull it out at a constant rate, the top of the ladder speeds up. The top of the ladder [spreading his fingers out covering the distances between the marks he made on the wall], every two inches, would be greater.

Upon looking at the incremental y values, he determined that the ladder was speeding up. When asked to graph any quantities illustrating the problem, time was explicitly stated in his description of the movement of the bottom of the ladder.

Ted: Um, well if you had a way to correctly, um, to accurately pull the ladder out for time. You could use those variables to find, um, the velocity. So you would um, move the bottom of the ladder out with a time variable, um. And then measure how fast the top of the ladder, um, slides down.

The change in vertical distance indicated the ladder sped up as the base traveled a constant distance per unit of time.

Discussion

George and Ted both discussed rate and time explicitly when gathering information about the task. George expressed his image of the ladder's movement by relating the speed of the top of the ladder and the speed of the bottom of the ladder. The horizontal speed was affecting the vertical speed. Once George was prompted to use the model, words describing rate were replaced with horizontal and vertical changes.

Ted focused on the time it took for the bottom of the ladder to move at a constant rate. He wanted to record the time. When prompted to use the model to assist with visualization, he defined the constant rate to be one centimeter per second. He pulled the bottom of the ladder in two centimeter increments and recorded the heights of the ladder.

Ted's measuring of fixed distances representing a unit of time can be considered a speed-length. A speed-length is defined as the "distance traveled in one unit of time" (Thompson, 1994; Thompson & Thompson, 1992).

George and Ted changed their language of rate and time to changes in vertical and horizontal distance when offered the manipulative. Reasons for this require further investigation.

How do the theoretical frameworks mentioned above connect with the data seen? How might the interviews be structured to understand why the language of rate and time changed to that of differences in distances?

A 14 – foot ladder is being pulled away from a wall at a constant, slow rate. Does the top of the ladder slide down the wall at a constant rate? Or, does it speed up or slow down?

Figure 1. Interview Question

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A Modern Look at the Cell Problem

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A models & modeling theoretical perspective has been suggested to supersede both constructvism (Lesh, Doerr, Carmona, & Hjalmarson, 2003) and problem-solving paradigms (Zawojewski, 2007). This is important to the RUME community because many of our foundational works are rooted in theoretical and methodological perspectives derived from these paradigms. In the case of problem solving and modeling, it is important to re-view the problem-solving research settings with a mathematical modeling lens. Without this glance backwards we cannot connect new ideas to old knowledge and we should not supplant a theory without ensuring the next can account for existing observations. The objective of this paper is to revisit a well-known problem setting to explore alternative interpretations of students' mathematical work.

The Task This report is part of a larger study to examine how students mathematically structure nonmathematical settings. The task considered is one of 17 modeling tasks used in the larger study which were designed to elicit the cognitive and mathematical activities attendant to mathematical modeling. The Cell Problem was selected for closer inspection because of its connection to and impact on the course of mathematics education in the US (see Schoenfeld, 1982a): *Estimate how many cells there are in the average adult human body. (How much faith do you have in this figure? What about a lower estimate? An upper estimate?)* The problem fits the criteria of a Fermi problem, a type of estimation problem championed for its ability to require the modeler to identify conditions, assumptions, relevant variables and parameters, and to estimate values for those parameters (Sriraman & Lesh, 2006; Ärlebäck, 2009). The Cell Problem provides an ideal setting to re-examine interpretation of students' problem-solving activity because the task lends itself to several readily realized mental models (e.g., arising from weight and density, partitioning, percentages).

Previous Findings The Cell Problem is well-known for its use in RUME in the early 80's (Schoenfeld, 1982a). These students sought increasingly finer estimates of the volume of the human body, and in other studies (e.g., Schoenfeld, 1982b, 1985) they carried on with such "wild goose chases" without pausing to evaluate their productivity. Schoenfeld concluded that students' metacognitive control was failing. Schoenfeld (1982b) further argued that even in methodologically "clean" laboratory settings, subjects' responses may be influenced by (i) a need to produce *something* due to the pressure o being recorded (ii) the expectation that some methods are "more legitimate" for solving problems than others, and (iii) the solver's own beliefs about the nature of mathematics. These observations and suggestions all point to why a student may have failed to produce certain expected mathematical behaviors and they contribute substantially to our understanding of how mathematical thinking is socially constituted in an interview setting. However, some aspects – such as the students' interpretation of the problem's context – are missing from the picture. From a models & modeling perspective, this aspect is central to interpretation of the student's behavior.

Research Questions The questions guiding analysis were: (i) How did the students make sense of the problem context? (ii) How did the students examine their own productivity?

Mathematical Modeling Theory A *model* is a simplified representation of some system. *Modeling* refers to both a sequence of behaviors and a way of thinking about a problem (Kehle & Lester, 2003). These behaviors include a proclivity to describe, explain, or interpret phenomena in mathematical terms (Lesh & Yoon, 2007). A mathematical model is the ordered triple (S, M, R), where S is a situation in the real world, M is its mathematical representation, and R is an invertible, idiosyncratic, cognitive link between the two (Blum & Niss, 1991). The modeler's success is a function of the information that hte modeler takes into account, how he accesses and harnesses conceptual models, choices in symbolization, and use of symbolic intuition (see Shternberg & Yerushalmy, 2003) to attribute meaning to the model. Mathematical modeling is theorized as a cyclical, iterative process that connects the Real World to the Mathematical World and can be seen in Figure 1. According to the theory, a modeler's activity can be resolved into stages (labeled by letters [a] - [f]) and transitions among those stages (labeled by numbers [1]-[6]). The stages and transitions are in Figure 1, Tables 1 & 2.

Methods For the larger study, four engineering undergraduates enrolled in differential equations were selected among volunteers to participate in a series of seven one-on-one, task-based cognitive interviews focused on mathematical modeling. Each interview session lasted an hour to an hour-and-a-half, with the Cell Problem taking between 16 and 23 minutes and demonstrating the full scope of the modeling cycle. The interview sessions were semi-structured and I interacted with students to ask follow up questions, to clarify my understanding of their statements, or to challenge their assertions. I also encouraged the students to use resources they felt might help them (e.g., textbooks, internet searches, calculators). Video recordings of the sessions were reviewed and transcripts of the sessions were segmented into statements containing a complete idea and these were tagged, via the method of constant comparison, with externally observable indicators corresponding to cognitive activities in the modeling cycle (activities [1]-[6]) (Borromeo-Ferri, 2007). The research questions were operationalized in terms of Figure 1: *understanding* and *simplifying/structuring* activities were associated with sense-making and *validating* with the students' verification activities. Instances of these activities were examined for technique of validation, the factors in S that were selected to become represented in M, and an emergent theme centered on the students' perceived importance of *accuracy and precision.*

Interview Results and Discussion The Cell Problem data is comprised of protocols from four male engineering undergraduates: Orys, Trystane, Mance, and Torrhen. The students spent extended periods of time in the *simplifying/structuring* and *validating* transitions, as is typical of a Fermi-type problem. Findings are organized thematically.

Sense-making. In this task, all students selected measurable parameters (eg, average human volume, size of the average cell) and all students were able to see multiple ways of structuring the problem multiplicatively, either as a weight or volume model. In contrast to Schoenfeld students, these students spent more time worrying about the impact of cell size than human size. Trystane and Orys were both concerned over cell size a function of its type. Since Orys found an estimate for one dimension of the average cell, he assumed the cells were spherical and viewed the task as a packing problem. Trystane solved it as a partitioning problem, breaking up the body in to different types of cells. Trystane iterated interpretations of the problem as changes in parameter sets. Using google, he adjusted M to reflect the parameters he had data for. In contrast, Mance's activity revealed a negative relationship among *simplifying/structuring*, *mathematization*, and *validation*: he created three situation models (based on weight, surface area, and volume) but could not disentangle one relevant variable set from the others during *validation*, leading to *competing conceptual systems* (see Lesh & Yoon, 2007). In terms of Figure 1, Mance was unable to produce a multiplicative mathematical representation M for any of the variable sets because the relationship *density* $\times$ *volume* $=$ *weight* persisted each time he tried to build R.

Validating The students primarily validated their models using empirical and experiential comparisons to their real results (model predictions) and also dimensional analysis. Orys and Torrhen commented that it was difficult to validate the results because the numbers were so large "that they are just *big* and the meaning is lost in the physical sense even though it still has meaning in the mathematical sense." Other than comparing the prediction to an empirical value or number sense, Trystane compared his mathematical model (a weighted average) against a real model (a diagram of a human divided into cell types according to organ or system). This kind of validating was not predicted by the modeling cycle.

Accuracy and Precision Torrhen was the only student who gave a "ballpark" estimate. Mance dismissed results of such models as being inaccurate and so not worthwhile estimates. Orys voiced his concern over his spherically-shaped cells and about averaging over cell types. Trystane lamented that the average of so may kinds of cells was not useful. He concluded, "I don't know the context of the question. Why would someone ask this?...I can't see why having the number of cells in the entire body would be useful." Thus, without an idea of what the answer was going to be used *for* (eg, a measure of health), it was impossible for him to determine error tolerance. Taken together these students' responses highlight a few issues in validating mathematical models: accuracy of the parameter estimates (and how errors compound), the representativeness of *average* and whether it can be substituted for a set of measurements, accuracy of the prediction based on these assumptions, and the precision needed to answer the question (determined by the *purpose* for posing the question).

Mance's case revealed that the struggle to articulate a mathematical model may be hindered by multiple competing situation models. These participants did not behave pathologically, but rather spent their time seeking information that had meaning according to their situation and real models, and then adjusting their models in light of available information. These students were wary of small variations in cell volume and its impact on cell count. Schoenfeld's students may have resorted to adjusting human parameters because they did not have information available about cell sizes, because human dimensions are more familiar to the human senses, or because humans can be replaced with sets of familiar shapes. The students in this sample tended to rely on empirical and experiential methods of validation.

Mathematical modeling theory predicts that both identification of relevant variables and relationships (*simplifying/structuring*) and *validating* activity critically affect model construction. However, in the case of The Cell Problem, aspects of the *sensibility* of the task were not considered, and these are important from a modeler's perspective: Who would want to know the answer? For what purpose? The students were left to guess answers to these questions, and therefore were unable to interpret the problem as intended. Thus, in order to observe this kind of mental activity in students' mathematical work, we must provide problem contexts that are commensurate with their means of carrying it out. This work is ongoing, but it is sufficient to show that mathematical modeling perspectives can provide for full and sensible reinterpretations of existing observations.

Questions for the Panel How can I explore the role of information requests and uses in mathematical modeling? Is *purpose* of the modeling task as important to accuracy when there is no numerical result? Are there other foundational works about which our understanding might benefit from re-visits with a mathematical modeling lens?

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Figure 1. Mathematical modeling cycle (Blum, 2011)

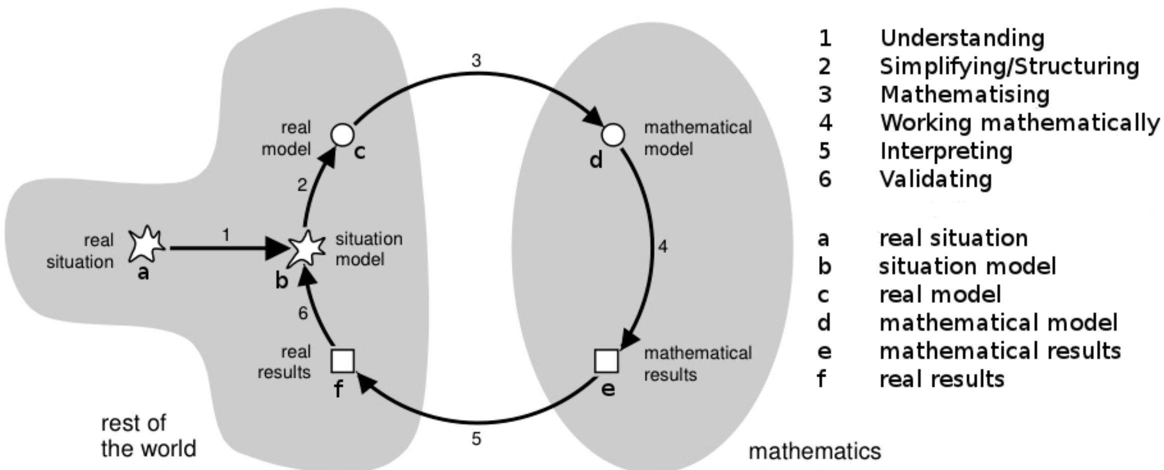

Table 1: Stages of Mathematical modeling

Stage	**Description**
real situation	problem, as it exists
situation model	conceptual model of problem
real model	idealized version of the problem (serves as basis for mathematization)
mathematical model	model in mathematical terms
mathematical results	results of mathematical problem
real result	answer to real problem

Table 2: Cognitive Activities of Mathematical Modeling

Activity	**Trying to Capture**
understanding	forming an idea about what the problem is asking for
simplifying/structuring	identify critical components of the mathematical model (ie, create an idealized view of the problem)
mathematizing	represent the real model mathematically
working mathematically	mathematical analysis
interpreting	recontextualizing the mathematical result
validating	verifying results against constraints

FORMATIVE ASSESSMENT AND STUDENTS' ZONE OF PROXIMAL DEVELOPMENT IN INTRODUCTORY CALCULUS

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One of the challenges of teaching introductory calculus is the large variance in student backgrounds. Formative assessment can be used to target which students need help, but little is known about why formative assessment is effective with adult learners. The purpose of this qualitative study was to investigate which functions of formative assessment as described by Black & William's 2009 framework help students progress through their Zone of Proximal Development. By regularly collecting information from low-stakes opportunities for students to demonstrate their current understanding, instructors were able to target subsequent class discussion on critical scaffolding for student growth. The formative assessments also enabled students to evaluate their own progress and ask clarifying questions and, provided students who would not ordinarily ask questions during class opportunities for legitimate peripheral participation.

Key words: approximation framework, formative assessment, self-monitoring, Zone of Proximal Development

Introduction

Formative assessments, low stakes assignments given to assess students' current level of understanding, increase student achievement (Black & Wiliam, 2009; Clark, 2011), but little is known about how implementing formative assessments facilitates this achievement gain. The purpose of this research was to study the impact of formative assessment on students' engagement in their Zone of Proximal Development (ZPD) in a calculus course designed with Oehrtman's (2008) approximation framework. Our central research question is: How does formative assessment impact students' engagement in their ZPD and conception of the limit structures as developed in Oehrtman's (2008) approximation framework for calculus instruction?

Understanding how the use of formative assessment affects college students' engagement in their ZPD and development of a particular conceptual structure can advance the theory of formative assessment, which has been most prominently influenced by research in European primary and secondary schools (Black & Wiliam, 1998; 2009). Black & Wiliam's (2009) framework of formative assessments suggests that there are five functions of formative assessment (Figure 1).

(1) Clearly communicating learning goals
(2) Allowing instruction to be based on students' current level of understanding
(3) Providing learners with feedback that scaffolds learning
(4) Giving peers a common experience for future collaboration
(5) Raising students' ownership of their learning process through increased metacognition

Figure 1. The five purposes of formative assessment

Theoretical Perspective and Methods

There are several characterizations of the ZPD (Vygotsky, 1987); this report will focus on the interplay between students' spontaneous and scientific concepts and scaffolding that supports students in deepening their conceptual understanding. Results from this study reveal ways in which formative assessment enabled instructors to better assess and target the areas in which critical scaffolding was needed and that this process increased both students' self-monitoring of their understanding and opportunities for peripheral participation in the classroom. Generally, the ZPD can be identified by determining what students can do but only with assistance. The learner is a peripheral participant in this assessment and subsequent scaffolding, because they are being assisted by a more central member of the learning community (Lave & Wenger, 1991; Smagorinsky, 1995). As the learner gains expertise, scaffolding may be reduced and the learner becomes a more central participant in the community of practice.

We recruited participants from classes utilizing Oehrtman's (2008) approximation framework as a coherent approach to instruction in introductory calculus. This framework is built upon developing systematic reasoning about conceptually accessible approximations and error analyses but mirroring the rigorous structure of formal limit definitions and arguments (Oehrtman, 2008, 2009). This study focused on the three multi-week labs developing the most central topics in the course: Lab 3 (limits), Lab 4 (derivatives), and Lab 7 (definite integrals).

This qualitative study centered on a document analysis (Patton, 2002). Our primary sources of data were student documents: formative assessments, homework assignments, and exams of all students in two sections of introductory calculus, with particular attention paid to ten students who each participated in at least one interview. The first author also observed the classrooms the day before and the day after the weekly formative assessment was distributed to the students and debriefed the instructors on a weekly basis to obtain their observations of student and classroom learning trajectories.

Figure 2 provides a portion of a typical formative assessment. These assignments were given prior to each lab (pre-lab) and after each day of lab work (post-lab). The first questions of our formative assessments were conceptual questions about important aspects of the approximation structures in the current lab (not shown in Figure 1). Two open-ended questions always appeared as the last two questions of every formative assessment. An analysis of students responses to the questions were used to plan a brief intervention in the next class addressing the problematic issues.

We coded the data chronologically. First, each action a student needed to take to successfully complete each pre-lab and final lab report was listed. Each student's assignment were then coded for each of these actions; we noted if the action, such as correctly identifying over-and underestimates, was present/absent or appropriate/inappropriate if an action was present. When coding the observation notes during labs we noted which points of difficulty groups asked for help on and made counts of how often those points of difficulty appeared in various groups. The post-labs were coded for three things: (1) mathematical errors students made on any calculational questions, (2) noting if the students identified the problems they had with calculations or parts of the lab accurately, and (3) coding all questions by the concept students found troubling. During the intervention, the first author observed the class using three minutes to count student behaviors (paying attention to the instructor, taking notes, texting or other off task behavior) and then spent three minutes recording impression. Those observation notes were coded for changes in participation patterns.

We recorded which concepts each student ($n = 46$) explicitly stated they did or did not understand and what, if any errors they made on computational questions on each formative assessment. Students' responses to the formative assessments were triangulated with field notes of the classes immediately before and after the lab, as well as their submitted lab reports. Each student's work was coded for particular areas of improvement after the intervention. This initial coding was then analyzed at three levels: by interview participants, by grade bands, and by assignment. At all levels we attempted to identify when a concept was a point of difficulty (entered the ZPD) and ceased to be a point of difficulty (left the ZPD)

Post-Lab 3a: Locate the Hole (Limits)
Directions: Answer the following questions to the best of your ability. Responses need not be lengthy, but should answer all parts of the question.

1. Which question is your group working on?
2. What have you figured out about the answer so far?
3. Write a short paragraph that answers the following two questions. What mathematical concepts or phrases used so far this week do you recognize from calculus? From other mathematics courses?
4. What questions do you have about the material we have covered so far in class?

Figure 2. A typical formative assessment

Findings

For the purposes of this paper, we will focus on Lab 4, the second of the three central approximation framework labs in the semester. During Lab 3, the first lab developing the elements and relationships of the approximation framework, students approximated the *y*-coordinate of a removable singularity in a given function. On the formative assessments, students were able to evaluate the function at nearby *x*-values to approximate to unknown *y*-coordinate with little difficulty. They struggled to use information about the monotonicity of the function to consistently identify over- and underestimates and to understand the difference between errors and error bounds. Students also needed significant scaffolding to draw graphs that were appropriately sized, scaled, and labeled to effectively represent the relevant quantities and relationships. On the formative assessments, students identified their difficulties with errors and error bounds, but not the other two areas mentioned. The scaffolding provided in class addressed all three points of difficulty, and students improved in all three areas from their post-lab to their final write-up. Students who earned B's or C's in the course showed the most improvement, which is consistent with the literature.

Although students submitted exemplary graphs for their final Lab 3 write-up and were given extensive instructions on how to construct high quality graphs for Lab 4, the graphs on pre-lab 4 were inappropriately small with little detail and labeling (Figure 3 is a typical example). The pre-labs allowed the instructor and undergraduate teaching assistants to immediately respond to students' need for additional assistance constructing their graphs. The other points of difficulty during the first day of lab 4 were the from applying context-specific concepts in lab 3 to lab 4; groups wanted to approximate slopes and identify over- and underestimates using the heuristic for Lab 3, using *y* values as approximations, rather than using average rates of change. On the post-lab that night, students repeated this mistake, but did not indicate any confusion; instead students asked about identifying over- and underestimates or the difference between errors and error bounds. The intervention in the next class discussed why *y*-values were not appropriate approximations, how to classify approximations, and the graphical, algebraic, and numerical representations of errors and error bounds.

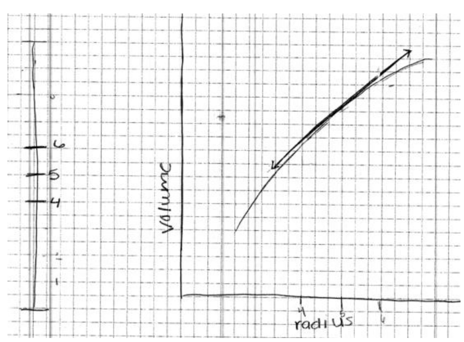

Figure 3. A typical pre-lab 4 graph

In the second week of Lab 4, students completed their problems with minimal assistance; although two groups in each class required additional help entering complicated functions into their calculator. The subsequent post-lab asked students to make connections between the algebraic and graphical representation of the derivative (Figure 4). Students had a median of 4 errors (out of 7 questions), but 66% of the students reported that they were sure some of their answers were wrong, evidence of self-monitoring.

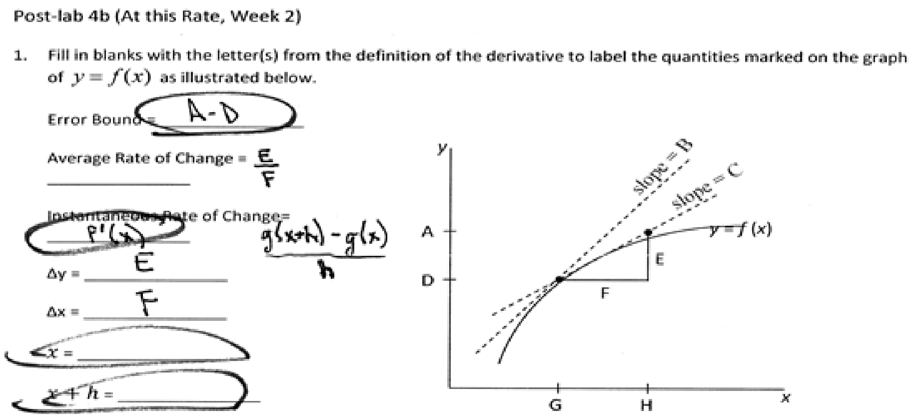

Figure 4. Post-lab 4b, Question 1

The formative pre-labs and post-labs allowed instructors to identify students' ZPD and provide the scaffolding they needed. Students were more likely to take notes, pay attention and refrain from texting in class during lab interventions than any other time in the class. Students were reasonably successful in identifying when they were making mistakes or did not understand a concept in their post-labs. For this lab, the second, third, and fifth functions of formative assessment were the ones that most identifiably helped students engage in their ZPD (Figure 1). In their formal lab reports, the only students who did not successfully improve their graphs, use the correct approximations, classify approximations, or distinguish errors and error bounds were those not in attendance during the instructor-led intervention. The students who did not complete post-labs but attended the intervention showed the same improvements as the students who completed post-labs, which suggests that their peripheral participation in the intervention was sufficient for students to progress through their ZPD.

Students showed the most improvements on their graphical representations, for example, the graph in Figure 5 was submitted by the same student who submitted the graph in Figure 2. On Lab 7, on definite integration, students turned in appropriately sized, scaled, and labeled graphs on their pre-lab, and the points of difficulty were all context-specific or calculator-based rather than difficulty with elements and relationships in the approximation framework, indicating significant progress since their difficulties with these issues in Lab 4.

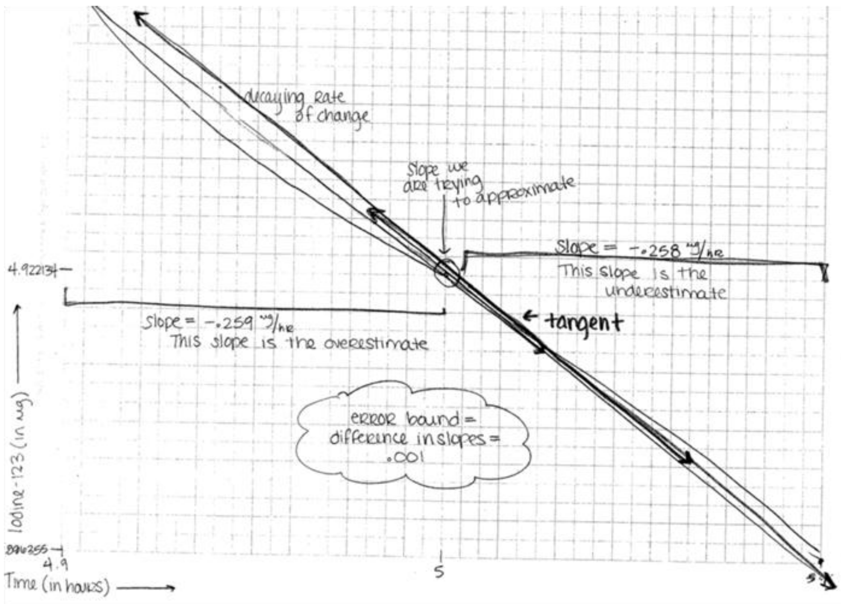

Figure 5. A typical final Lab 4 report graph

Discussion

While the formative assessments were intended to provide a snapshot of students' current understanding and allow instructors to make decisions on what scaffolding their class needed, the act of completing the formative assessment also helped students improve their self-monitoring skills and gave them opportunities to peripherally participate in class without becoming a central participant. Hence, the asynchronous formative assessments had both instructor-centered functions and student-centered functions. The pre- and post-labs gave the instructor a chance to evaluate students current understandings of the activity and target the scaffolding in the next class as precisely as possible. Students gained opportunities for ownership of the material through self-monitoring and peripheral participation opportunities. Although the completion rate of the formative pre- and post-labs was lower than for the labs themselves, students that did not complete formative assessments but attended the intervention still improved in the areas instructors scaffolded; this suggests that all students derived some benefit from the scaffolding based on the formative assessments. The next phase of analysis on this data will detail students' development aspects of the approximation framework from spontaneous concepts to scientific ones. Questions that we will pose to those attending our talk are: (1) What additional insights could be sought from analyzing interviews of students explaining their reasoning behind all of their written responses? (2) How might we more fully integrate multiple characterizations of the ZPD in our data analysis?

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PROOF STRUCTURE IN THE CONTEXT OF INQUIRY BASED LEARNING

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Data was collected from three sections of an introductory proofs course that was taught from an inquiry-based perspective. Inquiry-based learning (IBL) gives authority to students and allows them to present to their peers, rather than having the instructor be the focus of the class and authority on proof. Data from the final exams of 68 students was analyzed with a focus on proof structure. Proofs chosen to analyze included concepts considered "prior knowledge", as well as problems that required new concepts from class. This research utilizes an adaptation of Toulmin's method for argumentation analysis. Our goal was to compare the proof structures generated by these students to previous research also applying some form of Toulmin's scheme to mathematical proof. There was significant variety of proof structures, which could be a result of the IBL atmosphere.

Keywords: Proof, Inquiry-Based learning, Undergraduates, Structure, Toulmin

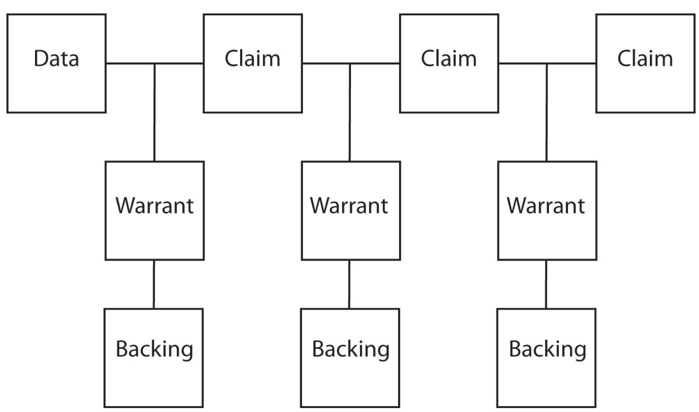
Stephen Toulmin analyzed argumentation and developed a new approach to formal logic. He classified six interrelated components he believed essential in constructing a sound and convincing argument: claim, grounds (data), warrants, backing, rebuttal, and modal qualifiers. In Toulmin's (1979) work, an argument was defined as "the sequence of interlinked claims and reasons that, between them, establish the content and force of the position for which a particular speaker is arguing" (p.13). This definition differs from that of mathematical proof. Argumentation relies on making claims and then justifying, while mathematical proof relies on making inferences of previous results to come to a claim (Barrier, Mathe, & Durand-Guerrier, 2009). In other words, argumentation relies on the content of each claim and proof relies on the function of each claim. (In the context of this paper, we use mathematical argument and proof to mean the same thing). Though Toulmin did not devise his scheme from the perspective of mathematical proof, it is valuable in this context because of the many parallels between argumentation and proof.

Many mathematics education researchers agree that a formal proof contains elements identified by Toulmin, as evidenced by the various proof analysis schemes adapted from Toulmin's work. Though much merit is given to Toulmin's scheme, it is often the case that a *restricted* version is applied in the context of proof. However, Inglis, Mejia-Ramos, and Simpson (2007) argue that you need Toulmin's *complete* model to analyze proof (i.e. consider all six elements). In their research, an interviewer interacted with a student in order to understand the process by which they came to their conclusion. This interaction enabled researchers to witness corrections of mistakes and possible uncertainty of each student. In this case, the *complete* scheme can be considered necessary. On the contrary, a researcher analyzing written proof only witnesses the final stage of the student's thought process; students have effectively already qualified their statements and considered plausible rebuttals and do not present any uncertainty. Hence qualifiers are no longer being stated, a *restricted* scheme is sufficient.

Toulmin's criteria for the structure of argumentation have been used in various contexts of mathematics education research: traditional lecture-based classrooms (Fukakawa-Connely, in progress), interviews (Inglis, Mejia-Ramos, & Simpson, 2007), and classroom discussions (Krummheuer, 2007). Toulmin's scheme has also been applied to everyday argumentation, such as that found in the workplace (Simosi, 2003). One common finding in this line of research is a lack of warrants and backing within an argument or proof. While the structure of mathematical proof and argumentation has been explored in varied contexts, little or no research exists about the structure of student proof in the context of an IBL mathematics course. This work attempts to fill this gap in the literature.

After careful consideration of Toulmin's (1979) definitions and those of past research on mathematical proof, we agreed on definitions to classify statements in student proof and scheme for coding proofs. The only major shift from past work is related to *qualifiers*. In this paper we propose that, by Toulmin's (1979) definition, a *qualifier* is not directly associated with proof and hence suggest a new definition to use within the context of proof. According to Toulmin, *qualifiers* determine the strength of an argument in that they restrict the situation in which the final claim is true. In a mathematical proof, the final claim should be true in every situation, but its validity can rely on sub-cases within the proof. Hence, we chose to identify each sub-case within a proof as a *qualifier* because the arguments that follow are limited to that specific case. After developing our scheme, we applied it to 16 proofs to see if any adjustments needed to be made before finalization. When coding the remaining proofs, we found that new situations arose and the coding scheme evolved. We took note of every evolution, and had to go back to previously coded work to apply the most recent coding scheme. The end result was a scheme that exhausted the coding of the 136 proofs.

The proofs of the statements in the coded problems required several steps. Thus proofs consisted of a string of claims, each with its own warrants and backing. For every student's proof, the arguments were mapped using a similar schematic to one often seen in research of this nature.



Two proofs from each final exam were coded. One proof was related to "past knowledge" as it pertained to a statement about divisibility (referred to as the Integer problem). The second proof was related to functions being 1-1 and onto both ideas that were new to students in this course (referred to as the Functions problem). After mapping each proof the following characteristics were recorded: length, existence of warrants, existence of backing, floaters, qualifiers, incorrect statements and incorrect implications. This presentation will focus on the structure codes of length, warrants, backings, floater and qualifiers in these proofs. We will also compare this coding to instructor grading of the coded proofs.

Length refers not to the number of words used in each proof, but rather to the number of steps taken. Thus the total number of claims was counted. When recording length, a designation of *short* (S – one or two claims), *average* (A – three or four claims), or *long* (L – five or more claims) was assigned to each proof. If a student did not write any claims, then the proof was called "other." With regards to warrants each proof was identified as *complete* (*c* – warrants given for 100% of the claims), *most* (*m* – more than 50%, but not all, of the claims are warranted), *limited* (*l* – at most 50% of the claims are warranted), or *none* (*n* – 0% of the claims are warranted). With regards to backing each proof was given a designation of *complete* (*c* –backings supplied for every warrant), *limited* (*l* – some backings provided, but not for every warrant), or *none* (*n* – no attempted backing). This coding code resulted in a three-letter designation for each proof. For example, a proof with *average* length, *complete* warrants, and *limited* backing would be given a designation of Acl.

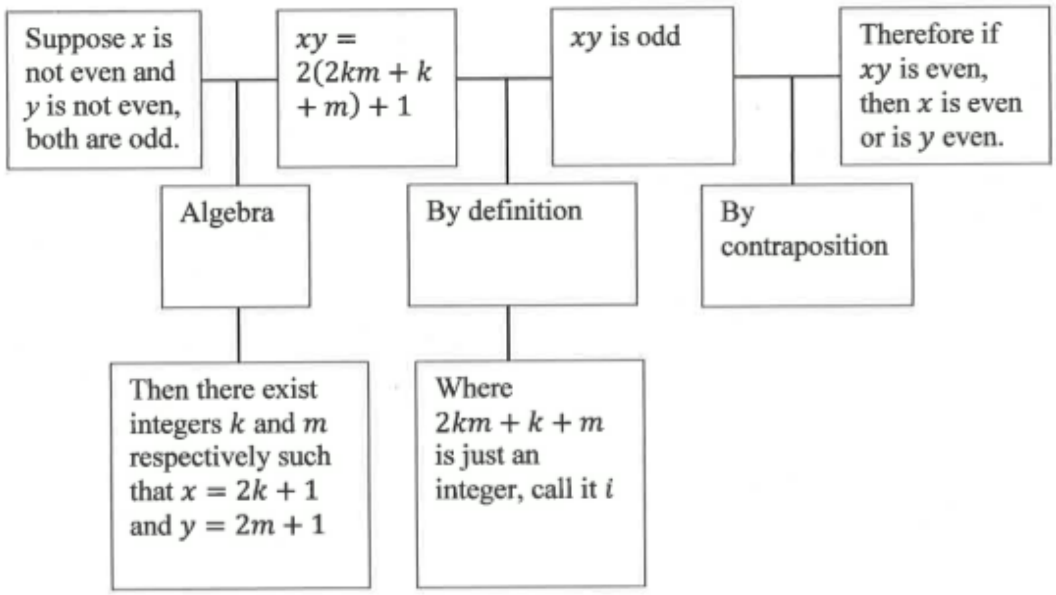
Since the coding scheme constructed for this project was based on proof length, warrants, and backings, there were 28 potential codes. 20 of these arose in the coding of this particular data set of proofs. When looking across the five most frequently used codes for Integer and Function problems, there were three common codes: Acl, Aml, and Lml.

Integer Problem						
Structure Code	Acl	Acc	Amc	Aml	Lml	
Frequency	21	10	9	9	5	

Function Problem						
Structure Code	Scc	Acl	Lml	Lcl	Aml	Lcc
Frequency	11	9	7	6	5	5

All Problems						
Structure Code	Acl	Aml	Scc	Acc	Lml	
Frequency	30	14	13	12	12	

The most common code was Acl. This was often a proof that was correct, but the very last step was not given any backing. An example of the coding of such a proof follows.



To prove the proposition, many students chose to use contraposition. For the final claim that the proposition is true, many of the students whose proofs were coded as Acl stated that they used contraposition as their warrant but did not give backing. In this case, the backing would have been writing out that the contrapositive is true.

With regards to length using three or four claims in the proof was deemed *average* (A). Here the term *average* is used somewhat loosely, for it refers to the most common length and not to the actual average number of claims. A proof with more than four claims was deemed *long* (L), and a proof with less than three claims was deemed *short* (S). It was discovered that the Function proofs have much more variety in length than the Integer proofs. The vast majority (76.5%) of the proofs in the Integer category were *average* length. On the other hand, even though the Function category had more *average* proofs than any other length, the percentages of each length are much closer. In both categories, *long* proofs were more common than *short* proofs. The *long* proofs had the highest overall average score, with 8.46 out of 10. The *short* proofs had the lowest average score (4.68), and the *average* proofs had an average score of 7.55.

Providing warrants for claims is one of the most important parts of a proof. It was thus encouraging to see that every coded proof had some kind of warrant. used, since every proof had at least one of its claims supported. Even further, in both Integer and Function categories, the majority of the proofs were *completely* warranted. While the numbers of proofs with *complete*(c) warrants were similar between the two categories (37 and 36, respectively), there were large differences in the distributions of the *most*(m) and *limited* (l) proofs. The Integer category has 23 more *m* proofs than *l* proofs, whereas in the Function category there is a difference of only 8. As expected, the average score of proofs with *complete* warrants was higher than that of the other proofs, however the difference is very small and there does not appear to be any relationship between score and proportion of warrants

As with warrants, all Integer proofs had some kind of backing, which gives validation to the warrants. The Function proofs, however, had 10 proofs with no backing present. In contrast to the results found concerning warrants, *complete* backing was not the most common occurrence. In both Integer and Function proofs there were more *limited* proofs than *complete* proofs. This most common code (*l*) also had the highest average score overall (7.64). The *completely* backed proofs, on the other hand, had a slightly lower average score (7.43), which was somewhat unexpected. However, the proofs with no backing still had the lowest average score (5.3).

Of the 136 coded proofs, only 15 proofs contained a *floater*. Recall that *floaters* are unnecessary pieces of information that do not follow from or directly connect to the logical sequence of the main proof. Twelve of these proofs were found in the function category. The distribution of codes associated with proofs containing *floaters* was somewhat varied. The most common structure code containing *floaters* was Acl (4) and the only other category with more that one floater was Lmc (2).

Recall that *qualifiers* identify sub-cases. Ten student proofs contained *qualifiers*, all of which were in the Integer category. All but two of these proofs were *long*; the other two were *average*. The proofs that contain *qualifier*s were widely distributed among the structure codes, but the most common was Lml (3) and two other categories had more than one proof that contained qualifiers, Lcl (2), Lmc (2).

A large part of this IBL classroom was student presentation and peer collaboration. It could be expected that this level of collaboration would influence students to approach and prove statements with similar structure, but the data from this project seems to provide evidence to the contrary. It was found that there was not a consistent proof structure across all analyzed proofs. The most common structure code appeared only 30 times (22.1% of all proofs) and the next most common code only appeared 14 times (10.3% of all proofs). This lack of consistency may be due to the fact that the class did not have one consistent person modeling formal mathematical proof but instead an entire classroom of presenters.

In Fukakawa-Connely's (in progress) work, it was found that in a traditional, lecture-based classroom, students modeled their proofs after an authoritative figure (i.e. the instructor). It may be the case that the IBL students did not model their proofs after their figures of authority because they did not fully trust each student presenter's mathematical competency. This lack of trust would likely force students to critically think and assess the validity of each proof instead of trusting that the professor is correct. Also, as students are being exposed to the many different proof structures provided by various presenters, they are able to judge which proof structure makes the most sense and works best for them. The wide variety of proof structures identified in this research analysis shows that IBL classrooms facilitate a flexible environment that encourages student creativity within formal proof. The fact that average scores were not drastically different between proof structures also supports this.

There was a wider variety of proof structures among student proofs in this research than we anticipated. This may suggest that students in the IBL class learned to take responsibility for their proof style and not solely rely on the authority of the instructor. Since every student provided some level of warrant, students seemed to understand the importance of justification in mathematical proof. Backing, on the other hand, was less common. It was found that lack of backing did not significantly affect score, which may mean that implicit backing is acceptable in mathematical proof and not simply in oral argumentation. Would applying our coding scheme to problems from lecture-based classes or different IBL classes yield similar results?

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STUDENTS' WAY OF THINKING ABOUT DERIVATIVE AND ITS CORRELATION TO THEIR WAYS OF SOLVING APPLIED PROBLEMS

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Previous researchers have examined students' understanding of derivative and their difficulties in solving applied problems and/or their difficulties in applying the basic knowledge of derivative in different contexts. There has not been much research approaching students' ways of thinking about derivative through the lens of applied questions. In this research, first I categorized the students' way of thinking about the basic concept of derivative by running a survey of questions addressing the different ways of thinking about derivative based on the existing research works. While analyzing these surveys, I used grounded theory and added more ways of thinking about derivative. I specially noticed very incomplete ways of thinking about derivative as described below. Since my goal was looking at the students' ways of thinking about derivative through the lenses of applied questions, I also piloted my applied questions survey with 51 multivariable calculus students. I noticed a lot of students struggling with defining variables (the initial translation as described below) and if they could define the variable, a lot of them struggled on applying their ways of thinking about derivative into solving the applied problem. These difficulties are great venues to study their ways of thinking about derivative using their struggle in the applied questions. This is a summary of my initial works on this ongoing research, the goal of which is to shed new insights into students' solving of applied problems.

Keywords: Derivative, Applied Problems, Ways of Thinking

Introduction

Derivative is one of the fundamental concepts covered in calculus. There is rich, extensive research on students' difficulties with the concept of derivative and their difficulties with applied derivative problems. However, not much is known about how students' conceptual understanding interacts with their work on applied problems. Sometimes students can think of derivative as the slope of a tangent line and sometimes as an instantaneous rate of change. They can even have incomplete ways of thinking about it for instance thinking of derivative as a slope of a function or thinking of it as rate of change. Students can also have combinations of these ways of thinking about derivative. Looking at correlations of students' different ways of thinking about derivative and how those understandings can impact their ability to solve applied problems can contribute to our understanding of undergraduate students' skills and conceptual understanding and generate insights into their thinking about application problems. Names used by researchers for these ways of thinking include "multiple representations," "contexts," or "layers of process object pairs" (Zandieh, 2000). Even our best students do not completely understand concepts taught in a course, and when faced with an unfamiliar problem, have difficulty solving the problem (Carlson, 1998; Selden et. al 1988; Selden et. al 2000; Bezuidenhout, 1996). The research question for this study is: Is there a relationship between students' multiple ways of thinking about derivative and their success in solving of applied problems?

Students' ways of thinking about derivative

Zandieh (2000) framed students' understanding of derivative in the multiple representations of graphical, verbal, physical, symbolical and other which were analyzed under the contexts of ratio, limit and function as the underlying concepts. Abboud and Habre (2006) used graphical, numerical, and symbolic views of derivative in assessing students' understanding of derivative. Kendal and Stacey (2003) used three representations of differentiation (graphical, symbolical, and physical) in creating what they called "Differentiation Competency Framework." In the application of these research works to the present study, I combined categories in some cases and added subcategories in others. This research uses the phrase *way of thinking* align with concept images or multiple representations of derivative which are based on constructivist cognitive theory.

Application Problems

In solving "real-world" problems, Tall (1991) wrote that the given problem is first translated from the context to the abstract level of calculus, the abstract problem is then solved, and the solution is translated back to the context. The first step obviously calls on students' conceptual knowledge of variables, algebra skills, and calculus concepts because it depends on the identification not only of the appropriate concepts in the given context but also of the relationships among them. The identification of appropriate concepts might involve the selection of one or more symbolized variables from among several concepts.

Research Design

Due to the complexity of the concept of derivative, we need to look at the students understanding of it from different perspectives. I used the cognitive variability and strategy choice as described by Stiegler (2003) as features of students thinking. Stiegler described how the students in different age use "multiple thinking strategies when solving problems of the same type" (P. 293). I describe these thinking strategies as ways of thinking about the concept of derivative. Therefore by identifying these multiple ways of thinking we can look at the students' problem solving strategies and its correlation to their problem solving strategies.

I used written surveys to collect data. Two separate surveys were created and administered to 125 differential calculus students and 51 multivariable calculus students at a large northeastern university. The first survey consisted of six questions three of which were used to look at students' fundamental ways of thinking about the derivative.

The second survey was focused on the applications of the derivative. This survey included one optimization problem where the students could either solve it intuitively or by just applying their basic understanding of derivative. The second question prompted the students to use derivative in solving a maximum/minimum question.

Some tasks came from existing research; others were created by the researcher. The first survey consisted of tasks addressing different possible “representations” or “concept images” held by the students. Most of these tasks were borrowed from existing research (Zandieh, 2000; Abboud & Habre, 2006; Kendal & Stacey, 2003; Carlson, 1998). The second survey was designed using tasks from White and Mitchelmore (1996).

Preliminary Data Analysis

I used the categories used by the other researchers and started analyzing the students surveys from differential calculus. While looking at the students responses there were a lot of responses where I could not fit their answers into any of the categories used by the other researchers. Using methods from Grounded Theory (Glaser & Strauss, 1990), I was able to add other categories in order to frame students’ multiple ways of thinking about derivative.
The preliminary analysis was done based on what differential calculus students had written in response to the tasks on the surveys. Question one was based on Zandieh (2000).

1. You are talking with someone who just started high school. In a sentence or two explain to them what is meant by the derivative of a function. (Feel free to use any graph, symbols, or words in your explanation.)

Question one from firstthe survey

Question 2 was borrowed from Kendal and Stacey (2003) and was the question that the majority of their participants had the most difficulty with. Other research findings indicate that students don’t define the derivative using the notion dy/dx and this question was designed to gather data on whether they would use it in their definitions.

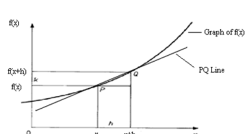
2. If y is a function of x, explain in words the meaning of the equation $\frac{dy}{dx} = 5$ when x = 10.

Question 2 from the first survey

It is known from previous research that likely not many students would use formal definition of derivative to express their ways of thinking therefore I added question 4 (Shown above) to the survey explicitly provided this as an opportunity for them to perhaps use it in their definition.

4. Here is a definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Why does the formula give us the derivative of the function? Provide an explanation for your answer.

Question 4 from the first survey

Analysis of the remaining questions of first survey will be presented during the conference.
To analyze the second survey a couple of White & Mitchelmore’s (1996) categories were used and other categories also emerged in the analysis. Table II is a sample of the analysis. As you can see this table includes four different categories of the students’ difficulties with solving the

applied problem. As explained earlier, these can be opportunities (venues) to explore the correlations of the students' multiple ways of thinking about derivative and their methods solving applied problems.

1. If the edge of a contracting cube is decreasing at a rate of 2 centimeters per minute, at what rate is the volume contracting when the volume of the cube is 64 cubic centimeters? (Provide an explanation for your answer.)

First question from the second survey

Results

The categories shown below are only based on what differential calculus students had written on their surveys. This provides one kind of window into their thinking but may not capture all of what they know. Some students might have used the same way of thinking several times in answering the survey questions however I counted them as once.

Symbolic category refers to the formal definition of derivative. Graphical ways of thinking refer to when the student uses slope of the tangent line in describing derivative. Incomplete Graphic refers to when students explain the derivative using only one term such as slope or only tangent line or when they say slope of a function. Numeric is referring to descriptions of the derivative using question 2 from the first survey. For instance when students define the derivative in terms of "derivative of y with respect to x is ten when x is 5, they are using a numerical example to describe the derivative which is why I used the title " numeric" for this group. Verbal refers to using Instantaneous rate of change in explaining derivative. Incomplete Verbal category refers to using just rate of change or rate of a function to explain the concept of derivative. Procedural is when the students talk about power rule or actually write an example of taking the derivative as a way of explaining it, for instance: f(x)=X^2 so f'(x)=2X. Category others refer to when students use area under the graph or accumulation to explain the derivative. For the purpose of this paper I used a very general term, "incomplete" in order to show that many students do not have a complete ways of thinking about derivative as we expect them do. Each one of this incomplete categories include several ways of thinking which are not complete in relation to the complete way of thinking about derivative as described by other research works or as defined in calculus books.

Ninety five out of 125 students had multiple ways of thinking about derivative however more than 70% of the students had incomplete ways of thinking about derivative as shown on the below table under "Incomplete Graphic", and "Incomplete Verbal" categories.

Table 1: Differential Calculus Students Surveys Results based on students answers to questions 1, 2 and 4

Categories of Student thinking*	Symbolic	Graphical	Incomplete Graphical	Numerical	Verbal	Incomplete Verbal	Physical In terms of Speed	Physical in terms of velocity/ Acceleration	Procedural	Others$
Number of Students thinking in this way	3 out 125	19 out of 125	91 out of 125	9 out of 125	18 out of 125	89 out of 125	2 out of 125	9 out of 125	31 out of 125	9 out of 125
%	2.4	15.2	72.8	7.2	14.4	71.2	1.6	7.2	24.8	7.2

40 out of the total of 51 students in multivariable calculus could not answer the question 1 from the second survey.

Table 2: Multivariable Calculus Students Surveys Results based on students answers to questions 1

Could not remember how to set it up, or did not try	Attempted but no symbolizing any variables for any quantities or wrong definitions of variables	Correct Variables but wrong modeling (the relationship between the variables)	Correct Translation but wrong calculus- x,y syndrome-Manipulation focus
15	25	8	3

*We had total of 51 surveys

Conclusions and Implications

As it was shown on the tables of results, majority of the students have multiple incomplete ways of thinking about derivative. They also have difficulties in using their knowledge about derivative to solve applied problems. If we can show that lack of complete ways of thinking about derivative impact students' abilities to solve applied problems, we can use that in enhancing our curriculum to ensure students' ways of thinking about derivative can be developed properly so they can solve the real world problems more effectively.

Future Plans

I am running the same survey this semester at three different differential calculus classes and then I am planning to divide the students based on their responses into three categories. These groups will be invited to participate in a task based interview. I am adopting a new approach for my interview similar to the method Selden, Selden, Hauk, & Mason (2000) and Selden, Mason & Selden (1994) used in collecting their written surveys. I am interested into investigating students multiple ways of thinking about derivative and how that affects their applied derivative problem solving. It seems from the preliminary data that students with multiple ways of thinking about derivative should be able to do better on the applied questions.

Questions for Discussion:

- The next phase of this project is to examine college mathematics instructors' knowledge of student thinking about derivative and application of derivative. What questions might be asked of these instructors to tap into their knowledge of the student thinking, including their knowledge of the impact of these differences on students' performance on tasks?
- What other interview protocols do you think can help us in investigating the correlations of students' ways of thinking and their methods in solving applied problems?
- Do you think these categories capture all the important aspects of thinking about derivative?

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CALCULUS STUDENTS' UNDERSTANDING OF VOLUME

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Researchers have documented difficulties that elementary school students have in understanding volume. Despite its importance in higher mathematics, we know little about college students' volume understanding volume. This study investigated calculus students' understanding of volume. Clinical interview transcripts and written responses to volume problems were analyzed. One finding is that some calculus students, when asked to find volume, find surface area instead and others blend volume and surface area ideas. We categorize students' formulae according to their volume and surface area elements. Clinical interviews were used to investigate why students might find surface area when asked for volume. We found that some students believe adding the areas of an object's faces measures three-dimensional space. Findings from interviews also revealed that understanding volume as an array of cubes is connected to successfully solving volume problems. This finding and others are compared to those for elementary school students. Implications for calculus teaching and learning are discussed.

KEYWORDS Student thinking, Calculus, Volume, Surface Area

1 INTRODUCTION

Many calculus topics involve volume: optimization and related rates in differential calculus, volumes of solids of revolution and work problems in integral calculus, and multiple integration, to name a few. Although volume shows up in these places, researchers have focused on elementary school students' difficulties with volume and little is known about older students' understanding of this idea. This study extends knowledge about students' understanding of volume and, in using non-calculus tasks, builds a foundation for studying volume understanding in calculus contexts.

We conducted this study within a cognitivist framework, giving students mathematical tasks and analyzing the reasoning underlying their answers. This is consistent with the cognitivist orientation toward focusing on "the cognitive events that subtend or cause behaviors (e.g., [a student's] conceptual understanding of the question)" (Byrnes, 2000, p.3). We collected written survey data and conducted clinical interviews to investigate the following research questions:

The research questions investigated are:
1. How successful are calculus students at volume computational problems?
2. Do calculus students find surface area when directed to find volume?
3. If calculus students find surface area when directed to find volume, what thinking leads them to do so?

Our major finding is that nearly all students correctly calculate the volume of a rectangular prism, but many students perform surface area calculations or calculations that combine volume and surface area elements when asked to find the volume of other shapes.

2 ELEM. SCHOOL STUDENTS’ UNDERSTANDING OF VOLUME

There is a paucity of literature about calculus students’ volume understanding, though it is known that some calculus concepts involving volume (e.g., related rates, optimization, and volumes of solids of revolution) are difficult for students (Martin, 2000; Tomilson, 2008; Orton, 1983). Research about elementary school students’ volume understanding provided a basis from which the researcher investigated calculus students’ volume understanding. Findings from research about elementary school students’ volume understanding suggest that this population has trouble with arrays, formulae, and cross-sections.

Volume computations rely on the idea of an array of cubes, a representation with which elementary school students struggle (Battista & Clements, 1996; Curry & Outhred, 2006). Two difficulties students have are (1) understanding the unit structure of an array and (2) using an array for volume computation. Battista and Clements (1996) found that only 23% of third graders and 63% of fifth graders could determine the number of cubes in a 3x4x5 cube building made from interlocking centimeter cubes. One source of this difficulty is not seeing relationships between rows, columns, and layers, leading students to double-count innermost and edge cubes or viewing the array “strictly in terms of its faces” (Battista & Clements, 1998, p. 229). In other words, it seems that some elementary school students are thinking about surface area when asked about volume.

Other findings indicate that some elementary school students use area and volume formulae without understanding them (De Corte, Verschaffel, & van Collie, 1998; Fuys, Geddes, & Tischler, 1988; Nesher, 1992; Peled & Nesher, 1988). For example, Battista and Clements (1998) found that some students’ strategies involved “explicitly using the formula *L x W x H* with no indication that they understand it in terms of layers” (Battista & Clements, 1998, p. 229).

Lastly, findings indicate that identifying the shape of a solid’s cross-section is difficult for students (Davis, 1973). This finding is important because some volumes can be thought of as V=Bh where B is the base of the solid and the base is, in fact, a cross-section. Students having difficulty finding the shape of a cross-section would thus have difficulty using the V=Bh formula. This finding carries particular importance if it is also true for calculus students, as volumes of solids of revolution problems require identifying the shape of a cross-section.

3 RESEARCH DESIGN

The data analyzed were from written surveys completed by 198 differential calculus students and 20 clinical interviews with a subset of those students. Data collection had two phases: first, students completed the written tasks (modeled after those used in research with elementary school students) and data were analyzed based on those researchers’ methods and a Grounded Theory inspired approach (Corbin & Strauss, 2008) where necessary. Clinical interviews (Hunting, 1997) were used to investigate patterns from the written data; that is, interview subjects were selected because their answers on written tasks represented an emergent category. This methodology allowed for a quantitative analysis of a large number of written responses and a qualitative analysis of student thinking about those written responses.

Written survey tasks consisted of diagrams of solids with dimensions labeled. Students were directed to compute the volume and explain their work. The rectangular prism task is shown below; the other tasks are included in Appendix A.

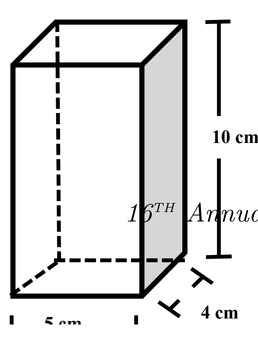


Figure 1. Volume of Rectangular Prism Task

Clinical interview tasks were the same as the written tasks. Interviewees were asked to re-work their solutions, thinking aloud as they did so. Questions were asked to probe student understanding, such as “Can you tell me about that formula? Why is the 2 there?” and “Can you tell me why that [formula] finds volume?” Interviews were audio recorded and transcribed.

Analysis of written surveys. Data were analyzed using both the methods of other researchers and an approach inspired by Grounded Theory (Corbin & Strauss, 2008). This entailed first looking for patterns in a portion of the data and using patterns to form categories, followed by forming category descriptions and criteria. Those criteria were then used to code all the data, testing the criteria until new categories ceased to emerge. The analysis resulted in three categories for students’ work: volume, surface area [instead of volume], and other. The categories and their descriptions are as shown in Table 1.

Found Volume	**Found Surface Area Instead of Volume**	**Other**
Magnitude is the correct magnitude of the object’s volume, or magnitude is incorrect for the object’s volume but the work/explanation is consistent with volume-finding (i.e., multiplication or appropriate addition)	Magnitude is the magnitude of the object’s surface area or the student work/explanation contains evidence of surface-area like computations, such as addition. To allow for computational errors, magnitude may or may not be the actual magnitude of surface area.	Student found neither volume nor surface area

Table 1. Categories of Students’ Responses

We used these categories to develop coding algorithms for the four volume computations. See Appendix B for a sample algorithm.

Analysis of clinical interviews. As interview data included both transcripts and students’ written work, there were two parts to the analysis. First, written data were categorized according to the aforementioned algorithms. Second, transcripts were used to investigate the thinking involved in answers for each category. This was useful because formulae themselves do not tell the whole story; for instance, some students explained the formula $2\pi r^2h$ as a V=Bh formula, mistakenly believing that the area of a circle is $2\pi r^2$. Other students described $2\pi r^2$ as accounting for the area of two circles. The former is a volume idea; the latter a surface area one. This led us to categorize students’ formulae according to their area and volume elements, with the categorizations based on how students talked about those formulae. We begin with this in the Results section, then present the frequency of volume and surface-area finding for the four shapes.

4 RESULTS

We believe there is an important link between students’ formulae and their reasoning: that is, our data leads us to believe that students’ formula are not (as is commonly assumed) remembered or misremembered, but are instead representative of ideas students have about volume. This finding, based on the synthesis of interview data with written work, led us to categorize students’ formulae according to their surface area and volume elements. What we mean by “surface area and volume elements” is what we alluded to in discussing how the 2 in $2\pi r^2h$ might be from an ill-remembered area formula and might be from accounting for two bases. Categorizing students’ formula in this way gave us the categories and component formulae shown in Table 3. Note the appearance of $2\pi r^2h$ in both the “incorrect volume, no surface area element” and “surface area and volume elements” categories, per the reasoning stated above.

Correct volume	Incorrect volume, no surface area element	Surface area and volume elements	Surface area	Perimeter
πr^2h	$2\pi r^2h$ $(1/3)\pi r^2h$ $(1/2)\pi r^2h$ $(4/3)\pi r^2h$ πrh $(1/2)\pi rh$ $h*d*r$	$2\pi r^2h$ $2\pi rh$ $2\pi r + \pi rh$ $\pi r^2 + 2\pi d$ $2\pi r^2 + 2rh$	$2\pi r^2h + 2\pi rh$ $2\pi r^2h + \pi dh$	d+h

Table 2. Categories for student responses to the cylinder task

This table includes all formulae that appeared in students’ written work and interviews. Interview data provided help in placing the formula, and interview data are the basis of our claim that students’ formulae are a reflection of their reasoning. For instance, consider Nell’s reasoning about the volume of the cylinder:

> **Nell:** I don’t know the formula for this one. Two pi r squared… times the height. Sure. We’ll go with that one. So you have two circles at the ends, which is two pi r squared… you have two pi r squared because that’s the area on the top and the bottom so you can just double it, then you have to times it by the height.
> **Interviewer:** Why do I have two areas?
> **Nell:** You have two circles.

The inclusion of the areas of the bases of a shape (what Nell calls the top and bottom) is part of finding surface area. However, Nell was not thinking about surface area, she was thinking about volume. This is evidenced by the following excerpt:

> **Interviewer:** What about this multiplying by the height? Why do we do that?
> **Nell:** It gives you the space between the two areas. Volume is all about the space something takes up so you need to know how tall it is.

Nell’s reference to the space between two areas is indicative that she was thinking about volume. However, as previously stated, her formula ($2\pi r^2h$) included a surface area idea. We thus put the formula $2\pi r^2h$ in the “surface area and volume elements” category (see Table 3). It is also included in the “incorrect volume, no surface area elements” because other students talked about this formula as *area of base times height* where the area of the base was $2\pi r^2$. In this case, the two is not a nod to two bases, it is an incorrect formula for area but correct reasoning for volume.

Nell was not the only student who thought about including both circles when finding volume: Jo went back and forth about whether she should use the formula $2\pi r^2h$ or πr^2h. The interviewer asked her to make the case for both one and two circles as a way to investigate her reasoning:

> **Jo:** The area of the circle is pi r squared times the height, but I can't decide if I need one or two circles.
>
> **Interviewer:** Convince me that you need two circles.
>
> **Jo:** You need two because you have the top and the bottom of the cylinder. But you don't actually need two… you just need the one. Because you get the area of the circle and you multiply it by the height… the circle is the same throughout the whole layer so you just multiply it by the height.

Jo's final reasoning was correct, but it's noteworthy that her initial response to the problem involved a surface area idea. Thus, despite her correct final response, we believe this is evidence that some students have mixed and combined surface area and volume ideas. An additional interesting result is that the frequency of this phenomenon appears to be shape-dependent. This is evident in the percentages of students who fell into each category, shown in Table 3.

	Rectangular prism (n=198)	Cylinder (n=198)	Triangular Prism (n=122)	Trapezoid (n=7)
Found Vol.	194 (98.0%)	172 (86.9%)	95 (77.9%)	5 (71.4%)
Found SA	3 (1.52 %)	10 (5.1%)	17 (13.9%)	2 (28.6%)
Other	1 (0.5%)	16 (8.0%)	10 (8.2%)	0 (0%)

Table 2. Percentage by task

Analysis of written data indicated that some differential calculus students find surface area when directed to compute volume. However, the percentage of surface-area-finding students varies by shape, with few students (1.52%) finding surface area for the rectangular prism, 5.1% of students finding surface area for the cylinder, and 13.9% of students finding surface area for the triangular prism. A much higher percentage of students (71.4%) found surface area for the trapezoid; however, as n=7, this may not be a representative sample.

5 CONCLUSIONS AND IMPLICATIONS

Findings indicate that student success with computational volume problems differs by shape. Students were extremely successful with objects such as the rectangular prism, but struggled with the assumedly less-familiar trapezoidal prism. This has implications for volume-finding in calculus; for instance, volumes of solids of revolution are rarely elementary shapes. A second finding is that some calculus students find surface area instead of volume, either thinking that adding the areas of faces finds volume, or having formed an amalgam of surface area and volume ideas.

One implication for instruction is that instructors might use student-generated formulae to diagnose their ideas. Viewing students' formulae in terms of surface area and volume elements may provide clues to the ideas students hold about surface area and volume, and asking students about the formulae they use may provide insight as to the ideas they hold. An additional implication for instruction is to provide opportunities for calculus students to revisit and strengthen their understanding of surface area and volume, including unpacking the formulae for each. Further, using the ideas of the shapes of cross-sections and bases could be useful not only in understanding the volume of geometric solids for which V=Bh can be applied, but might aid students when they learn volumes of solids of revolution.

A suggestion for further research is to find out if students' volume/surface area difficulties interact with their learning of related rates, optimization, and volumes of solids of revolution. It would be interesting to know what happens for a student with an amalgam of surface area and volume tries to optimize the surface area for some given volume and whether difficulties with this are related to underlying issues with surface area/volume, issues with the calculus, or both.

Appendix A: Tasks

2b. What is the volume of the cylinder? Explain how you found it.

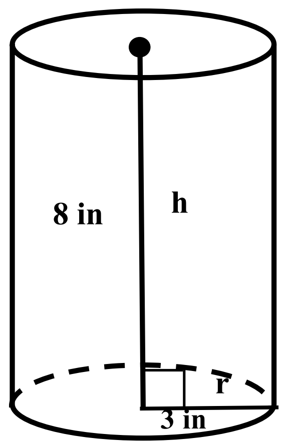


Figure 2. Cylinder Task

3. What is the volume of the object? Explain how you found it. <u>Note:</u> The figure is not drawn to scale.

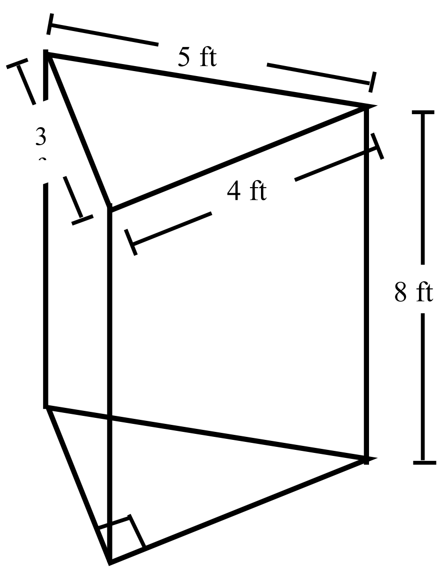
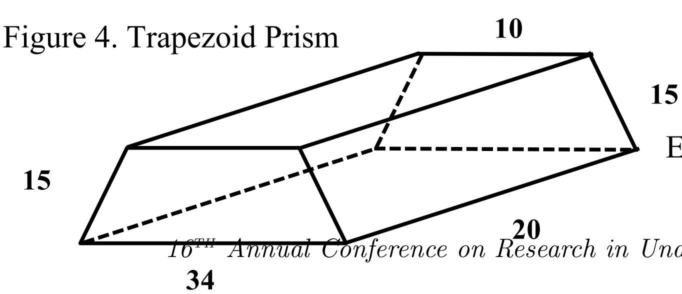


Figure 3. Right Triangular Prism Task

Figure 4. Trapezoid Prism



6.What is the volume of the prism? Explain how you found it.

Appendix B: Example Coding Algorithm

Cylinder coding algorithm:

Correct volume = $\pi r^2 h = \pi(3^2)(8) =$ *72π [units^3]*

Correct surface area = $2\pi r^2 + 2\pi rh = 2\pi(3^2) + 2\pi(3)(8) =$ *66π [units^2]*

1. Did the student write the formulae $\pi r^2 h$ or $2\pi r^2 h$? Did the student write 72π or 144π? If so, categorize as "found volume." If not, proceed to #2.
2. Did the student write πr^2 + ________ or $2\pi r^2$ + ____________where ________ is something that looks like it might be πdh or some other computation that looks like an area of a lateral face? Did the student write 66π? In either case, categorize as "found surface area instead of volume." If not, proceed to #3.
3. Categorize as "other."

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A CASE STUDY ON A DIVERSE COLLEGE ALGEBRA CLASSROOM: ANALYZING PEDAGOGICAL STRATEGIES TO ENHANCE STUDENTS' MATHEMATICS SELF-EFFICACY

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Shifting demographics show America rapidly diversifying, yet research indicates that an alarming number of diverse students continue to struggle to meet learning outcomes of collegiate mathematics curriculum. Consequently, recruitment and retention of diverse students in STEM majors is a pervasive issue. Using a sociocultural perspective, this study examined the effect of two pedagogical strategies (traditional instruction and cooperative learning) in a diverse College Algebra course on enhancing students' mathematics self-efficacy. Particular attention was paid to investigating the role student discourse and interaction play in facilitating learning, improving conceptual understanding, and empowering students to engage in future self-initiated communal learning. The goal is to develop an effective classroom model that cultivates advancement in content knowledge and enculturation into the STEM community, culminating in a higher retention rate of diverse students in STEM. Preliminary data analysis suggests that a hybrid model encompassing both traditional instruction and cooperative learning successfully enhances students' self-efficacy.

Key words: Diversity, Pedagogy, College Algebra, Sociocultural Theory, Self-Efficacy

Research indicates that an alarming number of diverse[1] students struggle to meet learning outcomes in mathematics (National Center for Education Statistics [NCES], 2009; Trends in International Mathematics and Science Study [TIMSS], 2007). This is a critical issue because according to the U.S. Census, the racial and ethnic population of the U.S. continues to expand considerably. However, many culturally diverse students struggle in mathematics, and therefore, elect not to major in and successfully complete degrees in Science, Technology, Engineering, or Mathematics (STEM) disciplines. For the U.S. to remain competitive in the world market of global technology, an increase in minority STEM leaders is essential and must be addressed.

Although mathematics educators (Boaler & Staples, 2008; Gutierrez, 1996; Gutstein, 2003) have performed extensive research on studying various techniques to help diverse students thrive at the K-12 level, significantly less research has been done at the college level. Analyzing college students is equally imperative because numerous high school graduates enter college mathematically underprepared (Bettinger & Long, 2006), lacking the confidence they need to succeed, and fail to complete degrees in STEM majors. Therefore, the specific problem I will focus on is how to more efficiently close the achievement gap for diverse college students by increasing their mathematics self-efficacy[2] (Bandura, 1977, 1995).

Research studies demonstrate that students with high self-efficacy perform better academically than those with low self-efficacy (Multon, Brown, & Lent, 1991; Pajares &

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[1] I use the term *diversity* as defined by The National Council for Accreditation of Teacher Education (2008): "differences among groups of people and individuals based on ethnicity, race, socioeconomic status, gender, exceptionalities, language, religion, sexual orientation, and geographical area."

[2] Self-efficacy shall be understood as a person's confidence in accomplishing a given task.

Graham, 1999). According to Hackett and Betz (1989), even when variables such as mathematics aptitude, gender, and anxiety are controlled, self-efficacy beliefs are predictive of students' choice of major and academic performance. I take the perspective that obtaining high self-efficacy will benefit students not only in the short-term (i.e. passing classes), but have significant long-term impacts as well. Students with high self-efficacy are more likely to persist and graduate in STEM-related fields, which will ultimately lead to diversifying STEM education. To accomplish that, teachers play an instrumental role in implementing effective pedagogical strategies and establishing a supportive learning environment.

Over the years, teachers have implemented various pedagogical methods in mathematics classrooms to heighten student learning and enhance self-efficacy. Commonly employed pedagogies include variations on *traditional instruction* (where teachers exclusively lecture and students work independently). Through independent learning, students can increase their confidence levels by solving problems without aid. Another type of pedagogy is *cooperative learning* or peer interaction in collaborative groups, which has since emerged as one of the prominent methods in developing student learning and confidence. The NCTM Standards asserted, "Whether working in small or large groups, [students] should be the audience for one another's comments – that is, they should speak to one another, aiming to convince or to question peers" (NCTM, 1991, p. 45). Peer interaction provides students the opportunity to collaborate with their classmates, explain their solution methods, and construct knowledge while participating in a community of practice (Lave & Wenger, 1991). I hypothesize that working in collaborative groups will promote student engagement in discussions, expand learning by combining each member's knowledge base, and encourage students to teach and learn from their classmates, which can ultimately increase self-efficacy such that students persist in STEM.

Participants (n = 10) for this study were enrolled in one College Algebra course consisting of diverse undergraduate students. This course was set up as a hybrid model where during each class period students learn in two settings: traditional instruction and cooperative learning. In the traditional setting, students learn independently (lecture-style) with little or no interaction with fellow peers. In cooperative learning, students work together in groups while freely interacting with one another to examine and complete discussion-provoking activities. I investigate the role student discourse plays in facilitating learning and improving conceptual understanding; subsequently I specifically analyze *how* students increase their self-efficacy in the classroom.

This study explores the following research questions:

1) What do individual students perceive to be the benefits and challenges of traditional and collaborative pedagogies, particularly with respect to enhancing their self-efficacy?
2) What evidence exists indicating the role of discourse in improving diverse students' learning outcomes?
3) What are the implications of addressing diverse perspectives when implementing either pedagogy?

I have framed this research study using a sociocultural theoretical perspective. Operating under a sociocultural perspective, I am able to reinforce the idea that learning is a co-constructed process that occurs and is influenced by multiple contexts (Vygotsky, 1978). A sociocultural perspective has crucial implications for informing the study design, type of data to be collected, and how the data is analyzed. For example, to capture the multiple contexts of students learning mathematics, it behooved me to collect data on *how* students perceived their learning in different settings (traditional vs. cooperative). Therefore, I meticulously deconstructed and analyzed group interaction and discourse as students worked to solve mathematical problem sets.

Given my goal of comprehending the highly complex phenomena of teaching and learning mathematics, I collected data using multiple sources in a mixed-method research design. The College Algebra course I studied with diverse students (n = 10) took place during the summer term of 2012. Data were collected at a medium sized public university in the Pacific Northwest. While this study is a mixed-method design, to most efficiently attend to the research questions, the majority of the data were collected qualitatively. Each data source (classroom observations, video data, interviews, surveys, journals, artifacts, and reflexivity) was analyzed using standard procedures of coding and/or statistical analysis. To support the qualitative data and strengthen findings, I administered the Mathematics Self-Efficacy Scale (Hackett & Betz, 1989) as a pre/post survey (reliability coefficient alpha of .96 (Hall & Ponton, 2002)) to determine whether students' self-efficacy increased or decreased throughout the study. This quantitative survey was used to support the qualitative data. I administered pre/post interviews with all 10 participants, observed them during class, and all 10 students completed the surveys.

To strengthen the validity and reliability of the study, I focused on three strategies: First, triangulation (Denzin & Lincoln, 1998) was used throughout the analysis process to understand each source in relation to each other, elucidate clear trends and themes, and fortify the credibility of the study. Second, I performed member checks (Lincoln & Guba, 1985) with participants to ensure the study's trustworthiness. Third, I completed inter-rater reliability (Mays & Pope, 1995) with an experienced colleague to establish reliability among coding.

Preliminary findings suggest that a hybrid course using both traditional instruction and cooperative learning is successful because it attends to the students' various learning styles. Class observations, video recordings, interviews, and journals indicate that through a hybrid model, different students gain from different parts of the course. A hybrid course is beneficial because it is as diverse as the students. Students pointed out that in cooperative learning, instead of diminishing learning by "relying on someone else" it was advantageous to "bounce ideas off of each other," which garnered "multiple viewpoints." In traditional instruction, students acknowledged the "possibility of getting behind if you don't understand what's going on," but it also "teaches you to learn how to do things individually." I performed a paired sample t-test on the survey data, indicating that students' self-efficacy increased significantly during the course.

The results of this study show the usefulness of a hybrid classroom model for attending to individual student needs and enhancing their self-efficacy. The outcomes also suggest the hybrid model would be beneficial to introductory courses, where students of varied backgrounds congregate to become proficient with foundational concepts. Moreover, this collection of student perspectives offers evidence that components of the hybrid can be optimized for full effect on learning outcomes. For instance, grouping students whose learning styles mesh well together could lead to heightened discourse and problem solving skill development. For diverse STEM students, this setting is a venue in which to strengthen content knowledge while participating in the discourse of the STEM communities to which they strive to gain membership. Increasing diverse students' self-efficacy in this way can lead to similar future interactions and identification with peers and authorities in STEM fields; an important consequence, as engagement and self-identification with STEM professionals has been reported to lead to an increase in the likelihood of retention (Carlone & Johnson, 2007).

Questions: 1) How can I conduct a similar study on a larger class? What challenges come with a larger class, and what can I do to overcome them? 2) How do the students' perspectives mesh with your understanding of the benefits and challenges of the two pedagogies?

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DETERMINING THE STRUCTURE OF STUDENT STUDY GROUPS

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Although students have reported on their study habits outside of the classroom, there has been little verification of their actual behavior during group study sessions. This project observed undergraduate students studying together outside of the classroom setting in order to determine what study groups formed and what structures described these groups. Elements of social network analysis were employed to identify the groups that students formed. Transcripts of the study sessions were coded and frequency counts were established for each type of student interaction in order to characterize the roles students assumed while studying. This paper discusses the process of identifying the study groups and sets the groundwork for sharing the student roles. One main finding of this work is that the presence of a student recognized as an authority or facilitator of the group impacts the type of conversations that occur in the group setting.

Keywords: Group Work, Study Habits, Discourse Analysis, Social Network Analysis

Introduction

Faculty expect students to spend up to 3 times the number of hours spent in class studying outside the classroom (Wu, 1999). However, little is known about what it looks like when students fulfill this part of didactic contract. Most of the information we do have about student studying behavior is drawn from anonymous surveys and interviews (Thompson, 1976; Shepps & Shepps, 1971; Hong, Sas, & Sas, 2006). Such studies shed some light on what activities students engage in during their study time, but do not verify this self-reported data through observation of the students at work.

Even less is known about student behaviors when the students assemble into self-formed study groups. Although instructors often advise students to study together outside the classroom, they rarely offer guidance to students on how they should structure their study time together or how they should interact with each other while studying. Recently Pazos, Micari, and Light (2010) attempted to develop instruments for characterizing student interactions in a group. Their study, however, relies on the involvement of a peer leader to act as an authoritative director for the group which could impose an underlying structure that might not be present when students form their own study groups and meet outside of class time.

The study presented in this paper undertook the development of a method for observing students as they worked together in self-formed mathematics study groups outside the context of the classroom and endeavored to create a description of what it actually looks like when students are left to their own devices to prepare themselves for exams and homework assignments. The preliminary findings from the data collected in this study will be used to address the following research questions:

1) How do we determine which subsets of students form a study group?
2) What patterns in interactions and roles arise in student self-formed mathematics study groups when they work on homework assignments together?

Framework

Observing students outside of the classroom required an ethnographic approach to the study. This led to the accumulation of hours of video recordings of students studying in addition to accompanying field notes and student journal entries reporting on the session's goals and accomplishments. The task of analyzing such qualitative data in order to address the relevant research questions led to two main methods of analysis.

Social Network Analysis

Although one may develop an intuitive sense of which students are working together consistently through direct observation, there is enough variation in attendance at study sessions that a mathematically defendable partition is necessary. To this end, the researcher turned to the theories that have been developed for social network analysis.

Treating each study session as an "event," an affiliation matrix can be developed which records which students were present at, or "attended," each event. From there a co-membership matrix can be developed that identifies the number of events students have in common. Co-membership matrices are susceptible to one-mode analysis techniques such as defining a *clique at level c* where, in this context, "a clique at level *c* is a subgraph in which all pairs of actors share memberships in no fewer than *c* events" (Wasserman & Faust, 1994, p. 320).

However, considering a co-membership matrix alone fails to take into account which events were attended by the participants. Thus we also explore the two-mode method of correspondence analysis, which accounts for both the students and the events they attended simultaneously, in order to develop a second perspective of which subgroups of students represent study groups.

Student Interaction Analysis

Identifying the student study groups is only half the battle. A different analysis approach is needed in order to identify the structure of the groups and the roles assumed by students. To develop descriptions of student roles, we created a list of all student interactions, both physical and verbal, and assessed the frequency of certain types of interactions.

Goos, Galbraith, and Renshaw (2002) describe one list of codes they employed while studying student-student interactions in small group work in a high school classroom. Blanton, Stylianou, and David (2009) built upon the work of Goos, Galbraith, and Renshaw by adding new codes in their attempt to develop a framework for analyzing teacher and student utterances in classroom discourse. Both of these sets of codes were used as a base for analyzing the student interactions observed during this study.

Data Collection and Analysis

Participants were drawn from an undergraduate course that blended topics from linear algebra, differential equations and multidimensional calculus. The students in this course represented a variety of majors, from engineering to physics to pure mathematics. Students were chosen from this particular class due to the collaborative emphasis the instructor included as a classroom norm. None of the groups that occurred in the classroom, either inside or outside of class time, were the result of assignment by the instructor or the researcher. All groups were self-formed and self-directed.

In order to capture students studying outside of the classroom, a study space was set aside and equipped with tables, chairs, internet access, and white boards. Students from the class were invited to utilize the space whenever they wanted to study with the understanding that their actions would be video recorded and recorded in field notes by the researcher. Students were also invited to complete journal entries reflecting on their activities during the study session and to participate in up to 3 follow up interviews.

At the end of each study session, students were asked to record both what they worked on and with whom they worked in their journal entries. From these entries, study sessions that focused on reviewing for an exam were separated from study sessions that focused on homework preparation. These attendance charts were then turned into affiliation matrices, co-membership matrices, and event overlap matrices in preparation for social network analysis techniques.

All study sessions and interviews were transcribed and coded using a combination of Goos, Galbraith, and Renshaw's (2002) coding scheme and Blanton, Stylianou, and David's (2009) coding scheme. While the combination of these two sets of codes encompasses most of the interactions that could be expected, the researcher also employed an open coding scheme in order to add codes as needed should an utterance defy categorization in either of the two schemes. After coding, the researcher determined the frequency of each code per student to establish profiles of what each student's presence contributed to the group and look for patterns in profiles across study groups.

Preliminary Results and Significance

Although the analysis phase is still underway at this time, several observations have been made so far and more are expected to be available soon.

Identified Student Study Groups

The first look at student connections generated from the co-membership matrix yielded 2 easily distinguished study groups along with a solitary individual seen in Figure 1. Using the one-mode analysis of finding cliques at level 3, or imposing the condition that all students in a clique must have at least 3 events in common, generates the colorized graph presented on the right in Figure 1.

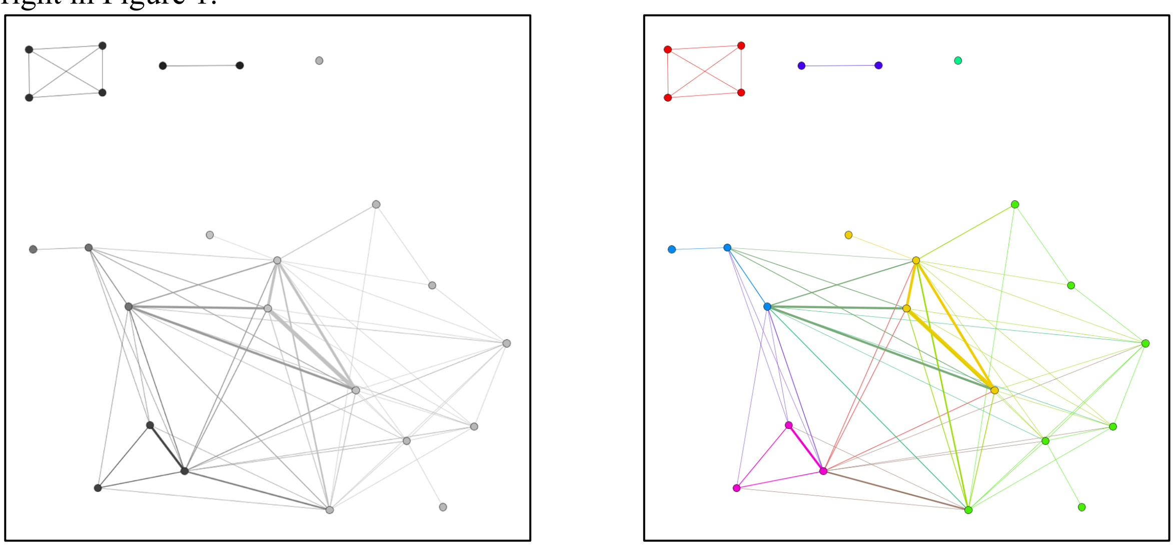

Figure 1. In greyscale, the unaltered graph generated by the homework study session co-membership matrix. On the right, the same graph colorized to show the cliques at level 3.

By requiring students to attend at least 3 events together we cut down on some connections in the largest group that were formed by students who may have dropped in to use the study space and discovered other individuals already present. From Figure 1 it is clear that the large mass of connections can be broken down further into at least 3 distinguishable subgroups depicted in the graph by yellow connections, pink connections, and pale green. Graphical representations of the two-mode analysis are not yet available, but are expected to be ready soon.

Patterns of Interactions and Student Roles

Although the group characterization instrument developed by Pazos, Micari, and Light (2010) is reliant on the presence of an assigned peer facilitator that has already passed the course, it has much that is in agreement with the data collected in this study. The data suggests that there are several different types of conversations that occur within groups that are dependent upon the presence of a facilitator-type individual, or an individual treated as an authority by the group.

When a student has assumed the role of facilitator for the group or has been designated as an authority by his or her peers, conversations mimic 3 out of the 4 group classifications offered by Pazos, Micari, and Light: *simple instruction* where the facilitator essentially lectures her peers on how to approach a problem, *elaborated instruction* where the facilitator incorporates more conceptual foundations to explain a solution strategy, and *supported discussion* in which the group strategizes a solution but defers to the knowledge of the designated authority among them (2010, p. 194). In the absence of a facilitator, the conversation is often more undirected and includes: *comparisons* of progress made toward a problem's solution either through strategy checks or answer checks, *stagnation* as students with similar understandings can't move beyond a particular issue, and *digressions* that improve social ties among the group but contribute little toward problem solving.

Questions for Discussion

While there are many questions that could be asked of the research at this point in time, I would like to focus on the following:

1) What information about the structure of the study group could be revealed by exploring the subgroup interactions that occur within the group?
2) What other ways are there to create characterization of the roles that may arise in study groups (beyond analyzing frequency of interaction types)?

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GESTURES: A WINDOW TO MENTAL MODEL CREATION

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Keywords: Classroom Discourse, Instructional activities and practices, Modeling, Problem Solving

Gestures are profoundly integrated into communication. This study focuses on the impact that gestures have in a mathematical setting, specifically in an undergraduate calculus workshop. We identify two types of gesture – dynamic and static – and note a strong correlation between these movements and diagrams produced. Gesture is a primary means for students to communicate their ideas to each other, giving them a quick way to share thoughts of relative motion, relationships, size, shape, and other characteristics of the problem. Dynamic and static gestures are part of the students' thinking, affecting how they view the problem, sway group thinking, and the construction of their diagrams.

Background

There have been numerous studies conducted on gestures and their presence in the educational environment especially in math and science related fields (Rasmussen, Stephan, & Allen, 2004; Chu & Kita, 2011; Goldin-Meadow, Cook, & Mitchell, 2009; Scherr, 2008). For example, Rasmussen, Stephan, & Allen (2004), studied gesturing in a differential equations class where they observed mathematical classroom practices become what they call taken as shared (TAS) ideas for the participants.

Johnson-Laird (1983) extensively discussed three types of mental representation of which two we considered "mental models which are structural analogues of the world, and images which are the perceptual correlates of models from a particular point of view." (p. 165) Studies conducted by Engelke (2004, 2007), found that many students fail to understand problems such as related rates because there is a lack of transformational/covariational reasoning, leading to incomplete mental models. It has been theorized that in order to understand word problems, students benefit from drawing a diagram and trying to understand the relationships their diagrams represent of the given situation (Engelke, 2007). Visual and analytical skills are essential for students to understand mathematical concepts and construct mental models (Haciomeroglu, E.S., Aspinwall, A., & Presmeg, N., 2010). Gestures are a form of visual-spatial representation and we investigate how such representations facilitate the problem solving process. We seek to answer the question: How do students' gestures facilitate contextual problem solving in calculus? Through observing students' use of gestures while solving related rates and optimization problems, we will better understand the mental models and diagrams being created during the problem solving process.

Methods

In this study, we used open and axial open coding to observe three different supplemental instruction (SI) workshops, which consisted of undergraduate students taking first semester calculus. Students were solving related rates problems chosen by the SI leader. We watched the videos multiple times specifically attending to students making hand gestures and the diagrams they drew during the problem solving process. Pseudonyms were given to the participants for privacy purposes.

Results

We identify diagramming as a visual tool, including drawing a picture, which is used during the problem solving process and is intrinsically linked to the construction of the mental model. The definition of gesture varies in the literature. For instance, Roth (2001) defined gestures as hand movements made with a specific form where "the hand(s) begin at rest, moves away from the position to create a movement, and then returns to rest." Although there are many definitions present today, we adopted Roth's definition.

There are two types of gestures we identified: dynamic and static gestures. Dynamic gestures consist of moving the hands to describe the action that occurs in the problem or movements made to represent mathematical concepts. Within dynamic gestures there are two subcategories: dynamic gestures related to the problem (DRP) and gestures that are not related to the problem (DNRP). Static gestures are done to illustrate a fixed value (length, constant radius, etc.) or to illustrate a geometric object. Static gestures consist of static gestures related to a fixed value (SRF) and gestures related to the shape of an object (SRS).

Dynamic gestures

We define DRP as hand gestures that consist of movements describing parts of the students' diagrams, whether it is the motion of an object or changing rates/values. DRP is further broken down into two subcategories. The first subcategory identifies hand gestures used to answer/clarify a concept/question to a classmate. A student may use a hand gesture to reason the problem out. For instance, a group of students, Jackie, Josh, and Cathy, were trying to solve the following related rates problem (the boat problem):

> A boat is pulled onto a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8m from the dock? (Stewart, 2009, p.132)

Although the problem in the textbook has an image depicting the situation, students were not provided with the image. They had to construct the diagram and solve the problem.

Josh: You know what a pulley is right?
Jackie & Cathy: No
Josh: It's a little thingy [HG*: raises his right hand in the air, rotates his right index finger inward, then raises left hand to the same height as his right, puts both hands together, and pulls downward*] you pull…so it's gotta be on top.
Jackie & Cathy: Ohh….
Jackie: So it's going to be like that?
Josh: So… [*Drawing diagram*]
Cathy: I did it like that [*laughs a little*]
Jackie: See, how do they expect us to not know what a bow is
Cathy: Yeah….
Josh: You don't know what a boat is?
Jackie: I know what a boat is…
Cathy: But not a bow…or whatever

Although his peers, who did not know what a pulley was, prompted Josh's gesture, the gesture revealed his mental representation of the problem. After he makes the gesture, he begins drawing his diagram and concludes that the pulley must be on top. The gesture influenced Josh's perception of the problem (i.e. placement of the pulley), which also had a role in how he labeled his diagram. After this exchange, Jackie constructs an appropriate mental model of the situation as is evidenced by her subsequent exchange with Cathy.

The second subcategory deals with hand gestures done in order to understand and reason about the problem. For instance, Jackie is trying to solve the boat problem by first attempting to understand the dynamic element of the problem.

Jackie: [HG_1*: Jackie moves her hands in a circular inward motion (Figure 1)*] Is the…when its being pulled, it's being pulled from there?

Cathy: Yeah

Jackie: [HG_2*: Jackie points her left index finger towards Cathy's diagram (Figure 2)*] so then it's approaching on *x*, so okay we are looking for *dx/dt*, but then we're looking for this rate…

Jackie's hand gesture is prompted because she is trying to understand the problem both conceptually and physically. With the second hand gesture, she references Cathy's diagram (*Figure 2*) to understand what is happening to the boat. Through the first and second hand gestures, she sees that as the rope is pulled, the boat moves closer to the dock. She recognizes that the change is occurring on *x*, and hence, she relates the change in *x* with *dx/dt*. The gesture made here, appropriated in part from Josh's earlier gesture, allows her to conceptualize the diagram described in the problem. The gesture acts alongside Jackie's constructive thinking as she tries to comprehend the situation at hand.

Figure 1: Jackie's gesture for boat

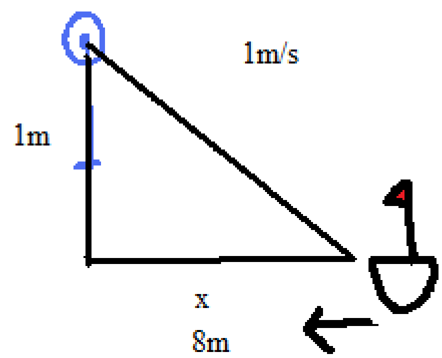

Figure 2: Cathy's drawn diagram for problem

For DNRP, students and/or SI leaders use hand gestures to refer to mathematical concepts. The SI leader assisted Susan, Brian, and Cesar with the trough problem which states:

> A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep? (Stewart, 2009, p.132)

SI Leader: Now, what do you do to find what… cause you're trying to find *dh/dt* [HG*: He first moves his right hand right to left, with his fingers curved in the shape of a c, and then he changes the position of his hand and moves it up and down*] right?

Susan: Yeah. So we have to take the derivative of each side.

SI Leader: and that is when you plug in what your paused… Do you see it Cesar?

When the SI leader makes the hand gesture, his motion depicts the fractional aspect of the derivative (i.e. *dh* over *dt*). Additionally, Susan seems to associate finding *dh/dt* with implicit differentiation because she immediately thinks about taking the derivative of both sides when the SI leader mentions *dh/dt*. In another clip, Jackie explains to Cathy the difference between taking the derivative in a related rates problem, and taking the derivative of a (more common) function with respect to *x*.

Cathy: Jackie I have a question (laughs). Remember how last time you said we always keep *dy/dt*

Jackie: Mhmm.

Cathy: Do we also keep *dx/dt* cause it's not the same?

Jackie: Yeah…no, because we're not solving for uhmm… [HG*: lifts right hand up then moves it downward in a diagonal manner (Figure 4)*] it's no longer like a just taking the straight out derivative of it, cause we have different properties that we need to relate together.

Along with her hand gesture and her explanation to Cathy, it can be seen that Jackie is able to discriminate between the derivatives in the context of a related rates problem and generally taking the derivative of some function of *x*. Although she does not explicitly state that the difference is based on taking the derivatives with respect to time, that underlying concept somehow triggered Jackie to separate the two. The gesture, in this case, did not act as part of Jackie's constructive thinking about the problem; rather the gesture was used as an explanatory aid.

Static gestures

Apart from dynamic gestures, we identified static gestures, which consist primarily of gestures that illustrate a geometric object or refer to a fixed value (length of one the sides, constant radius, etc).We will distinguish them as static related to the shape of an object (SRS) and static related to a fixed value (SRF). SRS is focused on students or SI leaders utilizing their hands to depict the geometric shape of an object; we also consider referencing the general diagram as SRS. Here we see Susan trying to process the trough problem. *Figure 3* shows Susan's initial image of the trough problem. She traces with her fingers the outline of a tip up standard equilateral triangle. As seen in *Figure 4*, however, Susan changes the orientation of the triangle to a downward position as she realistically reasons out the scenario. Although not much is said, Susan's facial expression and gestures illustrate her thought process of trying to understand the general shape of the trough.

Susan: [HG*: elbows are bent and on top of desk, wrists are touching, hands are open diagonally, and pointing in opposite directions, flickers her left hand as she moves her pencil between her fingers, puts hands together, then pulls them apart in a diagonal direction (Figure 3)*] so it's not the sides…it's the width [*SI Leader interjects and provides insight on the problem*]

Her first gesture illustrates a cross-section of the ends of the trough; as she continues to think, she forms the length of the trough by pulling apart her two hands. Susan, along with many students, struggled to understand the geometric aspects of related rates problems. Author indicated that students tend to adopt a procedural way of thinking when they approach related rates problems. This may be why students have a hard time understanding problems such as the trough problem, which deal with a three-dimensional object, as well as applying the concept of similar triangles. For instance in the trough problem, Susan automatically drew an upright equilateral triangle *(Figure 4)* as opposed to visualizing the triangle upside down or oriented in a different way.

Figures 3 & 4: Susan gestures to describe the trough

Students with geometric misconceptions tend to construct incorrect diagrams, leading them to the wrong solution. If the SI leader had not intervened, the students would have attempted to solve the problem with an incorrect diagram, hence leading them to the wrong answer. That said, we consider SI leader intervention necessary at times to provide students with guidance on challenging problems. The SI leader's help can start the students on the path to solving the problem correctly, without just giving them the answer. However, sometimes intervention by SI leaders, as also revealed in Scherr (2008), may actually interrupt a student's thinking.

For SRF we take into consideration the students' gestures done with respect to a fixed value, such as length or constant radius. These gestures are usually associated with students' diagrams. As mentioned above, SRF deals mostly with students referencing a fixed value. In this clip, Brad and Mark work on a related rates problem that deals with the distance between two cars moving in different directions.

Brad: [*diagramming*] Specific, its saying, one is traveling south at 60 miles per hour

Mark: [*mumbles something*]

Brad: I don't know. So you times it, so 2 times 60 [*labels*], cause the two hours, [HG: *moves his left hand top to bottom*] that's the length, and 25 is the top.

Brad's gesture was done in order to describe a fixed value, in this case, length. Although his gesture was quick, it was done in an effort to explain to Mark the values corresponding to their specific diagram. This type of gesture shows how students associate given values to their diagrams.

Conclusion

A strong relationship between diagramming and the two types of gestures identified in the study is evident. Static and dynamic gestures were often used in regards to the students' diagrams, but static gestures seem to have a stronger relationship to diagramming as they deal with the diagram itself. In order for the student to even attempt solving the problem he or she began by drawing a diagram. It appears that when the students were stuck with their diagram or parts of their diagram, they gestured while trying to reason out the part about which they were confused. Several students gestured because they were trying to obtain a better understanding of how the diagram corresponded to geometric terms in the given problem. The more challenging the problems were, the more the students gestured. Some gestures were influenced by prior gestures and students quickly adopted and adapted gestures made by their peers. For instance, in the boat problem with Jackie, Josh, and Cathy, before Jackie made the gesture to describe how the boat was being pulled by the pulley, Josh's gesture, which described the pulley, had to occur first. Other times gestures arose because there was a lack of knowledge that was needed to even begin the problem. An example would be the trough problem that Susan's group was assigned. Her gestures were caused partly because neither she nor any of her peers knew what a trough was. In her mind she pictured an object with a square bottom and triangular sides. Since she did not know what a trough was, most of her gestures were done to figure out what the trough looked like. Most gestures have one thing in common; they were made to solve the problem by first understanding the problem abstractly.

Little research has been done on the impact gesturing has on undergraduate mathematics students; one must wonder whether or not the gesturing that occurs is beneficial to students. There is no denying that gesturing does influence the way students approach a problem, but to what extent? Gestures are used to communicate ideas from one student to another, but do the gestures influence how the speaker is thinking about the problem? Can physically mimicking a problem space give greater understanding? This study provides some hints to the power of gesture, but more work is needed in this area.

Acknowledgements

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Student Difficulties Setting up Statistics Simulations in Tinkerplots™

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Introduction/Background

The past 20 years has produced a growing body of research on instructional technologies in mathematics classrooms, as well as a small, but growing, body of work on the impact of technology in introductory statistics classrooms (e.g., Chance, Ben-Zvi, Garfield, & Medina, 2007; Chance, delMas & Garfield, 2004). However, much of the work touting the benefits of using technology in the statistics classroom has not been empirically based (see Mills, 2002) and Shaughnessy (2007) has argued that there is a lack of research investigating the use of technology in statistics classrooms. Specifically, research is needed that explores how technology changes the way students think about statistics and the ways technology can be used to enable students to construct models to solve statistical problems. There are two fundamental reasons why such research is badly needed: (1) statistical literacy and computer literacy are vital skills in an information age; and (2) new dynamic educational technologies carry much promise for supporting student learning, but without knowledge of how these technologies impact student thinking they fall short of their potential. This proposal focuses on student challenges interpreting a single trial of a statistical experiment.

Methodology

Data was collected in an introductory statistics course at a large urban university in the the United States. The first author was the classroom assistant who helped with classroom activities and data collection during the quarter. The second author was the classroom instructor. This particular statistics course was designed for students prior to entering the traditional introductory statistics sequence (descriptive statistics, probability, inferential statistics). Students enrolled in this course as a prerequisite for the traditional sequence or to satisfy the required math elective needed to graduate. A total of 16 students enrolled in the course and all students consented to be participants in the study.

The second author implemented the CATALST curriculum materials (Garfield, delMas, & Zieffler, accepted) with some minor modifications. The CATALST curriculum consists of three units, and each unit begins with a model eliciting activity (MEA, see Lesh et al., 2000). Following each MEA, there are several activities in each unit that guide students through key ideas raised in the MEA (e.g., informal inference based on a single population, p-value). Data collection consisted of all student work on in-class activities, task-based semi-structured interviews, and student assessment items. This proposal focuses on one activity that will be used to illustrate the nature of student reasoning as they learned to construct a statistical model with the dynamical statistical software, TinkerPlots™.

The research team had students create simulation models to answer informal statistical questions with the modified CATALST curriculum. One such task involved the One Son Policy, a situation where families continue to have children until they have a son with the assumption that it is equally likely to have a boy or a girl. This task was the first time in the course where students needed to find a way to stop drawing from the sampler once the desired result was reached. Previous tasks only required students to choose the appropriate number of repeats for a particular trial. For example, students might be asked to model flipping a coin five times as a single trial and would flip (draw with replacement) one coin with a repeat of 5 for a single trial. We believe that the One Son activity created a cognitive hurdle because the students encountered a task with the necessity for the sampler to stop after a prescribed result was reached.

A correct way to model the One Son task using a spinner is shown in Figure 1. Notice the spinner shows two equal parts representing boys and girls, under the assumption each sex is equally likely, and the "Repeat" is set to "Repeat until pattern matched" is BOY. A single trial is completed after a boy is drawn and the variable of interest is the number of "spins" (births) until a boy is selected. Multiple trials can be run to investigate trends, such as the expected number of children under such a birth policy. The correct interpretation of a single trial would be the number of children a particular family has (i.e., the number of spins until a boy is produced).

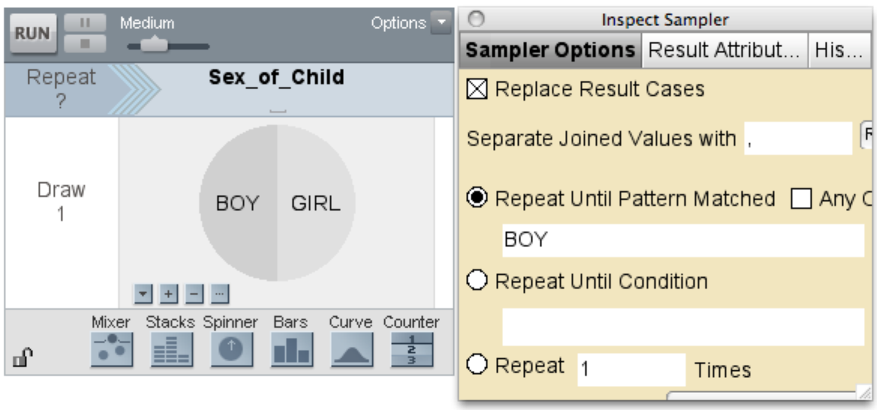

Figure 1. Tinkerplots™ One Son Policy Simulation Model

The instructors modified the CATALST curriculum by removing the instructions regarding how to setup the simulation in Tinkerplots™. Specifically, they removed instructions that detailed how to make the simulation stop after a boy was drawn from the sampler. The classroom instructor and assistant first had students brainstorm in groups how they might set up a model for the One Son Policy without the use of technology and then to try using the technology to build simulations from their mental models. After students investigated the technology they began to raise the question of how to "tell" the computer when to stop the simulation, at that point we gave them the Tinkerplots™ instructions. The next section details the results of student thinking during this investigation.

Initial Results

All 16 students created their simulation using a spinner[1], but had different ways of setting up their models, interpreting a trial and making an inference about the average number of children born under such a policy. Table 1 outlines the different student interpretations of a single trial in the One Son activity.

Characterizations of Student Interpretations of a Single Trial	**Evidence**	**Students (Pseudonyms)**	
A trial is the number of children a family has *until* a boy is born	The verbal model was expresseed to be the number of children that were born in a single family up to and including the son, so the sampler would need to stop	Arnold George Danika Kate	Bill Becky Sally Helen

[1] The spinner in TinkerPlots is automatically set at with replacement.

	drawing once the son was born	
A trial is the number of children a family has *before* a boy is born	The students collected counts on the number of girls born resulting in the average number of children to exactly one less than expected.	Ellen Starla Nils Andrea
A trial is each spin of a spinner (but only when the spin lands on a boy)	Knew when a trial would end, but did not account for the total number of children that would be born before a boy	Carolyn Kimberly
A trial is looking at the sex of the first born child in ten families	Used a "draw" value of one to represent the sex of the first child and a "repeat" of 10 to represent the ten families	Jennifer Jill

Jennifer and Jill reported the most idiosyncratic interpretation of a single trial in the One Son activity. These students worked together and reasoned that a single trial was identifying the sex of the firstborn child in ten families. Their samplers were set up with a "draw" value of one to represent the sex of the child with a "repeat" value of ten to represent the ten families in the trial. With the sampler set up in this manner, there was no way to answer the research question because they struggled with the definition of a single trial, making it impossible to simulate using technology or physical simulation.

After students spent time discussing possible models in small groups, the class discussed various ideas and converged on an appropriate model to carry out using Tinkerplots™. The strategy of choosing a draw value instead of having the sampler "repeat until pattern match" was not uncommon at this point in the activity. Helen, Carolyn, Andrea and Kimberly initially used the approach of choosing the number of "repeats", although for a different reason than Jennifer and Jill did. These students chose an arbitrary "repeat" value that they deemed sufficient enough to produce a boy. Andrea argued that a "draw" value of 5 was an adequate "draw" value to have a boy because the assumption was that it was equally likely to have a boy or girl. This approach could not be used successfully without more advanced mathematical logic statements necessary to count the number of children up to and including the point a boy is born and disregard any children after. While the students' intuition about the likelihood of the event of having a boy was correct, it was not sufficient as a method when incorporating Tinkerplots™ as an effective tool to help them successfully answer the statistics research question.

While some students struggled with the correct interpretation of a trial, eight of sixteen students identified a single trial and were able to correctly set up their model in Tinkerplots™. These students accurately collected statistics on the total count of children born in a family as shown in Figure 2. Figure 2 describes a single trial of three females born before a son, where statistics will be collected on the total count of Boys and Girls that are born.

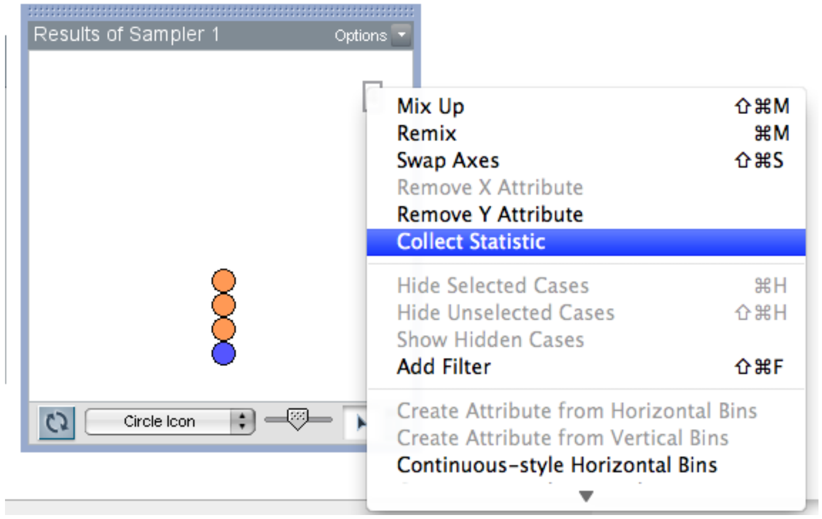

Figure 2: Students collecting statistics on the total count of children born in a single family.

Four of sixteen students also set up their model correctly in Tinkerplots™, but incorrectly considered the result of a single trial to be the number of females born. Their interpretation of a single trial was the number of children *before* a boy is born. This interpretation led these four students to have an average number of children per family to be exactly one less than expected because the count of the son was never included over many trials. These students found a solution to this issue without instructor assistance by simply adding 1 to their average children per family after many trials. However, this group never resolved the issue of how they collected statistics and were not able to prove that adding 1 to their average over many trials was the same as collecting statistics on the total number of children born.

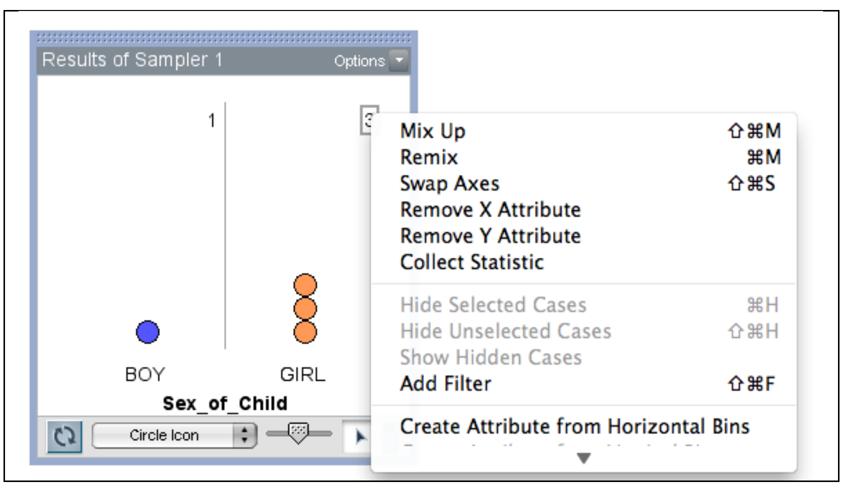

Figure 3: Students collecting statistics on the total count of females born in a single family.

Conclusions and Directions for Future Research

Students' interpretations of a single trial in the One Son activity fell into four different categories, three of which were incorrect or not useful in addressing the central question of the task. This suggests that the process of setting up a computer simulation to answer a statistical question is quite complex. Further, although the majority of students were able to correctly conceptualize the relationship between a single trial and the task situation, they still had difficulty using the technology to set up and interpret a simulation to address the question.

Future research should focus on identifying the source of the students' difficulties. In particular, it is unclear whether the students struggled primarily with understanding the statistics task, using the technology to conduct a simulation, or with the challenges of coordinating these. For example, future research could be designed to understand why students struggle to correctly use logical structures (e.g., If – Then) in order to set up a statistics simulation. Understanding more precisely the source of students' difficulties will support the design of more effective instructional approaches.

Questions for the Audience

- What kind of research design could help us tease out precisely where students' difficulties lie?
- Are there existing frameworks regarding use of technology in problem solving that could help us make sense of students' difficulties as they set up statistical simulations?

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MATHEMATICIANS' TOOL USE IN PROOF CONSTRUCTION

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This study sought to describe the tools and reasoning techniques used by mathematicians to construct and write proofs. Task-based clinical interviews were conducted with 3 research mathematicians in varying research fields. The tasks were upper-undergraduate and lower-graduate level proofs from linear algebra, basic analysis, and abstract algebra. Data were coded based on a framework constructed from Dewey's theory of inquiry and the characterizations of conceptual insight and technical handle. Preliminary results indicate the task of discovering a conceptual insight that can potentially lead to a proof can be problematic, and there are distinct moments in the construction process when the problem changes from "why should this be true?" to "how can I prove that?"

Key Words: Proof, Inquiry, Experts

Harel and Sowder (2007) called for a comprehensive perspective on proof; they stated that a central goal of teaching proof is to "gradually help students develop an understanding of proof that is consistent with that shared and practiced by the mathematicians of today. (p. 47)" Consequently, it is necessary to develop an accurate and comprehensive understanding of mathematicians' proof construction processes and the reasoning techniques by which they construct proof. This paper reports preliminary results from research intended to describe expert mathematicians' inquiry into proof construction. We specifically analyze experts' inquiry in regards to developing and implementing a conceptual insight and technical handle (Raman, Sandefur, Birky, & Somers, 2009).

Background/Theoretical Perspective

We apply Dewey's theory of inquiry and tool-use (Dewey, 1938; Hickman, 1990) to analyze mathematicians' proof construction process. Dewey defines inquiry as the intentional process to resolve doubtful situations, through the systematic invention, development, and deployment of tools (Hickman, 2011). A tool is a theory, proposal, or knowledge chosen to be applied to a problematic situation. Throughout the entire inquiry process, there is an "end-in-view" (Garrison, 2009; Glassman, 2001; Hickman, 2009). These ends-in-view provide tentative consequences which the inquirer must seek the means (tools and ways to apply tools) to attain. These ends-in-view may be modified and adapted as the inquiry process proceeds.

The process of active, productive inquiry involves reflection, action, and evaluation. Reflection is indeed the dominant trait. The inquirer must inspect the situation, choose a tool to apply to the situation, and think through a course of action. After this initial reflection of what could happen, the inquirer performs an action, applies the tool. In these actions, the inquirer operates in some way on the situation; she applies a tool to the situation, thus altering it. Reciprocally, during or after the fulfilling experience, the inquirer evaluates the effects and appropriateness of the application of the chosen tools (Hickman, 1990).

We attempt to focus on understanding the mechanisms that lead to new insights. Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009; Raman & Weber, 2006) have developed a model for describing student difficulties for proof production, including the moments of finding a conceptual insight (sometimes termed key idea) and a technical handle. Attaining a conceptual insight gives the prover a sense of conviction and why a particular claim is true. A technical handle is an idea that renders the proof communicable; discovering a technical handle gives the prover a sense of "now I can prove it" (Raman, et al. 2009). These constructs characterize moments when the prover creates a new insight, an instance of the invention and deployment of a tool. We believe that focusing on these specific moments distinguishes this work from others' research which focuses on rich, broad descriptions of the process of problem solving (i.e. Carlson & Bloom, 2005).

This report details preliminary findings for research into the following questions: (1) What tools and reasoning techniques are used by mathematicians in search of a conceptual insight and technical handles? (2) How do professional mathematicians use conceptual insights and technical handles as tools in constructing and writing proof?

Methods

The participants of the study consisted of 3 professional mathematicians. Each participant engaged in a task-based interview that included three proof construction tasks and follow-up questions. The tasks were upper-undergraduate and lower-graduate level proofs from linear algebra, basic analysis, and abstract algebra. Participants were video recorded, their work was recorded using a LiveScribe pen, observation notes were taken by the pair of interviewers, and interviews were transcribed. We developed a coding scheme based on Dewey's theory of inquiry (Dewey, 1938; Hickman, 1990) and Raman and colleagues' characterization of conceptual insight and technical handles (Raman, et al., 2009; Raman & Weber, 2006). In applying the coding scheme, we parsed transcripts into "major events," or individual actions or groups of actions involving one purpose or one problem. We coded each major event by type of experience, problem, tool-used, purpose of tools-used, type of evaluation and mode of thinking. We described problems and tools in context for clarity. We then further subdivided major events if we determined that more than one purpose or more than one problem occurred in its duration. Coders added additional codes for problems, tools, and purpose of tools as needed.

Results

From our coding, it was apparent that the participants began each problem first attempting to manipulate the premises they were given in order to get a sense of the mathematics they were engaging. Then, participants generally began applying tools with the purpose of looking for conceptual insight, or a sense of belief and insight into the reason why the statement is true.

After reading the analysis task, *[Let f be a continuous function defined on I=[a,b], f maps I onto I, f is one-to-one, and f is its own inverse. Show that except for one possibility, f must be monotonically decreasing on I.]*, Dr. K stated, "What I'm puzzled by is why it has to be decreasing." We interpret this as articulation of a problem of not seeing a conceptual insight. Dr. K applied the tool of turning "it into a geometry problem" by drawing a picture. He based his diagram on his conceptual knowledge of what it would mean for a continuous function to be one-to-one and onto: "it can't go up and down" and for a function to be its own inverse: "it has to be symmetric when I flip it over the line." He deemed his picture as fitting both the hypothesis and the conclusion, but he still wanted "to think about why that's true." He then drew the identity function and noted that it was the one exception. Finally he drew another picture including the line $y = x$ and a point located above the line $y = x$. He applied his previously developed knowledge of the unique exception and his conception of the graph needing to be symmetric about the line $y = x$, to reason "I have to have the geometric reflection of that point on my graph... and that forces it to be decreasing, because when I flip a point... well, if I flip a point above that *y* equals *x* line across that line, it moves to the right and down. And so there's a geometric argument that it has to be decreasing”. After this geometric reasoning, he asserted, “I think I’m done.” His inquiry then shifted into articulating an argument.

The linear algebra task asked participants to show that two similar 3x3 matrices would have the same characteristic and minimal polynomials. Dr. H set about the problem of determining why the statement should be true by sequentially proposing and assessing the following potential tools:1) if he had a theorem for determinants, 2) if he could argue that the polynomials are a property of the transformations in the change of bases, or 3) if he could “do something about row reduction preservation”. He did not deem any of the tools as being immediately helpful. He transitioned into working a numeric example where he generated two similar 2x2 matrices and went about computing their characteristic polynomials to see why they would be the same. Due to a computation error, his polynomials appeared different. Dr. H chose not to inquire into where the error occurred because “even without looking for my error, I don’t think that that’s promising to do it from the definition.” He then reflected back on the polynomials, recognized the roots would be eigenvalues, then decided he could construct a proof by reasoning about the eigenvalues. Dr. H then constructed an argument that required the eigenvalues to be distinct in order for his proof to work and claimed that he did not care enough to worry about the case they were not.

Once participants acquired conceptual insight into the problem, they switched to applying tools with the purpose of looking for technical handle, or a way to communicate the proof (Raman, et al., 2009; Raman & Weber, 2006). For example:

> Dr. N: Ok so, on the other hand if we start here and if we do something like that, can we make it be its own inverse? It just has to be symmetrical about that point. Ok, now at least I believe the statement... Ok so it must be monotonically decreasing, so now what could I do to give a proof of that? Well, I could try to... just kind of do a straightforward thing, say let *c* be less than *d*, and I want to show that, see decreasing... that *f* of *c* is greater than *f* of *d*.

For two of the participants, Dr. H and Dr. K, converting a pictorial or numeric argument to an analytic argument was not deemed necessary. Dr. H claimed he would not have continued with a proof if we had not asked for such; Dr. K claimed he was finished and did not need to write any more for a proof.

Discussion

In the two episodes where the participants searched for an insight into why the statement is true, we observed examples of the periods of “reflection, action, and evaluation” that characterize Dewey’s theory of inquiry (Dewey, 1938; Garrison, 2009). The participants reflected on the problem, reflected on tools that they may choose, applied a chosen tool, and reflected on the effectiveness of the tools and how their implementation changed the problem. We hypothesize participants’ lack of interest in producing an analytic argument may be a consequence of the situation of an interview or the mathematicians’ attitudes of where the actual mathematics happens in constructing proof.

Questions for Discussion:

(1) What distinctions in tool use might be important in differentiating between looking for any conceptual insight and looking for an insight into the reasons why the statement is true that can lead to communicable argument?

(2) How might we triangulate our hypotheses about mathematicians’ goals during proof construction in further studies?

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Identifying Change in Secondary Mathematics Teachers' Pedagogical Content Knowledge

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Like several other research groups, we have been investigating multiple measures for capturing change in middle and high school teachers' mathematical pedagogical content knowledge (PCK). This article reports on results among 14 teachers (of 16 enrolled) who have completed a distance-delivered master's program in mathematics education. The degree program seeks to develop content proficiency, cultural competence, and pedagogical expertise for teaching mathematics. Analysis included pre- and post-program data from classroom observations and written PCK assessment. Results indicate significant changes in curricular content knowledge on the observation instrument and significant changes in discourse knowledge on both the observation instrument and the written assessment. Path analyses suggest teacher discourse knowledge as measured by the written assessments is significantly related to discourse knowledge as measured by the post-program observation.

Key Words: Pedagogical content knowledge, in-service teachers, professional development

Background

In response to the call for advanced professional education accessible to in-service teachers, the Mathematics Teacher Leadership Center (Math TLC), an NSF-funded Mathematics and Science Partnership project, has developed and is researching a distance-delivered master's program in mathematics education. The primary goals of the program are to develop content proficiency, cultural competence, and pedagogical expertise for the teaching of secondary mathematics (grades 6 to 12). To document the development of mathematics teaching expertise, project research investigates the pedagogical content knowledge of participants before and after the master's degree program. This report is the first to include pre- and post-program data for the first cohort of graduates.

Pedagogical content knowledge (PCK) is a construct described by Schulman (1986) and subsequently refined by others. It encompasses the unique collection of discipline-connected knowledge needed for teaching. As PCK has become widely utilized in research on early grades (K-8) teacher development, a model based on "mathematical knowledge for teaching (MKT)" has emerged (Ball, Hill, & Bass, 2005). Many challenges in measuring PCK have been reported (Hill, Ball, & Schilling, 2008) and most framing of MKT includes some algebra and little in the way of proof-based understandings, such as are found in college mathematics. For the purposes of this research, we use an expanded model of PCK, based on the work of Ball and colleagues, which includes algebra and proof-based advanced mathematics. Working from the foundational three components proposed by Ball et al., the model adds a fourth node of knowledge needed for teaching, discourse knowledge (this aspect brings to the modeling of PCK the mathematical semiotics that was part of Shulman's original description). One way of visualizing the model is as a tetrahedron whose base is the MKT model with apex of discourse knowledge (see Figure 1). Our attention has focused on discourse knowledge and the three "edges" connecting it to the components in the MKT model (Hauk, Jackson, & Noblet, 2010). ***Discourse knowledge*** (**DK**) is knowledge about the culturally embedded nature of inquiry and forms of communication in mathematics (both in and out of educational settings). This collection of ways of knowing includes syntactic knowledge, “knowledge of how to conduct inquiry in the discipline” (Grossman, Wilson, & Shulman, 1989, p. 29). ***Curricular content knowledge*** (**CCK**) is substantive knowledge about topics, procedures, and concepts along with a comprehension of the relationships among them and conventions for reading, writing, and speaking them in school curricula. In its most robust form, this part of PCK contributes to what Ma (1999) called “profound understanding of mathematics” (p. 120). ***Anticipatory knowledge*** (**AK**) is an awareness of, and responsiveness to, the diverse ways in which learners may engage with content, processes, and concepts. Part of anticipatory knowledge growth involves “decentering” – building skill in shifting from an ego-centric to an ego-relative view for seeing and communicating about a mathematical idea or way of thinking from the perspective of another (e.g., eliciting, noticing, and responding to student thinking). ***Implementation knowledge*** (**IK**) is about how to enact in the classroom the decisions informed by knowledge of content and teaching along with discourse understandings. This includes adaptive, in-the-moment, shifting according to curricular and socio-cultural contexts.

This paper describes our efforts to gather evidence of PCK using this four-part framework. We report here on our progress to date in addressing the following research questions:

(1) Does teacher-participant PCK differ pre- to post-program as measured by
  (a) an observation instrument?
  (b) a written assessment?

(2) What is the nature of the relationship between PCK as demonstrated reflectively on the written assessment and in practice on the observation instrument?

While acknowledging the limitations of this non-experimental, small-*n* study, it is valuable in building a foundation for larger scale work in the future. Note that the intention is not to make causal claims. Rather, we are in the early work of testing predictive validity for instruments and exploring potential avenues for capturing PCK and documenting change in it.

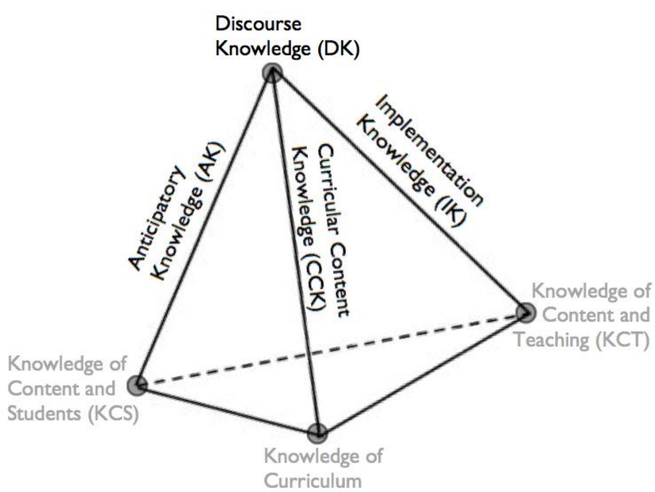

Figure 1. Tetrahedron to visualize relationship among PCK model components. Corners of the base are the aspects of PCK articulated in Hill, Ball, and Schilling (2008).

Methods

Setting: The setting was a blended face-to-face and online delivered master’s degree program in mathematics for in-service secondary teachers. Designed to reach urban, suburban, and isolated teachers in rural areas, the program is conducted using a variety of technologies (e.g., *Collaborate* for synchronous class meetings, *Edmodo* for asynchronous communication). Offered through a joint effort at two Rocky Mountain region universities, cohorts of 16 to 20 new students each year complete a 2-year master's program in mathematics with an emphasis in teaching (about half of course credits in mathematics, half in mathematics education).

Participants: Participants for this study were in-service secondary teachers who teach grades 6 to 12 mathematics. Of the 16 who started, 14 completed the coursework of the master's program. All 14 completed the pre- and post-program written assessment. While data included pre-program observations for all 14 teachers, as of this writing there are 10 for whom we have pre- and post-program observations.

Instruments: The development of the written and real-time observation instruments is reported elsewhere (Hauk, Jackson, & Noblet, 2010; Jackson, Rice, & Noblet, 2011). The most important things to note here are that the written assessment included: released items from the LMT (Ball et al., 2008), new items with more complex mathematical ideas modeled on the LMT items, some secondary *Praxis* items, and open-ended extensions to these limited option items. Multi-year test development has included cognitive interviews with in-service teachers and mathematics teacher educators as they did individual items or collections of items. The research team created an alignment of the four PCK constructs of interest across items (e.g., one item might present both curricular content and discourse knowledge challenges while another might foreground curricular content and anticipatory knowledge). These "loadings" of multiple PCK constructs to items is a purposeful part of the non-linear model underpinning test design. Each item on the written test loaded on at least two of the four PCK constructs. Consequently, factor analysis was not appropriate given this confounding of variables. In addition to the established face validity of the tests, tests of the constructs' internal consistency (Cronbach's alpha) indicate, for the pre-test, good overall reliability ($\alpha = .81$), good reliability on CCK ($\alpha = .81$), acceptable reliability on DK ($\alpha =.76$), and marginal reliability on AK ($\alpha = .55$). The PCK post-test had acceptable reliability overall ($\alpha = .75$), acceptable reliability on CCK ($\alpha = .75$), and DK ($\alpha = .73$) (George & Mallery, 2003). However due to an unacceptably low reliability on the post-test for AK for this first cohort of teachers ($\alpha < .5$), we cannot deem the written PCK tests as validly measuring anticipatory knowledge for the group. The observation instrument, based on the LMT video observation protocol (see LMT website; development reported elsewhere) showed good reliability overall ($\alpha = .85$); good reliability on CCK ($\alpha = .84$), DK ($\alpha = .89$), and IK ($\alpha = .85$); and acceptable reliability on AK ($\alpha = .78$). Like the LMT video protocol, the observation tool used samples (6 minutes each: 3 minutes observed, 3 minutes to identify presence/absence of each protocol category in the observed segment; each class visit had 7 to 12 segments). An "observation" was three consecutive classroom visits.

We did pre-program classroom observations in the spring term prior to teachers entering their first course of the master's program. Post-program observations were in the spring term two semesters after the teacher completed the program. Each included three separate classroom visits by the same researcher(s). Experienced observers trained new observers to use the instrument; new raters practiced using the protocol on video data, conducted their first observations of teachers in tandem with an experienced observer, and team members met to calibrate ratings and reconcile disagreements. Inter-rater reliabilities were greater than 0.8.

Teacher-participants completed the written pre-test at the beginning of their first class session in the program. Of the 14 teacher-participants who completed the program, 9 completed the post-program written test at the program closure meeting. For the 5 unable to attend the meeting, members of the research team administered the test at the teachers' school of employment. For each administration of the test, members of the research team created answer keys for multiple-choice items and a scoring rubric for short answer items. The rubrics were informed both by expected responses identified by item developers as well as cognitive interview data. The procedure for developing the rubric was (1) write a desired response, (2) list other anticipated responses, (3) read the responses from a subsample of participants, (4) come to consensus on a scoring rubric. Two or more research team members scored tests separately, compared scores, and met to reconcile any disagreements.

To date the research team has observed 10 teachers after completion of the program. The counts for each of the observation variables were summed and divided by the number of segments observed to report a relative frequency for each variable for each teacher. A teacher having a score of 23.25 on "Explicit Talk about Math" means that the rater(s) noted the teacher exhibiting explicit talk about mathematics during 23.25% of the segments observed. Similarly, on the written test, researchers calculated relative frequency percent scores for each of the four PCK constructs by summing teacher scores on items coded for the construct and taking the percent out of total points possible on each construct. To answer the research question of the impact of the master's program on teachers' PCK, we compared entrance and exit data from the written assessment and the observations using paired-samples *t*-tests.

To model the relationship between teachers' PCK as measured with the written items and in practice as observed, we conducted a path analysis on each of the four PCK constructs. The model considered the pre-test and pre-observation scores as exogenous variables. Thinking that change in knowledge leads to change in action, the model examined the effects of the exogenous variables (pre-scores) on the written post-test; then examined the effects of those three variables on the post-program observation scores. The results report path analysis for CCK and DK (the AK construct was not robustly reliable and the written test did not measure IK).

Results

Observations

Table 1 (see appendix) gives information on pre- and post-program observations for the 10 teachers for whom complete data are available. The table presents the means, standard deviations, differences from pre- to post-program, and results of paired samples *t*-tests on each variable. Because of the number of statistical analyses performed, a cutoff p value of 0.0015 (rather than 0.05) is appropriate, based on a Bonferroni correction (Bland & Altman, 1995).

With this threshold for alpha, there are two statistically significant results. One was in the observation category "General language for expressing mathematical ideas (overall care and precision with language)." While such use of general language was seen, on average, in about 49% of pre-program classroom segments, by the end of the program it was present in more than 80% (M=80.34, SD=19.71). The other significant result was in the category "Mathematical descriptions (of steps)" (i.e., segments where the teacher or students accurately used explicit language to describe the steps of some process). On average, across pre-program observations, this was seen in about 40% of class segments (M=40.28, SD=21.94), increasing to almost 70% of the time, post program (M=68.10, SD=19.31). Three other observed variables appear to be approaching significance (i.e., p<.01): the percent of segments where (a) student voices were present in the room (increasing from 80% to 90% of segments), (b) teachers were observed to use conventional notation (increasing from 54% to 90% of segments), and (c) fewer mathematical errors occurred (decreasing from about 4% of the time to nearly 0%).

Table 2 presents the results of aggregating observation variables associated with each of the PCK constructs. Based on the Bonferroni correction, none of the results were statistically significant. Two approached significance: curricular content knowledge (increasing from 45% to 57% of segments) and discourse knowledge use (increasing from 48% to 61% of segments).

Table 2. Paired samples t*-tests for PCK Constructs from Observation Instrument*

	Pre-program (N=10)		Post-program (N=10)			
PCK Construct	*M*	*SD*	*M*	*SD*	*t*	*p*
Curricular Content Knowledge (CCK)	45.12	13.18	56.64	10.66	4.31	.002@
Discourse Knowledge (DK)	48.25	13.47	61.27	10.27	3.92	.004@
Anticipatory Knowledge (AK)	44.18	12.97	54.26	17.56	1.95	.083
Implementation Knowledge (IK)	59.16	15.13	66.12	9.97	1.54	.159

@ indicates approaching significance, with a $p < .015$

PCK Observations and PCK Test

We conducted a variety of path analyses on the data. In the presentation we focus on the analysis of the Discourse Knowledge (DK) construct, examining the relationships and potential predictive power of the written instrument. Figure 1 shows the full model for discourse knowledge. There was a significant effect of the pre-test (β=.78, *SE*=.26, *p*<.05) and no significant effect of the pre-observation on the DK post-test. There was no significant effect of either the pre-observation or pre-test on the post-observation DK score, although, like CCK, the effect of the pre-test was negative (β=-.58, *SE*=.19). Finally, there was a significant effect of the post-test on the DK post-observation (β=.92, *SE*=.17, *p* < .05).

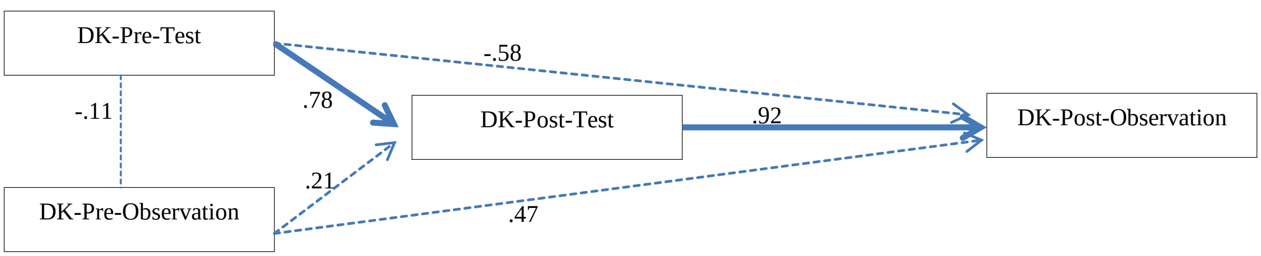

Figure 2. Path diagram for discourse knowledge (DK) construct.

Discussion/Applications/Implications

Because of the small sample size, the study is underpowered for full validation of the assessment and observation scores. Additionally, the small sample size makes generalizing the results problematic. What is apparent is that pre- to post-program written test score changes suggested positive potential outcomes of the master's program in the target area of investigation: development of pedagogical expertise for teaching secondary mathematics, particularly in the communication skills of responsive classroom discourse. The significant increase in curricular content knowledge (CCK) from pre- to post-program teacher observations may reflect the master's program emphasis on increasing participant understanding of advanced mathematics and deepening secondary-school-level-appropriate conceptual connections. This is evident in some of the significant increases on individual variables in categories. For example, the significant increase in the use of conventional notation may indicate that the master's program supported teacher-participants in the habit of using conventional notation to communicate. In addition, the mathematics courses required participants to be explicit about their thinking, reasoning, and justification of answers, which may help explain the significant increase in mathematical descriptions category. However, there were no significant increases in the mathematical explanations or the mathematical justification of the reasoning process, so more work needs to be done in the program to support teachers' attention in these important realms of mathematics teaching and learning (perhaps as they challenge the prescribed curricula, which tend *not* to foreground these things). Finally, the reduction in observed errors may indicate a stronger content knowledge for teaching secondary mathematics.

The significant increase in discourse knowledge (DK) on the written test and in observations may indicate the effectiveness of the master's program mathematics education courses. In particular, the program's emphasis on mathematics pedagogy that made explicit the research-based evidence of student-centered classrooms that support the construction of knowledge of students rather than the transmission of knowledge by teachers. For example, observers saw significant increases in the percent of small group work and in students' voices in the classroom. This may indicate that the teachers' practice shifted to a decentered (or some forms of "learner-centered") approach. Additionally, the program included several credit hours of reading and writing about mathematics education research focused on the NCTM process standards. There was a concomitant significant increase in teachers' explicit talk about reasoning. Finally, the increase in discourse knowledge in general may be attributed to the pedagogy courses that allowed participants to read research and experience what good mathematics discourse "sounds like and feels like" (Cohort 1 participant, personal interview, October 8, 2012).

The path analyses relating PCK as demonstrated on the written test and in practice provide interesting results that need further investigation. As noted, the path diagram for discourse knowledge, Figure 1, suggests that the written test may have predictive value in capturing classroom practice. If this turns out to be a robust result, across populations of teachers, it could reduce or eliminate the need for expensive classroom visits when attempting to determine impact on practice. Researchers need to conduct further investigation into the ways to measure these constructs and to extend the research to larger, more generalizable samples to verify these results. Additionally, researchers need to investigate the negative, albeit not significant, direct effect of the pre-test on the post-observation.

Acknowledgement

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Table 1. Paired Samples t*-tests for Observation Variables.*

	Pre-program (*N*=10)		Post-program (*N*=10)			
Observation Item	*M*	*SD*	*M*	*SD*	*t*	*p*
Format for Segment						
Whole Group	51.61	22.16	63.39	18.40	1.170	.272
Small Group	23.79	22.51	39.26	22.51	2.832	.020
Individual	41.70	26.03	28.98	12.15	-1.610	.142
Lesson/Segment Type						
Review	26.46	18.03	22.46	14.17	-0.549	.596
Introducing tasks	7.23	4.74	10.64	5.31	2.262	.050
Student work time	45.00	24.16	50.04	16.76	0.586	.572
Direct instruction	24.15	15.27	33.00	16.53	1.201	.260
Synthesis or closure	5.77	4.91	8.10	6.02	1.147	.281
Math Teaching Practices						
Voices – Students	79.82	18.32	89.29	15.88	3.375	.008@
Voices – Teacher	80.77	21.98	93.81	8.23	1.949	.083
Real-world Problems	26.55	28.47	36.50	32.44	.826	.430
Interprets Students' Work	63.33	18.39	73.01	13.75	2.120	.063
Explicit about Tasks	82.20	16.20	87.52	12.42	.916	.384
Explicit Talk about Math	59.03	27.18	75.59	11.91	1.801	.105
Explicit Talk about Reasoning	29.93	23.18	49.48	17.94	2.821	.020
Instruction Time	86.10	10.20	87.02	6.81	.249	.809
Encourages Competencies	67.07	26.83	45.04	40.52	-1.420	.189
Knowledge of Math Terrain						
Conventional Notation	54.39	21.38	79.95	15.25	3.353	.008@
Technical Language	72.59	17.38	77.67	13.53	.760	.467
General Language	49.06	13.88	80.34	19.71	4.528	.001*
Selection for Ideas	87.17	8.31	91.16	5.68	1.989	.078
Selection to Represent Ideas	31.70	23.97	43.64	25.51	1.892	.091
Multiple Models	17.80	14.90	33.69	24.20	2.138	.061
Records Work	59.67	28.25	52.01	20.20	-.585	.573
Math Descriptions	40.28	21.94	68.10	19.31	5.003	.001*
Math Explanations	40.65	23.26	55.80	16.29	1.782	.108
Math Justification	14.32	16.13	23.09	11.05	1.928	.086
Math Development	84.50	16.57	88.67	6.11	.753	.471
Errors – Not Present	96.27	2.67	99.78	.69	3.858	.004@

@ $p < .015$, * $p < .0015$

SELF-INQUIRY IN THE CONTEXT OF UNDERGRADUATE PROBLEM SOLVING

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Self-inquiry is the process of posing questions to oneself while solving a problem. The self-inquiry of thirteen undergraduate mathematics students was explored via structured interviews requiring the solution of both mathematical and non-mathematical problems. The students were asked to verbalize any thought or question that arose while they attempted to solve a mathematical problem and its nonmathematical logical equivalent. The thirteen students were volunteers who had each taken at least four upper division proof-based mathematics courses. Using transcripts of the interviews, a coding scheme for questions posed was developed and all questions were coded. While data analysis of the posed questions is ongoing, initial analysis suggests that the "good" mathematics students focus more questions on legitimizing their work and fewer questions on specification of the problem-solving task. Additionally, the self-inquiry of "fast" problem solvers mirrored that of the strong students with even less focus on specification questions.

Keywords: Problem solving, Proof, Self-Inquiry, Logic, Questioning

Many teachers refuse to simply answer a student's question; instead, these teachers insist on responding to the student's misconceptions with other related questions that the student can answer, slowly scaffolding the student's responses until the student has answered (knowingly or unknowingly) their own question. This method, when done correctly, allows the student to recollect related knowledge, receive a confidence boost in their own knowledge of the subject, and receive a lesson in problem-solving strategies that could be utilized to solve future problems. This method of answering questions with other questions seems to work extremely well for student ownership of material, but the question remains as to why students don't ask themselves some or all of these leading questions. Since the student is capable of answering the posed questions that lead them to the solution, what is stopping the student from posing these questions themselves? Is effective self-inquiry a mark of a "good" student? What types of questions do these "good" students ask themselves while problem solving? More importantly, how can we foster pedagogical knowledge from these "good" students' questions so that teachers can guide all students toward productive self-inquiry?

A goal of this study is to explore these questions by trying to document and code questions that students ask during the problem solving process. While related research has been conducted in secondary education and reading comprehension (Kramarski & Dudai, 2009; King, 1989) and in general mathematical thinking (Schoenfeld, 1992), it seems that the self-inquiry of undergraduates in the process of mathematical problem solving has not been explored. Therefore another goal of this project is to begin this line of inquiry and add to the current mathematics education research related to problem solving.

Thirteen mathematics majors were interviewed with the instruction to verbalize every thought process and question that arose as they attempted to solve two problems. The first was a problem in a mathematical context with contrived terminology that was new to all students:

Mathematical Context* (Pleasant Sets)**: A set *S* of real numbers is called ***pleasant if each element of *S* has both a unique immediate successor in *S* and a unique immediate predecessor in *S*.

Let *S* be a pleasant set. Suppose the numbers a, b, c, d, and e belong to *S* and satisfy

(i) b is greater than or equal to the successor of d and less than or equal to the predecessor of e;
(ii) a is the successor of d;
(iii) c is greater than or equal to the successor of e.

Put a, b, c, d, and e in numerical order.

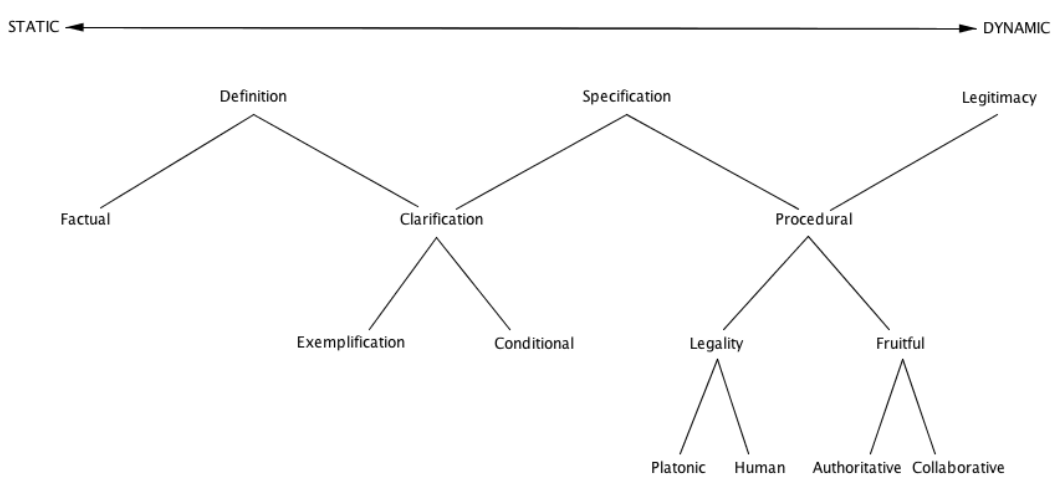
The second problem, though posed in a nonmathematical context, was logically equivalent to the first:

***Nonmathematical Context* (FeedingTime)**: Zookeeper Jane feeds the animals in 15 minute intervals every morning such that

(i) the giraffes are fed after the monkeys but before the zebras;
(ii) the bears are fed 15 minutes after the monkeys;
(iii) the lions are fed after the zebras.

What is the feeding schedule?

After completing the interviews the transcripts were analyzed by all three authors and the questions posed by each participant were agreed upon. This process led to a classification of the questions being posed during the interviews. The following question tree was developed to exhaust the coding of all questions posed. The question tree will be explained in detail during the presentation.



The authors then individually coded the posed questions using the question tree and met to discuss any discrepancies and agree on codes for all posed questions. In order to explore the self-inquiry of "good" students the authors defined the statistic RSQ (Relative Success Quotient) and calculated an RSQ for all students. To calculate the RSQ the authors focused on the 11 upper division courses that had been taken by at least 7 of the participants. For each course the average GPA and standard deviation of grades were calculated for the last 5 years of course offerings. A participant's RSQ is then calculated as the average number of standard deviations their grades are away from the mean for the courses they had completed from the 11 chosen. More details of the RSQ calculation will be described in the presentation.

Participants clearly fell into three RSQ categories, deemed low, middle and high, and data has been organized accordingly for the pleasant sets question.

	Low	**Middle**	**High**
RSQ	*-0.167*	*0.479*	*.879*
Average time to solve	0:12:55	0:15:16	0:06:01
Average # of Q's	13.75	16	9.2
Frequency of Q's	0:01:01	0:01:10	0:00:47
Definition Q's	29.09%	18.75%	28.26%
Specification Q's	29.09%	46.875%	26.09%
Legitimacy Q's	41.82%	34.38%	45.65%

It is interesting to note that the students with a high RSQ were more efficient problem solvers who asked fewer questions. More interesting, though, is that these fewer questions focused on legitimizing their problem solving efforts.

Similar data analysis was completed based on the time it took participants to solve the pleasant sets problem. Again participants fell into three groups deemed slow, medium and fast, and the table below depicts the data organized by these groups.

	Slow	**Medium**	**Fast**
Average time to solve	*0:21:02*	*0:10:07*	*0:04:13*
RSQ	0.117	.645	0.425
Average # of Q's	20.75	11.6	6
Frequency of Q's	0:01:01	0:00:52	0:00:42
Definition Q's	26.51%	22.41%	25%
Specification Q's	42.17%	36.21%	8.33%
Legitimacy Q's	31.33%	41.38%	66.67%

While the medium group had the highest average RSQ, the self-inquiry of the fast problem solvers was most similar to that of the participants with a high RSQ as the questions focused on definitions and legitimizing their work. It is interesting to note that the fast problem solvers posed a very low percentage of questions that served the purpose of specifying the problem-solving situation.

This preliminary report will focus on the development of the question tree used for coding and will present further data related to both the pleasant sets and feeding time examples. The goal of the presentation will be twofold. First, to address the questions posed at the beginning of this proposal with relation to this set of data. Second, we hope to receive feedback from the RUME community about the initial direction of this project and the questions below will be helpful in framing our conversation.

Questions

1. Is the question tree a useful tool for analyzing questioning in the context of mathematical problem solving?
2. Is RSQ the appropriate statistic for defining “good” students? Are there other options for defining “good” students that may shed a different light on the data?
3. What other options are there for research design to try to identify self-inquiry in undergraduates?
4. Is self-inquiry a topic of interest to the RUME community?
5. Should we expect different levels of self-inquiry throughout a math major’s undergraduate career?

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JUMP MATH APPROACH TO TEACHING FOUNDATIONS MATHEMATICS IN 2-YEAR COLLEGE SHOWS CONSISTENT GAINS IN A RANDOMIZED FIELD TRIAL

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Abstract

Many first year college students struggle with foundational mathematics skills even after one semester of mathematics. JUMP math, a systematized program of teaching mathematics, claims that its approach, though initially designed for K-8, can strengthen skills at the foundations college math level as well. Students in sixteen sections of Foundations Mathematics at a college in Canada were randomly assigned to be taught with either the JUMP math approach or a typical teaching approach. Students were measure before and after on their competence (Wechsler test of Numerical Operations) and attitudes (Mathematics Attitudes Inventory) to identify any improvements. Results showed that students in JUMP classes had modest, but consistently higher improvements in competence when compared to students in non-JUMP classes, even after controlling for potential confounding variables, while improvements in Math Attitudes showed no differences.

Keywords: Randomized Field Trial, College Math, JUMP Math, Effectiveness

Introduction

The College Math Project (CMP) has confirmed what many involved in math education at the post-secondary level in Canada and elsewhere already knew: despite the importance of math skills in predicting success for students within the college system and beyond, too many students continue to experience poor outcomes, with over 30% considered “at risk” after completion of their first college math course (Orpwood et al, 2010). One potential solution is to teach foundations college math using the JUMP approach. The Junior Undiscovered Math Prodigies (JUMP) approach to teaching mathematics has a strong theoretical foundation and has shown some dramatic successes at the elementary school level (Lambeth, 2006). It uses guided discovery in order to help students improve their math skills alongside a careful scaffolding of ideas. Its potential for college math had not been thorough assessed, particularly through randomized trials.

The Randomized Controlled Trial method has provided strong evidence for assessing the effectiveness of interventions in many fields including math education (Scheaffer, 2007, Clements, 2008). Even though it is considered the ‘gold standard’ in educational research it is a rarely used tool (Golfin, 2005; Cook in Boruch, 2001). The goal of this research project is to use a Randomized Field Trial in order to investigate the effectiveness of teaching using the JUMP approach in as natural a setting as possible. This presentation will describe the design and results of such a randomized field trial of the JUMP math program in a college setting.

Background

The JUMP approach to teaching mathematics is founded on the principal that everyone has the ability to do well in mathematics (Mighton, 2003), that mathematics can be successfully decomposed and decontextualized for learning (Anderson, 2000), and that unassisted discovery does not benefit learners, whereas feedback, worked examples, scaffolding, and elicited explanations do (Alfieri, 2011). JUMP has been showing signs of success both anecdotally and in unpublished research (Brock, 2005; Lambeth, 2006; Solomon, 2011) in improving mathematics skills in the Grade 1-8. However, there has yet to be a peer reviewed study published confirming these findings. Maciejewski (2012) conducted a pilot study on the JUMP math program at a Canadian College. Though findings were inconclusive, they provided the impetus for redesigning the JUMP materials for the foundations mathematics course in the college setting and for further examining the effectiveness of JUMP therein.

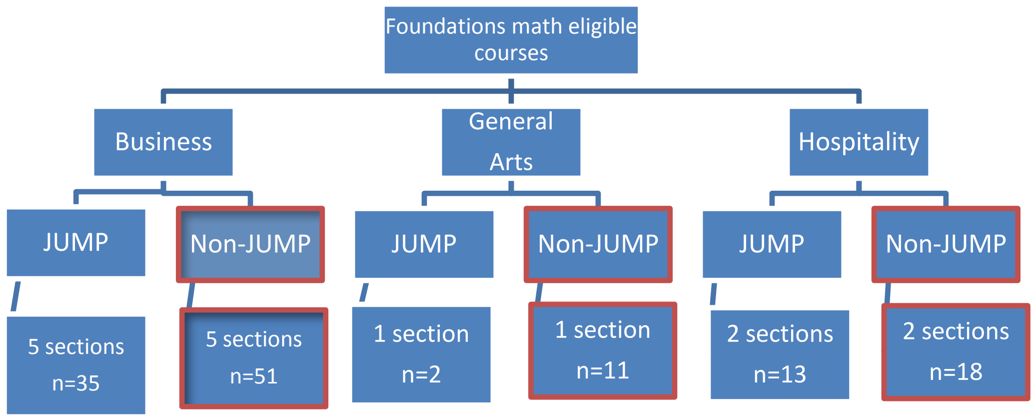
Improving math education is important as math, specifically basic numeracy, is an essential skill for college graduates and all citizens. Human Resources Skills Canada (HRSDC) is interested in supporting Canadians in developing skills that will help them lead quality lives, especially in times of transition. In particular, there is an interest in ensuring that Canadians develop the numeracy skills needed in our highly volatile economy. With the support of JUMP and HRSDC funding, this research study was initiated with the stated aim and title *Understanding Individual Numeracy: How are we doing? Does it matter?* This presentation will focus on one aspect of this HRSDC funded research project: What is the impact of the JUMP Math numeracy education intervention program on student achievement at the college level?

Context

The setting: Business, Hospitality and General Arts and Science (GAS) divisions within a large urban college in Canada. This college has no math department and no standard first year foundations math course. The JUMP approach was implemented in the Fall semester of 2011 during which time pre- and post-data was collected.
Courses: Each division has its own approach to teaching foundations mathematics: e.g. in Business, students typically receive three or four hours of instruction per week, while in Hospitality, all students have only two hours of instruction per week. All foundations math courses in this study were pass/fail in nature.
Course Content: Course content varies by division as well. However, the level of difficulty would not be far beyond the Grade 8 level in the province in which the college was located.
Materials: JUMP math classes were provided with a set of three professionally bound booklets, published by JUMP. Based in part on the findings of Maciejewski (2012), the JUMP math Grade 8 workbooks were revised to suit the more mature nature of college students and were based on course outlines. Non-JUMP students had to purchase their textbooks as usual.
Students: Participants in the study were first year students from the three aforementioned divisions: Business, Hospitality, and GAS. Each registered student had demonstrated a need for mathematics upgrading either through failure of an assessment test or by a self-declaration of unpreparedness.
Teachers: Teachers were assigned to JUMP vs. non-JUMP sections randomly in all but two of eight pairs of classes: one pair of Business classes and the pair of GAS classes. All teachers in participating sections signed letters of informed consent. JUMP math teachers received a two-day training session and a third two-hour session during mid-term. Non-JUMP teachers received training in other contemporary teaching approaches, equivalent to the JUMP training in terms of duration. Teachers teaching the JUMP approach were not passionate and experienced JUMP teachers, and with only two days of training before start of semester cannot have been fully aware of all of the nuances of teaching using the JUMP approach.
Sections (Classes): The randomized field experiment approach included a total of 16 sections (classes), with eight sections in which the JUMP approach to teaching foundations math was used by the teacher and eight sections in which a non-JUMP approach was used (see Figure 1). Student placement in their respective sections was controlled by the registrar's office at the

college, and varied by division. However, neither the Registrar's office, nor the students knew about the teaching approach in various sections before the first class.

Figure 1: Visualization of Structure with number of participants by division and approach.



Design

Unit of Analysis: The unit of randomization is the section. (Scheaffer, 2007. pg. 37) Nevertheless most of the exploration and discussion will focus on comparing JUMP vs non-JUMP students. Potential confounding of section will be accounted for by stratification and use of ANCOVA analysis.

Randomization: Randomization was accomplished through assigning of a teacher to JUMP vs non-JUMP using a coin flip. Six of eight pairs of classes had teachers assigned to JUMP vs non-JUMP randomly. Of the two not assigned randomly, one teacher in hospitality had developed his own non-JUMP materials and asked to teach non-JUMP one in General Arts and Science had some experience with JUMP and asked if he could be assigned to a JUMP section. Thus randomization is partial.

Population: There were 31 'eligible sections' i.e. sections for which the JUMP math content would be suitable ($n_{students} \approx 1000$). That number was reduced to 16 individual sections (See Figure 1) through purposeful selection based on logistics and scheduling.

Population frame I: Potential student participants were those registered on class lists of the 16 selected sections in both Week 2 (during pre-testing) and Week 14 (during post-testing). $n_{students} = 430$

Population frame II: Subset of frame I - 'active students'. Active students were those that were on the class list at the end of semester and had not dropped out. Students who had a zero GPA at the end of semester, failed all courses in that semester and did not return in the next semester were deemed not active. $n_{students} = 412$. This list was used in order to establish generalizability of results.

Dependent variable 1: Improvement in math achievement was captured using the Wechsler test of Numerical Operations (Wechsler, 2009). The Wechsler test is norm referenced for adults and is validated for pre-post test use with a minimum 12 week time span. Sixty-two test questions cover numerical computations from basic arithmetic to simple derivatives. Pre and post-test scores were collected, entered and verified, then raw scores were converted to standardized scores from which percentile improvement was calculated, based on Wechsler (2009) norms. Improvement was also calculated as gain scores to account for the fact that students who scored higher in the pre test had less room for improvement (Scheaffer, 2007 pg. 41). The standardized gain score is calculated as $(x_2\text{-}x_1)/(\text{max score} - x_1)$.

Dependent variable 2: Minnesota Mathematics Attitude Inventory (MAI) pre- and post-test scores in 6 dimensions: Perception of Mathematics Teacher (Te), Anxiety Toward Mathematics (Anx), Value of Mathematics in Society (Val), Self-concept/confidence in Mathematics (SeC), Enjoyment of Mathematics (Enj), and Motivation in Mathematics (Mot). (Welch, 1972)

Independent variable: JUMP Math teaching approach was used in 8 sections with a control set of 8 non-JUMP sections.

Possible confounding variables: Categorical: division, day of week, time of day class held, number of times class held per week, sex, highest education completed, work status, years since most recent math course, highest level of math taken prior to Sept. 2011.

Measurement: age, Baseline standardized Wechsler and baseline percentile rank (from Wechsler pre-test), baseline MAI for each of 6 dimensions.

Institutional data: Pass rates, GPA scores, year of birth (YOB), and admission status were collected through the Office of Institutional Research. For those with zero GPA, Winter 2012 semester academic standing was collected in June 2012.

Data Collection: Letters of informed consent, demographics, Wechsler pre- and post-tests and MAI pre- and post-tests were all obtained on paper in both Weeks 2 and 14 (one post-test took place in Week 15). Teachers were not involved in the data collection process. An incentive draw was held in Week 14 in each of the 16 sections with the winner receiving a $50 gift certificate to the college bookstore.

Ethics: Research Ethics Board (REB) approval was received for all aspects of the project.

Analysis: Data Analysis was conducted using SPSS-19. Similarities between JUMP and non-JUMP sections at baseline were established to account for incomplete randomization. External validity was assessed by comparing participants and non-participants in relevant characteristics. Comparison of JUMP vs non-JUMP with respect to outcomes was examined alongside comparisons by Division and correlations with possible confounding variables. Finally ANCOVA analysis was used to assess the level of statistical significance of any differences in student outcomes between JUMP and non-JUMP.

Findings:

Participation rates in the pre-test was 295/433 = 68.13%; pre and post-tests: 130/433 = 30.02%. Section specific participation rates for both pre and post-tests ranged from 0% to 70.6%

Student participation was voluntary for the pre- and post-tests, thus the possibility of self-selection bias was investigated. Participants (pre and post) were compared to non-participants in the following: pass rates {participants were 1.25 times more likely (95% C.I. = 1.16, 1.36) to pass the course than non-participants}, semester GPA scores {participants had a significantly higher GPA (mean 2.46 vs 2.16, p= 0.006, 95% C.I. for the difference = 0.1, 0.5)} age (mean 21 vs 20.7, p = 0.468), and admission status { rates of 'directly from high school' were similar in participant vs. non-participants (86.2% vs 87.2%, p>0.05)}.

GPA was not found to be correlated with percentile improvement (r=0.049, p=0.584) nor with standardized gain scores (r= 0.033, p=0.706), thus any self-selection bias with respect to GPA was minimal at best.

It seems that there is a slight bias towards participants being marginally stronger students than non-participants, but there is no evidence that being a stronger student is related to higher improvement scores.

Students' demographics and baseline characteristics in JUMP and non-JUMP math sections were equivalent (all p>0.079 as shown in Table 1), as were students across divisions (all p>0.08) and course sections (all $p>0.08$).. This similarity of JUMP and non-JUMP groups allows for comparison despite less than perfect randomization procedure.

Table 1: Equivalence of JUMP and non-JUMP students in selected baseline characteristics

approach used by teacher		age	highest edn	#years since math taken	Std score pre	Percentile pre	Te	Anx	Val	SeC	Enj	Mot
non-JUMP	Mean	21.35	1.29	3.70	87.77	28.50	23.92	14.50	21.52	14.45	16.56	8.39
	Std. Dev	5.24	0.73	5.71	14.96	26.36	3.33	3.49	3.55	3.91	4.65	2.46
JUMP	Mean	22.53	1.21	3.53	86.10	22.12	24.74	13.28	22.30	14.76	16.98	8.93
	Std. Dev	7.27	0.53	3.74	10.34	18.35	4.27	4.00	3.81	3.38	4.32	2.52

Correlation of baseline measures with outcomes of interest: Three baseline measurements were found to be weakly correlated with outcomes. Correlation of Attitude dimension Value of Mathematics in Society (Val) with Percentile improvement was positive ($r=0.26$, $p<0.05$) and with gain scores ($r=0.23$, $p<0.05$). Correlation of pre-test predictors with outcomes were weak and negative, with percentile improvement: ($r = -0.213$, $p=0.015$) with standardized gain scores: ($r = -0.213$, $p=0.015$). This indicates a possibility of covariance between the MAI dimension Val with improvement scores, and between Wechsler pre-test scores and improvement scores, which suggests the use of ANCOVA for controlling these potential covariates when examining the impact of JUMP on these outcome scores.

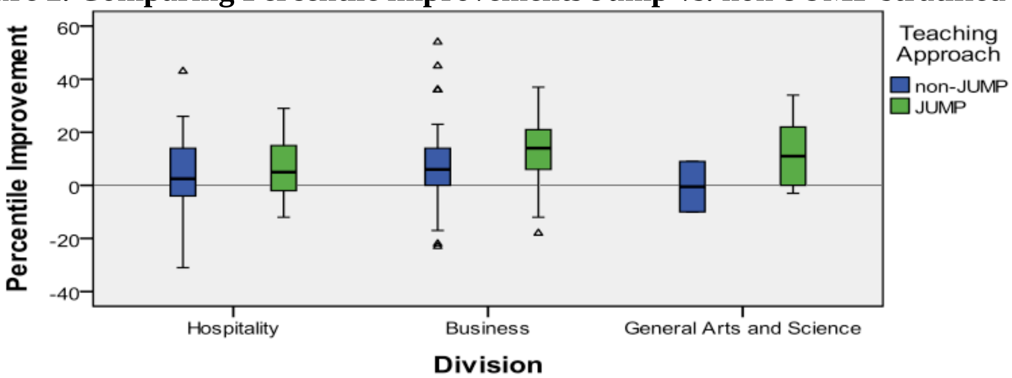
Comparing improvements in Wechsler test of numerical operations: JUMP vs non-JUMP
Mean scores for improvement in the Wechsler score were greater than zero both in percentile (95% C.I: 6.66, 11.68) and in standardized gain score (95% C.I: 0.042, 0.075). Comparison of improvements in Wechsler test showed that JUMP students' mean improvements were higher than non-JUMP students (see Table 2, and figure 2) overall, and when stratified by division.

Table 2: Comparison of percentile improvement JUMP vs non-JUMP stratified by Division

Division	approach used	Mean	JUMP – non JUMP differences % (cohen's d)	Std. Deviation	Median	N
Hospitality	non-JUMP	3.56	109% (0.27)	18.1	2.5	18
	JUMP	7.46		12.0	5.0	13
Business	non-JUMP	7.95	75% (0.41)	14.6	6.0	51
	JUMP	13.91		12.8	14.0	35
General Arts & Sciences	non-JUMP	-.50	n/a (0.91)	13.4	-0.5	2
	JUMP	12.64		12.6	11.0	11
Total	non-JUMP	6.60	85.6% (0.39)	15.5	6.0	71
	JUMP	12.25		12.6	12.0	59

Results for gain scores were similar to those demonstrated for percentile improvements herein. Furthermore, results were similar when stratified by other potential confounders (day of week, time of day class held, number of times class held per week, sex, highest education completed, work status, years since most recent math course, highest level of math taken prior to Sept. 2011).

Figure 2. Comparing Percentile improvements Jump vs. non-JUMP stratified by Division.



All correlations of improvement in Wechsler scores with possible confounding variables had r<|0.1|, except Val (r = 0.26, p = 0.004) and pre-test scores (r = -0.213, p = 0.015). These variables were included as covariates in the ANCOVA.

ANCOVA analysis to control for variation in division and section/teacher confirms that teaching approach has a modest effect on percentile improvements in the Wechsler test (see Table 3) Results for the effect on gain scores were similar.

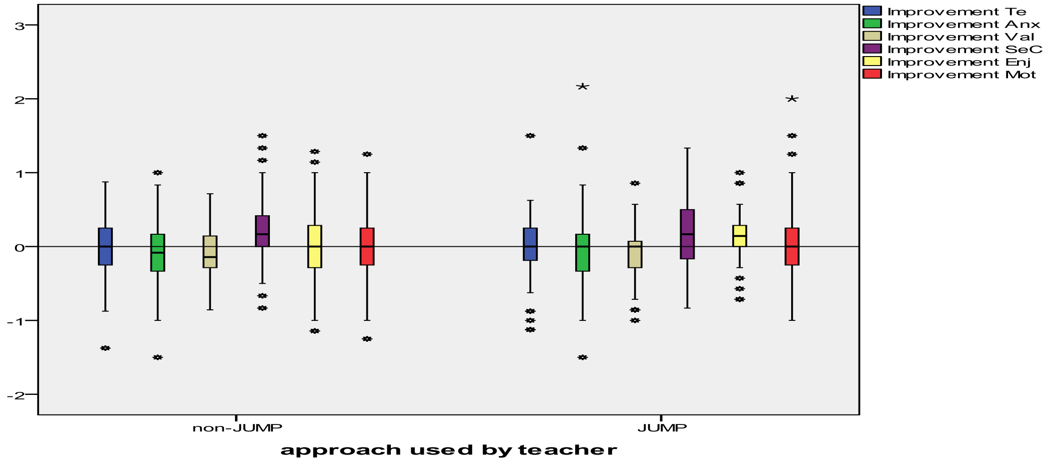
Table 3: output from ANCOVA.analysis with outcome: percentile improvement

Factor	F-value	Degrees of Freedom	P-value	η^2
Approach	6.384	1,22.6	0.019	0.22
Values Math in Society (from MAI)	5.955	1,103	0.016	0.055
Percentile_pre-test	3.936	1,103	0.050	0.037

Teaching approach was significantly related to percentile improvement in the Wechsler test of Numerical operations F(1, 22.59) = 6.384, p=0.019, η^2= 0.22. This confirms the modest and consistently higher improvements that were found in exploration.
2 covariates also demonstrated statistically significant, but weak effects: results for Val: F(1, 103) = 5.96, p=0.016, η^2= 0.055, and percentile pre test score: F(1, 103) = 3.94, p=0.050, η^2= 0.037. Results for gain scores were similar

Comparison of improvements in Math Attitudes Inventory in JUMP vs non-JUMP students:
There were no notable differences in improvement in any of the dimensions of the Math Attitude Inventory (Figure 3). In fact, only two of the Dimensions of the MAI had mean improvements that are non-zero: Val 95% C.I.=(–0.16, –0.02), and SeC: 95% C.I.=(0.09, 0.27).

Figure 3: Comparing JUMP to non-JUMP in MAI improvement scores



Discussion:

Though the differences are modest, the JUMP math teaching approach has shown that it was more effective in helping foundations math students improve their competence in numerical calculations than current approaches used by teachers at George Brown College.

Baseline comparisons showed that JUMP and non-JUMP groups were sufficiently similar before the intervention to allow for comparisons. Furthermore, with exploration techniques and ANCOVA analysis the threat of potential confounding variables has been diminished and results demonstrate that of all the potential factors teaching approach had the strongest effect (p=0.019, η^2= 0.22).

Despite the natural setting, partial randomization, and minimal training of teachers, improvements in standardized gain-scores and in percentiles on the Wechsler of numerical operations testing are consistently higher in students with the JUMP approach.

These differences in improvement in competence (Wechsler test) were not seen in measurements of attitudes (MAI).

This study took place in one college institution. Generalization beyond this college requires additional research demonstrating either similar patterns across institutions or evidence that foundations math students in other institutions are similar to those at this college. Secondly, despite accounting for many possible confounders, there remains the potential of a self-selection bias. Methods for collecting data that fit within the Canadian ethics policy guidelines TCPS2 and yet increase the participation rate are needed for these kinds of trials in the future.

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INTERPLAY BETWEEN CONCEPT IMAGE & CONCEPT DEFINITION: DEFINITION OF CONTINUITY

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This study looks at the interplay between the concept image and concept definition when students are given a task that requires direct application of the definition of continuity of a function at a point. Data was collected from 37 first year university students. It was found that different students apply the definition to different levels, which varied from formal deductions (based on the application of the definition) to intuitive responses (based on rather loose and incomplete notions in their concept image).

Keywords – Continuity, Concept image, Concept definition, Concept definition image, Cognitive conflict

Among others, functions, limit, derivative and continuity have been widely recognized as some of the advanced mathematical concepts that not only students but also teachers find somewhat hard to grapple with. In addition to research carried out on the understanding of these concepts individually (Bezuidenhout, 2001; Vinner, 1987; Cornu, 1991), there has also been research done on understanding of the relationships between some of these concepts (Aspinwall et al., 1997; Duru et al., 2010). Further, the presentation of these concepts in a particular text book is discussed by Tall & Vinner (1981). This paper aims to look at how students work with the concept of continuity. Concept image and concept definition by Vinner (1991) will serve as a theoretical framework for the analysis of the data. This study is driven by the following questions: To what extent do students recall and apply the definition of continuity when handling tasks involving continuity? What notions of continuity are present in their concept images?

Research Method

Thirty seven student responses to the following question were collected and analyzed for this study.

Let

$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & ; x \neq 1 \\ a & ; x = 1 \end{cases}$$

Which value must you assign to a so that $f(x)$ is continuous at $x = 1$?

The students were in their first year of undergraduate studies specializing in the biological and medical sciences and were taking a Calculus course. They had covered the topics functions, limits, limit laws and continuity at the time of data collection. Based on the definition that was taught in this course "A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$", a complete answer to the above question may include three distinct points.

- Identifying the condition that must be satisfied for $f(x)$ to be continuous at $x = 1$. For $f(x)$ to be continuous at $x = 1$, $\lim_{x\to 1} f(x)$ must be equal to $f(1)$ which is a.
- Finding the limit of $f(x)$ when x approaches 1.

$$\lim_{x\to 1} f(x) = \frac{x^2 + x - 2}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x\to 1} x + 2 = 1+2 = 3 \quad ; x \neq 1$$

- Concluding that a must be 3.

$f(1) = 3$ and $f(1) = a$ hence $a = 3$.

These steps need not be in this same exact order but there must be some logical sequence in the way the students organize their answer. The consultation of the definition in the first step requires them to proceed to the second step where they need to find the limit of $f(x)$ when x approaches 1. A student may do this step first 'knowing' it needs to be done in their head and may state the condition afterwards. Because without calling on the definition, there will not be a necessity to find the limit. The second step is a matter of finding the limit of a function where the function is a rational which produces an indeterminate form with direct substitution. This step hence, may not call on the definition of the limit but only on the procedures of finding the limit. Last step is the conclusion of the answer.

Results

Four different types of answers could be identified. The four categories are listed in a certain order which is from a poor answer to a good answer from a marker's perspective.

Type 1 - *The correct answer for a is obtained but taking the limit of $f(x)$ when x approaches 1 is not explicitly shown.*

Four (out of 37) students in the group gave the answer in this category as shown in figure 1. It is hard to say whether these students are *thinking* of taking the limit but not showing it or they are merely doing an algebraic manipulation of the expression. The line, $f(1) = \big((1) + 2\big)$ can be interpreted at least in two ways.

Case 1 : 'plugging a value into the function'

$f(x) = (x + 2)$ and hence $f(1) = ((1) + 2)$ or

Case 2 : applying the condition for continuity and hence stating an identity

$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x + 2) = \big((1) + 2\big)$ & this must equal to $f(1)$, $f(1) = ((1) + 2)$

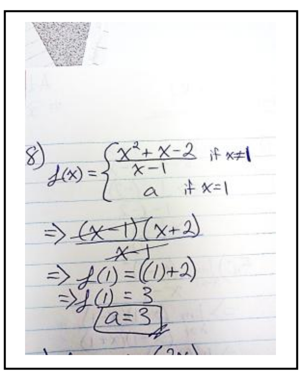

Figure 1: Type 1

The way they have presented their answer it appears as though the students meant the first case rather than the latter. This is because if they meant the second case, the way the argument is ordered, it should be written as $\big((1) + 2\big) = f(1)$, not as $f(1) = ((1) + 2)$.

The concept definition of 'continuity of a function at a point' *contains* the concept of 'limit of a function'. If students have trouble understanding the concept of limit and hence possess a blurred concept image of limit, then, this has a significant impact on the concept image of continuity. The portion of their concept image which is evoked by this problem does not seem to contain or have any overlap with the concept of limit. Their working can be best described as an effort to merge the two pieces of the function. This can be pointing to the notion that students were found to have by Tall and Vinner (1981) too, of the need for a function to be in *one piece* to be continuous. It appears that they simplify the case when $x \neq 1$ which is $\frac{x^2 + x - 2}{x - 1}$ to $(x + 2)$ and then assign it to the case when $x = 1$.

The features of the responses of this type also suggest that the task has not made these students to consult the concept definition but that they have worked on certain notions in their concept image of continuity. This intuitive response is modeled by figure 2 as illustrated by Vinner (1991, pg. 73).

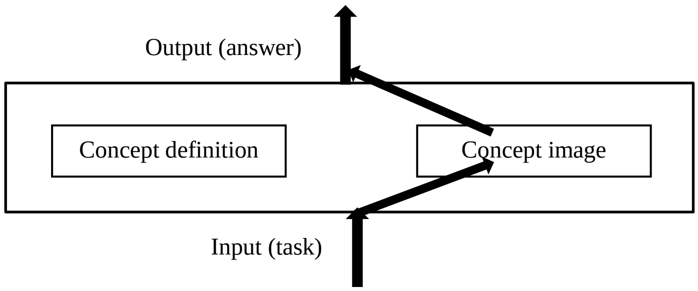

Figure 2: Intuitive response

Type 2 – *The limit is taken and the value for a is given without noting that the limit must equal to $f(1)$.*

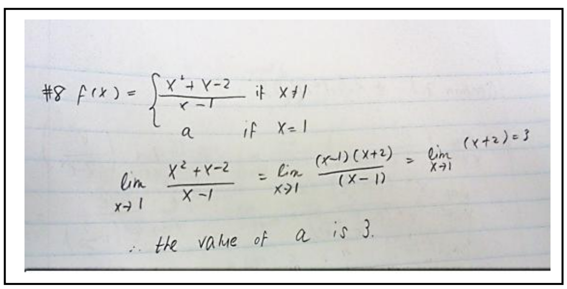

Figure 3: Type 2

In this category (12 out of 37) the students have taken the limit and have just concluded that it is equal to a (see figure 3). This kind of an answer can come from a correct reference to the definition. What is lacking in terms of writing is, not explicitly showing or stating that the calculated limit must equal the function value at $x = 1$. And it is not acknowledged that $f(1) = a$. However, this may have been thought through to obtain the answer as $a = 3$.

Another possible process that may be on work here is a rote memorization of a procedure rather than any attention given to the definition. Since this is a familiar and 'routine' kind of question, students may have developed an algorithm for it, as part of the concept image. It may be a rule like 'find the limit of the function given and assign it to the letter'. Only this procedure, in that case, may be evoked when presented with this style of a question.

Type 3 - *The limit is taken and notes that it should be equal to $f(1)$ and hence to a.*

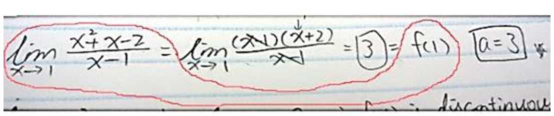

Figure 4: Type 3

These students (4 out of 37) have explicitly stated that $f(1)$ is equal to the answer they obtained for the limit and hence have exhibited an important part of the definition before concluding the final answer for a (see figure 4). And as shown in figure 4, the definition is embedded in their answer. It can be concluded that, in their concept image they have a complete concept definition image which they have been able to appropriately apply in this task. Based on the presented written work, students in this type are a step ahead of the students under type 2. Even if one argues that these students too can be applying a mere memorized algorithm, it is evident that their 'algorithm' is more closely grounded to the definition.

Type 4 – *A complete logical answer with all reasons is given.*

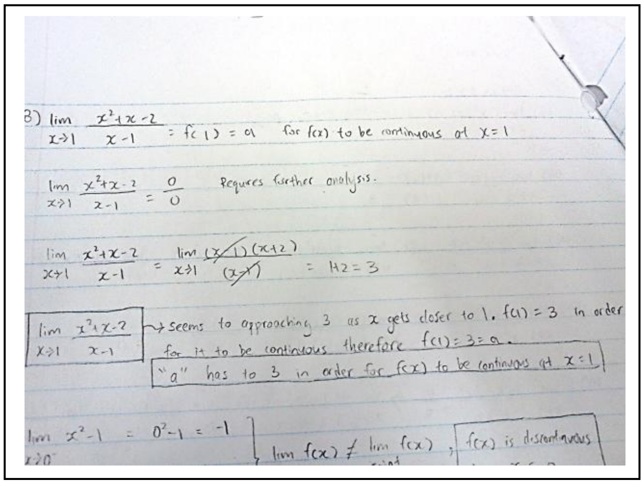

Figure 5: Type 4

The answers were with a good logical sequence of reasoning without missing any points as shown in figure 5. Thirteen of the students had given answers in this category.

It is clearly demonstrated how they formulate their answers by consulting the concept definition. And no sign of side tracking or being disturbed or intervened by unnecessary notions that *may* be present in the concept image is visible. Hence, this can be modeled by figure 6 as illustrated by Vinner (1991, p. 72) of a purely formal deduction.

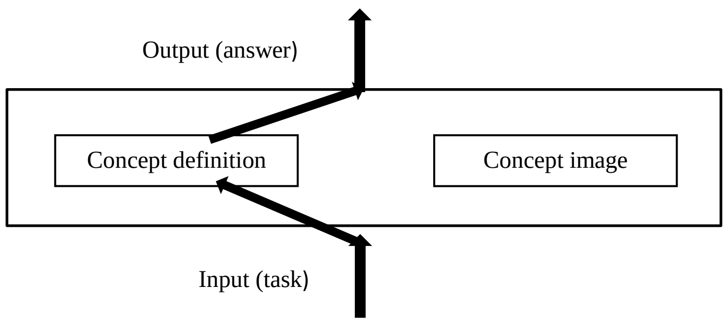

Figure 6: Formal deduction

Discussion & Conclusions

Response types 2,3 and 4, show clear attention given to the definition in different degrees. Vinner(1991) claims that the majority of students do not use definitions when working on cognitive tasks in technical contexts and that college courses do not develop in the science students, not majoring in mathematics, the thought habits needed for technical contexts. However, as far as using definitions goes, this study suggests that, majority of students who are not majoring in Mathematics do refer the definition but in different levels. They seem to have a concept definition image developed to different levels as part of their concept images. Or, if the assumption- that their writings reflect their cognitive processes- is removed, this can be pointing to a different category of levels in transforming their cognitive processes into writing.

What seems to emerge from type 1 is the tendency of some students to tackle problems in ways that they have built for themselves with little rigor which works and produces the correct answer. Vinner says that ‘as long as referring to the concept image will result in a correct solution, the student will keep referring to the concept image since this strategy is simple and natural’ (Vinner, 1991, pg. 80). Can this be overlooked as they produce the correct answer and be satisfied about their performance, as these students are not majoring in Mathematics? Or should these be resolved by creating cognitive conflicts that make students confront these erroneous methods?

Discussion Questions

1. Are there any other ways in which you can interpret student thinking/reasoning corresponding to the type 1 (figure 1) answer?
2. The text book uses the technique of cancellation of factors in examples and does not mention that what is obtained after cancelling the common factor is a different function that agrees with all but one point of the original function. What effect does - not knowing what is going on behind this technique - have on future learning of students, if any? How important is it for students to know this?
3. What other kinds of questions would be more effective in finding out erroneous concept images of continuity in students?

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CONCEPTUALIZING VECTORS IN COLLEGE GEOMETRY: A NEW FRAMEWORK FOR ANALYSIS OF STUDENT APPROACHES AND DIFFICULTIES

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This article documents a new way of conceptualizing vectors in college geometry. The complexity and subtlety of the construct of vectors highlight the need for a new framework that permits a layered view of the construct of vectors. The framework comprises three layers of progressive refinements: a layer that describes a global distinction between physical vectors and mathematical vectors, a layer that recounts the difference between the representational perspective and the cognitive perspective, and a layer that identifies ontological and epistemological obstacles in terms of transitions towards abstraction. Data was gathered from four empirical studies with ninety-eight total students to find evidence of the three major transition points in the new framework: physical to mathematical coming from the first layer, geometric to symbolic and analytic to synthetic from the second layer, and the prevalence of the analytic approach over the synthetic approach while developing abstraction enlightened by the third layer.

Key words: vector, geometry, representation, vector representation

The complexity and subtlety of the construct of vectors motivate the necessity of the new framework that permits: differentiating abstraction from physical embodiment, intertwining the representational perspective and the cognitive perspective on vectors, and revealing cognitive development on geometric representations. These needs guided us to deliver a new framework for conceptualizing the construct of vectors. By building a new framework and validating it, I explored a complex construct of vectors in mathematics with respect to mathematical abstraction, multiple representations, and cognitive development.

The primary goal of the framework is to give a new way to discuss the complexity of vectors, both conceptual and pedagogical, that students may grapple with in order to understand vectors in geometry effectively.

Complexity of the Construct of Vectors

Vector as a Translation

When representing a geometric translation with a vector in an arrow form, the complexity of vector representations comes out in a cognitive sense. Recent critical studies on students' experiences with vectors focus on physics education (Aguirre and Erickson, 1984; Aguirre, 1988; Hestenes et al., 1992; Heller and Huffman, 1995; Knight, 1995; Savinainen and Scott, 2002; Nguyen and Meltzer, 2003; Flores et al., 2004; Coelho, 2010). These studies focused more on the interrelationship among physical quantities, not on vectors themselves. These physical embodiments of vectors help students understand vectors initially, but soon block the progression to advanced and abstract understanding of vectors.

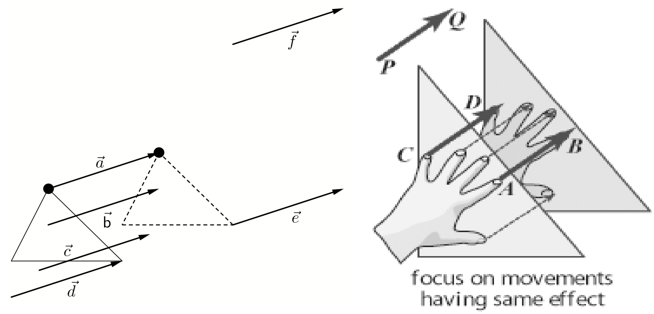


Fig. 1.1 Vector as a Translation Problem

For example, in Figure 1.1 physically all the vectors can show different meanings such as push, pull, moving vertices, and penetrating etc. However, all the vectors represent the same translation of a triangle even though their locations are quite scattered and different. The figure also shows all the vectors are equivalent just as all the vectors represent the same translation, although their equivalence is not clearly explained by the translation action with the given triangle and the same effect those vectors would bring. This difference between physical motion and mathematical motion (Freudenthal, 1983), and the idea of the vector equivalence relation with 'action-effect' approach (Watson et al., 2003; Watson, 2004) show the difference between physical embodiment and mathematical abstraction.

For a better observation and description of this complexity and subtlety of vectors, I need a new framework that integrates and balances these intuitive understanding and abstract understanding of vectors in college mathematics.

Vector as a Point, Point as a Vector

The complexity of vectors does not allow students an easier translation/conversion from one representation to the other. A combined view of the representational perspective and the cognitive perspective helps us understand this translation/conversion from one representation to the other.

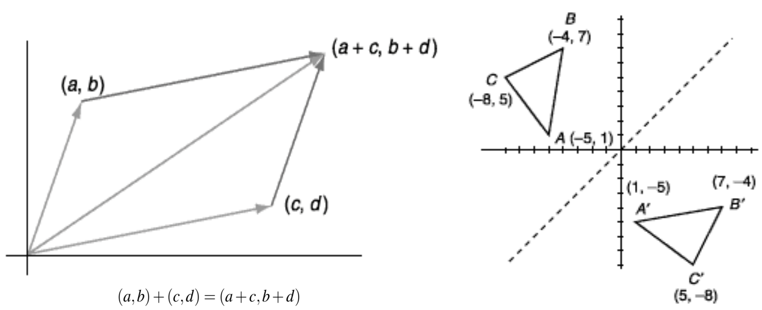


Fig. 1.2 Vector as a Point, Point as a Vector Problem (NCTM 2000)

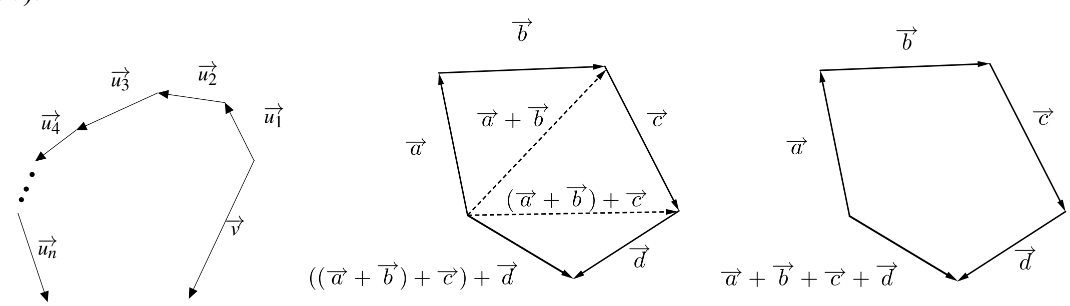
In Figure 1.2, the vector sum in arrow forms is usually accompanied by 2-tuple of numbers. However, it is not clear if this 2-tuple represents the terminal point of the arrow or the arrow itself. When geometry meets linear algebra, this problem of multiple representations gets more complicated. The term 'vectors', 'points (vertices)', and 'arrows' are not easily distinguishable in these two examples. Hillel (2002) criticized that in practice, most instructors tended to shift back and forth between the arrow and point depiction of vectors 'implicitly' and 'unconsciously' with three modes of description. The study of modes of thinking by Sierpinska (2002) reveals an expansion of the discussion as a problem of how students think about representations with focusing more on students' thinking and reasoning about epistemology. These classifications of vectors motivated me to consider multiple perspectives on vectors: representational and cognitive.

Research in the mathematics education community posits that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (Janvier, 1987; Kaput, 1987a; Keller and Hirsch, 1998). In this context, several standards documents have advocated K-12 curricula that emphasize mathematical connections among representations (NCTM, 1989, 2000; CCSSM, 2010). They suggest that students use graphical, numerical, and algebraic representations to investigate concepts, problems, and express results. However, these discussions are very focused on the way to talk about functions, not vectors, and on the way to discuss connections between representations as separate entities from representations themselves. Studies on multiple representations of vectors are few and focused on the views from linear algebra (Dorier, 2002; Harel, 1989; Dorier and Sierpinska, 2001). The construct of vectors is more complex than functions, so that graphical, numerical, and algebraic representations are not enough to describe this complexity (Pavlopoulou, 1993 as cited in Artigue et al. 2002).

These observations on translation/conversion bring a need for a unified, inclusive, and multidimensional framework to discuss a combined view of representational and cognitive perspectives on the complexity of the construct of vectors.

Geometrical Vector Sum

What the classical representations cannot provide from the complexity of vectors is the cognitive development of geometric representations. Specifically, translations/conversions in terms of the cognitive perspective are portrayed in considerable detail as cognitive development theories for symbolic representations. Pavlopoulou's research (as cited in Artigue et al., 2002) studied translations/conversions from one representation to the other (Duval, 2006). However, it is very restricted to certain forms of vectors: graphical, table, and symbolic representations (registers). The Action-Process-Object-Schema theory (Asiala et al., 1996) and reification theory (Sfard and Linchevski, 1994) are based on the duality of the mathematical concepts and on the assumption that process conception precedes object conception. Sfard (1991) calls process conception operational outlook and object conception structural view. However, these studies are restricted to discussions of symbolic representations. In terms of geometric representations, Figure 1.3 describes interesting observations about process-object duality (Sfard, 1991; Gray and Tall, 1993, 2001; Forster, 2000).



(a) Perform Operations (b) Represent the Structure

Fig. 1.3 Geometrical Vector Sums

When students calculate a vector sum with the tip to toe triangle method, the process view of a single vector is described as shifting or moving a particle. As a result of the sum, students can put the resulting vector on the appropriate position, and make the sum itself as an object in the structure of a triangle. On the other hand, in parallelogram method, vectors are objects and moving the object with equivalence relation to draw a parallelogram is the process of sum. This example shows that process and object are preceders and succeedors of each other and shows further need for cognitive development as a part of the framework.

The problem of identifying cognitive development in geometric representations of vectors poses the need for a new framework that can show the representational and the cognitive obstacles more clearly in terms of the transitions towards mathematical abstraction. This new framework brings the complexity of vectors to the surface so that one can capture the whole picture of encapsulation or reification both happening in a symbolic way and a geometric way simultaneously in the construct of vectors.

Construction of the Configuration

Those needs that I discussed in the previous chapter grounded this development of the framework that allows a layered view to see the complex construct of vectors. Three layers of progressive refinements are introduced sequentially. They have different scales of focus from aggregate and global dealing with the difference of physical and mathematical vectors (first layer) to individual and local dealing with the representational and the cognitive obstacles (third layer). The final construction of a configuration of vectors as a new framework based on the process of refinement grounded by the needs in the previous chapter is in Figure 2.1 (Kwon, 2011).

In the domain of mathematical vectors, each axis was hypothesized to have two important transitions that can be identified in the configuration. On the epistemological axis, there are (1) one from arithmetic to algebraic, and (2) one from analytic (procedural) to synthetic (structural). On the ontological axis, there are (1) one from geometric to symbolic, and (2) one from concrete to abstract. Among those four transitions, we will only focus on two transitions: analytic to synthetic and geometric to symbolic in this article.

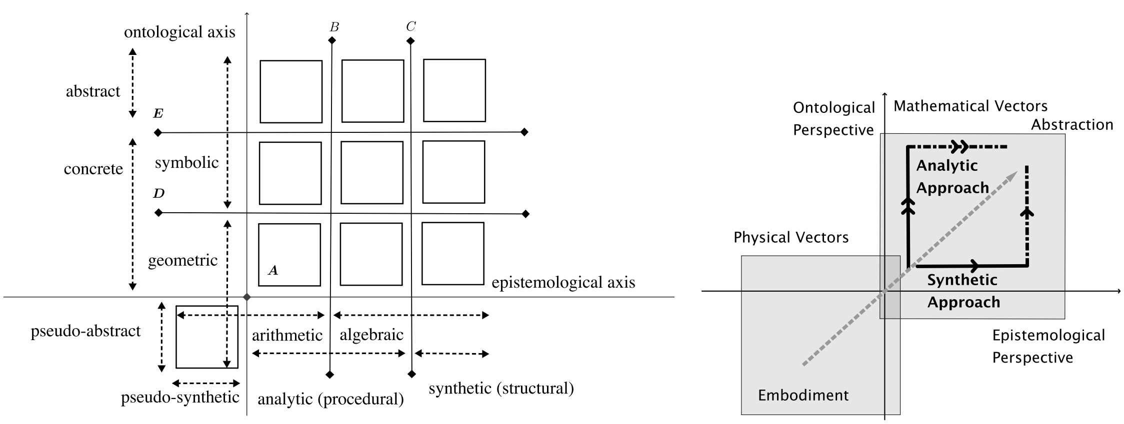


Fig. 2.1 The Configuration of Vectors and Two Approaches towards Abstraction

In this framework, three layers of progressive refinements are comprised. Assuming a difference of mathematical vectors and physical vectors, I first set up two categories: mathematical vectors and physical vectors. Direction of intended movement is the direction of the development towards mathematical abstraction wanted from students. The second layer describes the difference between the representational and the cognitive perspectives on vectors as the difference between the ontological and the epistemological perspectives. The third layer identifies the representational and the cognitive obstacles in terms of transitions towards abstraction. See Figure 2.2.

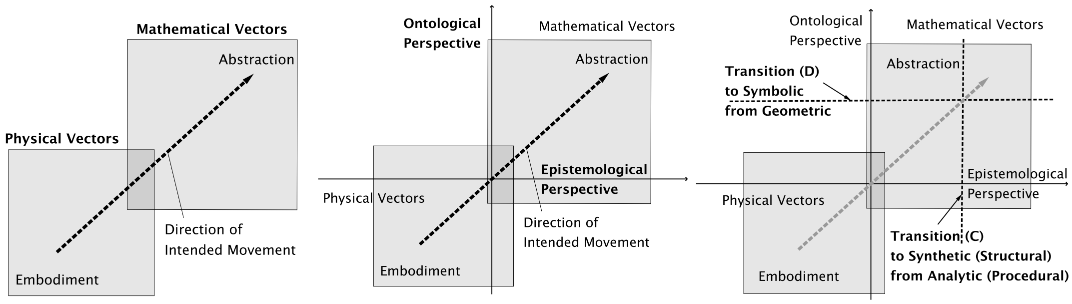


Fig. 2.2 Three Layers of Progressive Refinement

The configuration also shows the analytic approach and the synthetic approach towards mathematical abstraction described by the second and third layers. In terms of the configuration, I can describe the analytic approach as a trend of changing explicit representations along the ontological axis quickly from geometric representations such as arrows, to symbolic representations such as coordinate/column vector forms. Epistemological development is postponed and following symbolically after the ontological change. This analytic approach in the configuration can be illustrated as an upside down ‘L’ shape route towards abstraction. The synthetic approach is defined as the trend of changing the views/thinking of a geometric representation while maintaining arrow forms, from analytic (procedural) to synthetic (structural) first. Ontological development is postponed until the achievement of the change in epistemological perspectives such as from analytic (procedural) to synthetic (structural). This approach in the configuration marked as a reversed ‘L’ shape route towards abstraction. See Figure 2.1.

Empirical Studies

I will discuss the method for gathering evidence from student data to see if this layered view of the construct of vectors as a result of progressive refinements is reasonable.

Research Focuses

The focus is on evidence in student data of the following features that are hypothesized by the configuration of the construct of vectors: (1) the three major transition points: (A) physical to mathematical coming from the first layer, (C) analytic to synthetic, and (D) geometric to symbolic coming from the second and the third layers of the framework, (2) process-object duality in geometric representations, and (3) the prevalence of the analytic approach to the synthetic approach while developing mathematical abstraction.

Method and Participants

Four surveys and interviews were carried out to gather evidence on the important features suggested by the configuration. The results are a synthesis of the data gathered from total ninety-eight students who are pre-service secondary and elementary/middle level teachers. Multiple administrations were used to: (1) test appropriate survey questionnaire and interview process, (2) gather deeper knowledge of student background and idea on vectors, and (3) modify surveys and interviews in order to avoid any confusion derived from the questions. All data were collected from students located in the Midwest public university.

After administering each survey, students were selected for interviews based upon their response. The selected students signed up for a one-hour block of time for their interviews on the day and time that was most convenient for them. Interviews were held in a neutral location away from the students’ classrooms and were audio-recorded for further analysis. Transcribed interviews were coded and analyzed in order to find evidence in student work. Descriptive statistics were used for the second study and the third study on which a sufficient number of participants were available.

Design and Construction of Surveys and Interviews

For the in-depth discussion of the research focus, I chose the following questions from the surveys and synthesized the results. See Table 3.1. Interviews were conducted with repeating the survey questions and asking further about what the students were thinking. Because questions for examining the prevalence of an approach to the other in the surveys were only asked to students to choose the representations, I asked students to proceed and finish the proof in the interview sessions.

Key Features	Related Layers	Transitions	Questions
Physical vs. Mathematical	I	A	Translation, (Translation of Polygon), (Geometric Translation), Rainy Day, Robot Arm
Epistemological Diff. & Ontological Diff.	II	A, C A, D	Translation, Polygon Rainy Day, Robot Arm
Epistemological Obst. & Ontological Obst.	III	(A), C (A), D	Polygon, Very Long Sum Origin, Robot Arm
Process-Object Duality in Geometric Representation	III	C	Very Long Sum
Prevalence of Analytic Approach	II & III	C, D	Cube, △ Midpoints, Associativity

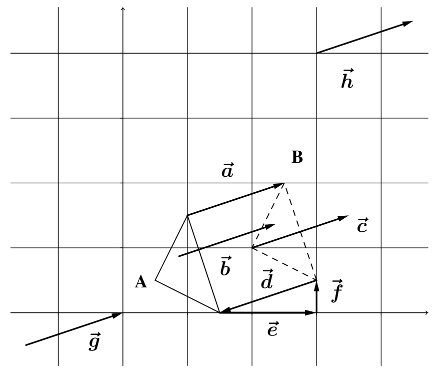
Table 3.1 Questions for Layered View of Configuration

Findings and Discussions

In this chapter, I provide evidence of some focal points (Transition A, C, and the prevalence of the analytic approach) that I hypothesized in Research Focus after careful analyses of the four consecutive empirical studies.

The global difference between mathematical abstraction and physical embodiment is evident in student work (Transition A). Student data showed evidence of the different interpretations between 'the same translations' and 'the equivalent vectors'. We usually assume that the concept of the vector equivalence relation in physics is the same with that in mathematics, because 'directions' and 'magnitudes' of vectors are used to verify equivalent relations in both fields. This means that the equivalent vectors are always representing the same translations and vice versa both in physics and mathematics. However, student work for 'Translation' question (Fig. 4.1) shows the difference of the interpretation between the same translations and the equivalent vectors. The cluster tree diagrams (Fig. 4.2) from hierarchical clustering with Euclidean distance are supposed to show similar categorizations assuming 'the same translations' and 'the equivalent vectors' are the same concept. The first cluster tree was made from the student responses to question (a), and the second cluster tree was made from those to (b). They show two different categorizations in Figure 4.2.

Translation: A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point P.



(a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

(b) List all vectors that are equivalent to $\overrightarrow{a}$.

Fig. 4.1 Translation Question

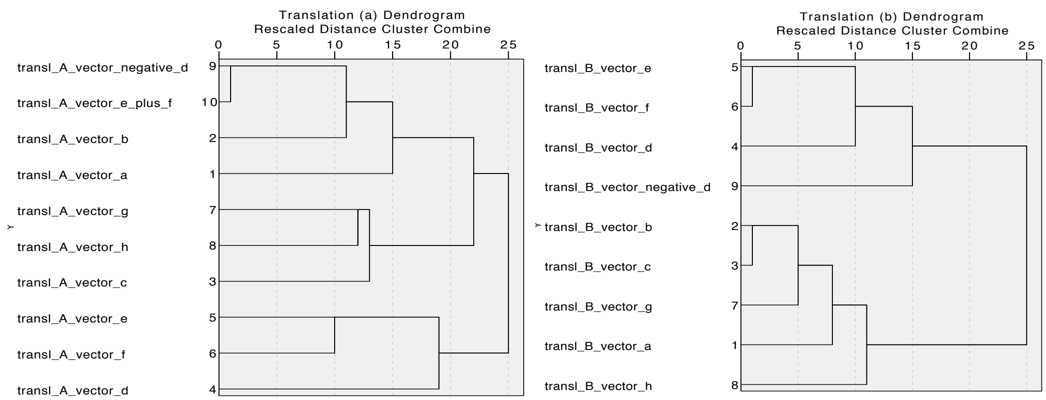


Fig. 4.2 Different Interpretation between Same Translations and Equivalent Vectors

The existence of an epistemological obstacle as Transition (C) from procedural (analytic) to structural (synthetic) is also evident in student work for ‘Polygon’ and ‘A very long sum’ questions. This obstacle prevents students to continue calculating the binary vector sum. For ‘Polygon’ question, students’ responses told us that question (a), (b), and (d) based on a triangle, a parallelogram, a rectangle figure were easier for students, but question (c) with a pentagon was harder to solve than other questions (Fig. 4.3). The following interview also shows evidence of Transition (C).

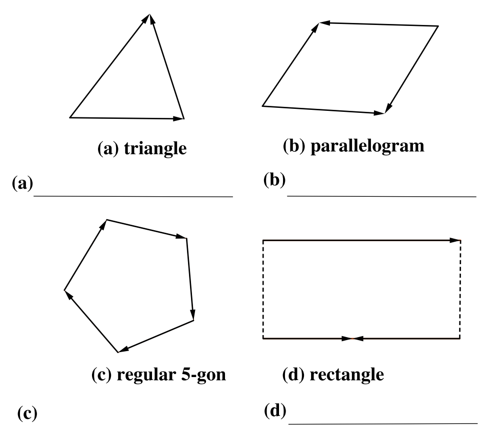
R: “Why did you draw these middle line segments? What are they?”

S: “I think I was trying to do the vector addition and [...] I couldn’t really find, based on the method I was trying, couldn’t find the way to express the relationship together from a polygon or from a pentagon. [...] Normally I’ve never seen vectors arranged in that kind of relationship. I’ve seen them in the triangle, [...] usually in many of these, a parallelogram, a four sided figure, but nothing like this one.”

From the interview with the student above, I could see that the student drew a parallelogram to figure out the sum. Thinking the sizes and the directions of arrows compared to thinking the structure that vectors lie on can be regarded as procedural thinking, because the sum was a binary operation and we needed those information for a binary operation. It is evident that there is an obstacle that prevents students using synthetic vectors or structural thinking.

I could also see that students tended to use particular representations more and confine their understanding and using vectors in one approach rather than having flexibility of using both. This tendency was identified in the responses as the prevalence of the analytic approach to the synthetic approach. Because this prevalence is studied and regarded as a trend, and not a specific student's preference, I used the collective data of twenty-nine students rather than concentrating on specific cases. 'Cube', 'Triangle Midpoints', 'Associativity' are questions specially designed to look into the prevalence of the analytic approach to the synthetic approach. The results in Table 4.4 show that the prevalence of the analytic approach to the synthetic approach that I hypothesized is evident in student work.

Polygons: label each vector and write an equation of the relations.



question	(a)	(b)	**(c)**	(d)
correct	72.4%	65.5%	**48.3%**	69%
not correct	27.6%	34.5%	**51.7%**	31%
total	100 %	100 %	**100 %**	100 %

Table 4.4: Polygon Result from Survey II

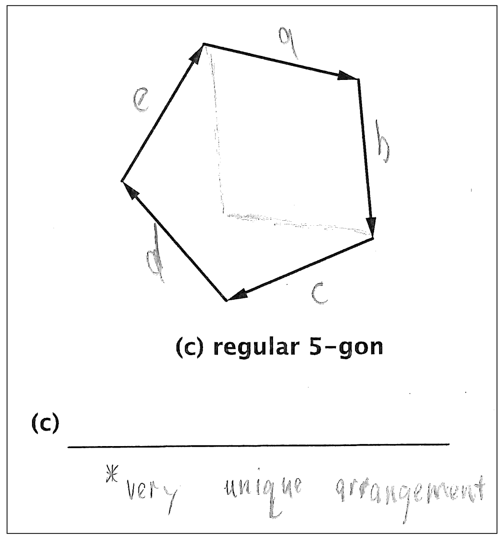


Fig. 4.3 Polygon Results and "Very unique arrangement"

	Analytic Approach	Synthetic Approach	No response
Cube:	**41% (12)**	31% (9)	28% (8)
$\triangle$ Midpoints:	**55.5% (16)**	34.5% (10)	10% (3)
Associativity:	**72.4% (21)**	24.1% (7)	3.4% (1)

Table. 4.4 Prevalence Results

In summary, I saw that the following are evident in student work when discussing the complexity/subtlety of vectors: (1) the difference between physical and mathematical vectors, (2) the multiple perspectives: ontological and epistemological, and interplay between those two, (3) the prevalence of the analytic approach to the synthetic approach, (4) an epistemological obstacle defined as Transition (C) and an ontological obstacle defined as Transition (D), (5) process-object duality on geometric representations of vectors. What stands out most from these empirical studies is this new framework is very helpful when talking about the complexity and subtlety of the construct of vectors. However, these tentative conclusions with the configuration await further refinement and correction in the light of further research.

While conducting empirical studies, unexpected evidence from the three progressive refinements of the construct of vectors is also shown up as limitations. These limitations include: (1) non-empty intersection between mathematical vectors and physical vectors, (2) unreasonable levels of sophistication reflected in the direction towards abstraction, and (3) problems in repetition of transitions and revered transitions in different contexts. These limitations suggest further refinement and correction of the framework as well as the implications for teaching and learning of vectors.

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Characteristics of Successful Programs in College Calculus: Pilot Case Study

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Abstract

The CSPCC (Characteristics of Successful Programs in College Calculus) project is a large empirical study investigating mainstream Calculus 1 to identify the factors that contribute to success, to understand how these factors are leveraged within highly successful programs. Phase 1 of CSPCC entailed large-scale surveys of a stratified random sample of college Calculus 1 classes across the United States. Phase 2 involves explanatory case study research into programs that are successful in leveraging the factors identified in Phase 1. Here we report preliminary findings from a pilot case study that was conducted at a private liberal arts university. We briefly describe the battery of interviews conducted at the pilot site and discuss some of the themes that have emerged from our initial analyses of the interview data.

Key Words: Calculus, Explanatory Case Study, STEM Student Retention

Issues Explored & Relation to Research Literature

The CSPCC (Characteristics of Successful Programs in College Calculus) project is a large empirical study investigating mainstream Calculus 1 to identify the factors that contribute to success, to understand how these factors are leveraged within highly successful programs. Calculus 1 is *the* critical course on the road to virtually all STEM majors. However, while more students are taking more advanced mathematics in high school than ever before—including over half a million each year who study calculus while in high school (Bressoud, 2009)—the percentage of all college students in 4-year undergraduate programs who are enrolled in mathematics at the level of calculus or above has decreased steadily from 8.93% in 1990 to 6.36% in 2005, a decrease of 29% (Lutzer *et al*, 2007). In Seymour's (2006) testimony to Congress, she noted that, contrary to what is commonly assumed, students do not leave STEM majors primarily for financial or academic reasons. Instead, they leave STEM majors because of poor instruction in their mathematics and science courses, with calculus instruction and curriculum often cited as a primary reason for students' discontinued STEM course taking (Thompson *et al.*, 2007).

Phase 1 of CSPCC entailed large-scale surveys of a stratified random sample of college Calculus 1 classes across the United States. Phase 2 involves explanatory case study research into programs that are successful in leveraging the factors identified in Phase 1. This second phase will lead to the development of a theoretical framework for understanding how to build a successful program in calculus and in illustrative case studies for widespread dissemination. Sixteen institutions have been selected as case study schools based on the results from the survey phase. The set of case study schools includes four community colleges, four bachelors granting institutions, four masters granting institutions, and four PhD granting institutions. In preparation for this case study phase of the project, the team assigned to the bachelors granting institutions conducted a pilot case study at a private liberal arts university. This university was selected because the bachelors degree was the highest mathematics degree offered by the university and because the institution's calculus pass rate was comparable to (actually higher than) the institutions identified as successful during the survey phase. Note that he institution was not part of the sample that participated in Phase 1). In this preliminary report, we will share some of the findings from the ongoing analysis of this pilot study.

Research Methodology & Conceptual Framework

While crafting the proposal for the CSPCC project, the PI's developed initial hypotheses as to the factors that could impact student success in Calculus I:

Instructor attributes: professional status (e.g. rank), professional preparation to teach calculus, awareness of common difficulties and misconceptions, and attitude toward institution, students, and teaching.

Departmental focus: placement exams, explicit learning goals, use of standardized exams, professional development opportunities, monitoring of student retention.

Classroom variables: size of class, text and curriculum, use of recitation or laboratory sections, incentives for attendance, format and mix of presentation (lecture, small group interaction, question/answer), use of calculators, frequency and nature of assessments, use of pedagogical strategies to increase active participation by students.

Out of class expectations: homework policy (including use of web-based tools to grade and/or provide feedback on homework), hours spent studying each week, encouragement of study groups, use of Learning Center, writing assignments, group projects.

Starting from these hypotheses, the CSPCC team worked collaboratively to develop a series of interview protocols. The data collection plan calls for interviews of instructors, calculus coordinators, administrators, students, tutor center personnel, and representatives of client disciplines. Additional data collection includes collection of assessment and program evaluation documents, exams, and placement tests. The data collected (not including collected documents) for the pilot case study is detailed in Table 1.

Data Collection Protocol	Interview Subject
Department Chair Interview	Chair of Mathematics Department
Instructor Interview	Instructor A, B, and C
Calculus Coordinator	Instructor A
Placement Sub-Interview	Instructor B
Student Focus Group Interview	Students of Instructor A (n=3) and B (n=7)
Dean Interview	Dean of the College of Arts and Sciences
Client Discipline Interview	Associate Dean – School of Engineering
Learning Center Interview	Tutor Center Director
(Modified) Learning Center Interview	Freshman Resource Center Director
Tutor Interview	Undergraduate Tutor
Teaching Center Interview	Director of Center for Teaching and Learning
Classroom Observation Protocol	Classroom of Instructor A, B, and C

Selected Preliminary Results of the Research

Analyses (of audio recordings) of interviews are being guided by the initial hypotheses described above and by a set of analytic codes that emerged during an initial open coding phase. Note that the goal of the study is to produce explanatory case studies. Our preliminary analysis is an initial step toward this goal and has resulted in the identification of several factors that the pilot study participants consider to be crucial to the success of their calculus program. These include:

- Placement
- Coordination
- Culture
- Resources and Administrative Support
- Faculty
- Freshman Support System
- Regular Auditing and Review of Progress (Resulting in Action)

Here we will briefly discuss placement and culture:

Placement. One year prior to our data collection, the department adopted a placement exam developed by the Mathematical Association of America. Since the adoption of this placement exam, the pass rate in Calculus 1 has increased from 78% to 89%. The exam is taken during the students' senior year of high school and the results are sent to the freshmen scheduler at the university. Every student must take the placement exam, including those who have passed Advanced Placement (AP) exams. The department also recently implemented a rule establishing a C- as the cutoff for passing Calculus 1 (previously it was D-). One important impact of these changes is that the previous stigma attached to taking pre-calculus at the institution has been greatly diminished as pre-calculus enrollment has increased dramatically.

"One of the things that the placement test has done is it's made it ok to take pre-calculus. When I first taught it, I had nine students in pre-calculus. Teaching that class was horrible since all the students felt like it was shameful and they were embarrassed. Now we have two classes of 30 and it's wonderful." –Instructor C

Culture. Teaching is the top priority at the university (even Deans are required to teach a course). The calculus professors (almost always tenure track faculty) have very generous office hours and constantly encourage students to come for help, using strategies such as requiring them to come to office hours to pick up their graded exams. The calculus instructors have semi-regular meetings and frequent informal conversations about teaching and learning. This is an observation that was made by almost everyone that we interviewed including the Dean, the department chair, the faculty and students.

"From what she's mentioned in the class, and the few times I've seen her in office hours, it seems like they're like a little math family. They're a little math community. Everyone's always talking... I've seen them go into other people's rooms and you hear like laughing coming down the hallway sometimes. – Student of Instructor A

Implications for Practice or Further Research

The CSPCC project has aims to have a significant impact on practice by providing models of successful calculus programs. One form these models will take is that of explanatory case studies of programs identified as being successful. The research reported here represents the beginning of this process. In the immediate future, we will need to continue triangulating the various data sources within the pilot study data set. For example, it is mentioned above that informal communication among the calculus instructors was something that was seen as important by instructors, administrators and students. However, we also identified factors that were mentioned by some interview subjects (e.g. good communication between the client disciplines and the math department) but were not mentioned (and in some cases strongly disputed) by other subjects. This kind of cross-checking across interview protocols will allow us to determine whether given factors actually have a significant presence in the program and the extent to which they are seen to be crucial by program participants. As the project continues to unfold, future research activities will include cross case analyses (across institutions within a single type and across institution types). In particular, such analyses will help the research team to recognize which factors appear to be essential and which may be helpful but are not essential (e.g., those that are not present in a number of successful programs).

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NOT ALL INFORMAL REPRESENTATIONS ARE CREATED EQUAL

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Some mathematics educators and mathematicians have suggested that students should base their proofs on informal reasoning (Garuti et al. 1998). However, the ways in which students implement informal representations are not well understood. In this study, we investigate informal representations made by undergraduates during proof construction. Their use of informal representations will be compared to mathematicians' use of informal representations as described in Alcock (2004) and Samkoff et al. (2012). Further, an analysis of different types of informal representations will investigate the necessity to treat these different representations more carefully in the future.

Key words: Informal representations, proof, proof construction

Mathematicians use informal representations, including reasoning from graphs, diagrams, and specific examples of more general concepts, to guide their proof construction (Thurston, 1994; Burton, 2004; Hadamard, 1945). For this reason, mathematics educators argue that students should base their proofs off of informal representations as well (Garuti et al. 1998). However, mathematics educators have also documented that students are not always successful when basing their proofs off of informal representations (Alcock & Weber, 2010; Pedemonte, 2007). In order to learn what distinguishes students who are successful when using informal representations to guide their proof production from those who are not, students' attempts to use these informal representations must be studied. This study is intended to address the need to better understand how students use informal representations in their proof productions.

Theoretical Perspective

We define a representation in mathematics to be a visual or algebraic portrayal of a concept or a situation. A formal representation would consist of a formal, rigorous definition, whereas an informal representation is any representation that is not formal. As such, example objects and visual representations of mathematical concepts are both considered informal representations in this study.

Garuti et al. (1998) argue that students' proofs should be based on informal reasoning, describing a cognitive unity that should exist between the way a student comes to informally understand the veracity of a theorem and the formal proof. This is consistent with Raman (2002) and Weber & Alcock (2004), who suggest that it is desirable for students' proofs be based on informal representations. In order to investigate students' use of informal representations, we compare students' proof productions against the ways in which mathematicians use informal representations based on the typologies of Alcock (2004) and Samkoff et al (2012).

Alcock (2004) addresses the ways in which mathematicians use example objects in producing a proof. Mathematicians were found to use examples to understand statements by (1) instantiating example objects, (2) generate arguments both directly and indirectly, and (3) check arguments by considering counterexamples. Alcock (2004) describes how mathematicians generate arguments directly by "arguing about or manipulating a specific example and translating this to a general case" and indirectly by "trying to construct a counterexample and attending to why this is impossible" (p. 21).

Samkoff et al. (2012) noted the ways in which mathematicians used diagrams in producing a proof. Mathematicians used diagrams to (1) notice properties and generate conjectures, (2) estimate the truth of an assertion, (3) suggest a proof approach, (4) instantiate or represent an idea or assertion in a diagram, and (5) verify the theorem using the diagram.

Research Questions

Within the framework of this study, which suggests that informal representations should be used in proof production, a stronger understanding of how students use informal representations can help mathematics educators to understand why students are sometimes unsuccessful in their proof mathematics courses. We hope to address the following questions:

1. Do students use informal representations, including examples and diagrams, for the same purposes that mathematicians do (as claimed by Alcock, 2004, and Samkoff et al, 2012)?
2. Do different types of informal representation lead to different uses?
    a. Do visual representations lead to different uses than algebraic representations?
    b. Do representations of specific mathematical objects lead to different uses than representations of general mathematical objects?

We note that this study is in the scope of linear algebra and calculus. As such the results of our analysis may not be generalizable to all areas of mathematics.

Methods

Data Collection

Participants were mathematics majors from a large public university in the northeastern United States who had completed their mathematics requirements for graduation. Data was collected in form of individual task-based interviews. We designed the tasks such that they could be approached both syntactically and semantically, so that informal representations had the opportunity to play a significant role in proof production. Students completed 14 proof tasks (of varying difficulty with 7 in calculus and 7 in linear algebra) during two meetings. Students were asked to think aloud and were given ten minutes for each task. Definitions and examples for relevant concepts and computer graphing software were accessible to the participants. From these interviews we have both the video data and the student work.

Analysis

Coding. Using the data, we identified over 270 informal representations. Each informal representation is analyzed for subsequent actions that followed the construction of a representation. Such actions are coded as one of the following: making an inference, giving an explanation, modifying a visual representation, verifying the theorem, verifying a statement (that is not the theorem), suggesting a proof approach, constructing counterexamples, or using a counterexample strategy. These actions are described in further detail in the Appendix. Actions were coded only if they were either immediately located or had explicit reference to the informal representation.

Question 1: Comparing mathematicians and students. In order to investigate this research question, we analyze students' use of informal algebraic representations in relation to Alcock (2004) and use of informal visual representation in relation to Samkoff et al. (2012).

Use of informal algebraic representations. To investigate students' use of informal algebraic representations, we employed the typology as described in Alcock (2004). To affirm that students and mathematicians use informal algebraic representations for the same purposes, we would expect to find students using informal representations to understand statements, generate arguments, and check arguments. For evidence of understanding statements, we would expect to find codes of verifying statements that are not the theorem to be proved; for instance, verifying particular concepts in the given statement in the proof, statements made by the student, and definitions are examples of understanding statements.

Next, in generating arguments there are two situations as described by Alcock (2004). First, students directly generating arguments would entail that we find codes of giving explanations. Second, students indirectly generating statements would suggest that we find counterexample strategy codes. Finally, to find evidence of students using informal representations to check arguments, we would expect to find codes of students constructing counterexamples with the intent of checking an argument.

Use of informal visual representations. To investigate students' use of informal visual representations, we employ the typology described in Samkoff et al. (2012). To affirm that students and mathematicians use informal visual representations for the same purposes, we would expect to find: a) inference codes as evidence of students noticing properties and generating conjectures, b) verifying statement codes to show students estimating the truth of an assertion, c) students suggesting a proof approaches, d) modification of visual representations as evidence of students instantiating or representing an idea or assertion in the diagram, and d) codes of verifying the statement to show students validating the theorem using a diagram.

Question 2: Different types of informal representations. During our analysis, we noticed that there were four types of informal representations, as exemplified in Table 1. Informal representations were classified not only as specific or general, but also as visual or algebraic.

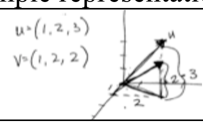
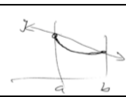
	Sample representation	Description
Specific-Visual		Graphs and diagrams representing specific situations/objects.
Specific-Algebraic	$\begin{pmatrix}0\\0\\2\end{pmatrix} \in W$	Algebraic representations using numbers to exhibit specific situations/objects.
General-Visual		Graphs and diagrams representing classes of situations or general objects.
General-Algebraic	$f(x)=x^{2n}$ $f'(x)=2nx^{2n-1}$	Algebraic representations using variables to exhibit classes of situations or general objects.

Table 1. Four types of informal representations.

In order to investigate whether different types of informal representations lead to different uses, we will analyze the total counts of each type of action code that follows each type of informal representation (specific visual, specific algebraic, general visual, and general algebraic). Considering the percentages of the occurrences of each action code per type of informal representations, we would expect there to be differences in the percentages to affirm that different informal representations lead to different uses.

Discussion

This study is in its preliminary stages and analysis has not yet been completed. However, we have coded one quarter of the data and have some tentative preliminary results. We have thus far not found evidence for students using informal algebraic representations to generate arguments or to check arguments in the sense of Alcock (2004). In contrast, we have found evidence that students use informal visual representations for purposes similar to the mathematicians in Samkoff et al. (2012).

Further, our analysis to date also suggests that students do use specific visual, specific algebraic, general visual, and general algebraic informal representations for different uses. For example, students so far appear to use specific algebraic representations to understand statements more frequently than they do with generic ones. Students also appear to use specific visual representations to estimate the truth of an assertion, whereas they have not done so with generic visual representations in our analysis so far.

These results may shed some light on why some students are unsuccessful in their proof productions. Thus far, it appears that students may use visual representations in the same manner that mathematicians do, but that this is not so with algebraic examples. If students are not in the habit of generating and checking arguments using informal algebraic representations, as our analysis to date suggests, they can perhaps improve their proof construction if they can be encouraged to do. However, these results are preliminary so we make no claims of generality.

If these trends continue throughout the analysis, then these results would suggest that more attention should be paid to the instruction of students' use of algebraic representations. Moreover, the result that students use visual representations in ways that are similar to mathematicians would lead to further questions. For example, if students and mathematicians use visual representations in their proof productions in similar ways, why do students continue to struggle in their proof productions?

Next, if the analysis on the different types of representations continues to show that these four different types of representations lead to different purposes, the results would highlight the need to treat these informal representations differently – both in future studies and in presentation to students.

Questions for the Audience

Are there other frameworks for mathematicians' use of examples or diagrams that we did not consider? Are there other ways to code our data?

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Appendix

Description of Actions that Follow Informal Representations

Action	Description of Action
Making an inference	A participant notices something that is true about an example or diagram that was not yet under consideration.
Giving an explanation	A participant mentions what he believes to be general properties of the informal representation that make a statement true.
Modifying a visual representation	A participant modifies or transforms an informal visual representation that has been considered previously.
Verifying the theorem	A participant constructs a graph or example and explicitly comments that he/she thinks the theorem to be proved is or is not true.
Verifying a statement	A participant constructs a graph or example and explicitly comments that he/she thinks a statement (not the theorem to be proved) is or is not true.
Suggesting a proof approach	A participant notes a proof approached based on the informal representation.
Constructing counterexamples	A participant constructs counterexamples for the purpose of verifying a statement or an argument.
Using a counterexample strategy	A participant attempts to construct counterexamples for the purpose of seeing why such counterexamples cannot exist.

RETHINKING BUSINESS CALCULUS IN THE ERA OF SPREADSHETS

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Abstract: The author is writing an electronic "book" to support the teaching of calculus to business students with the assumption that they will use a spreadsheet as their main computational engine. With the change in technology, it is appropriate to rethink the content of the course, as a different technology makes different tasks accessible. This study looks at what the content of the course should be. It compares the official learning objects of the course, with de facto learning objectives obtained by analyzing final exams from 20 sections of the course, and with the results of a survey of the faculty of the business school, the client discipline. It is intended that this preliminary study will establish a baseline that can be used to evaluate the effectiveness of the new approach to the course.

Key Words: Applied Calculus, Spreadsheets, Technology And Syllabus, Client Discipline Expectations

This report represents an attempt to look at the first issues of the scholarship of teaching and learning that arise in an effort to rethink business calculus. The underlying project looks at reforming a course that is typically a one semester, multiple section, terminal mathematics, service course offered for business students and which is typically taught by adjuncts or teaching assistants. A central premise in the rethinking of the course is that there are pedagogical advantages of teaching a terminal service course using the discipline standard software of the client discipline. For a business calculus class, that would call for using spreadsheets as the main computational engine. From a practical point of view, it is noted that in the last 5 years it has become reasonable to assume that business students will come to college with a laptop computer and office software. However, the teaching staff for business calculus typically has little or no training in business or in software used in a business setting. From a practical point of view, it is unrealistic to assume that schools will devote substantially more material to the course either in credit hours in the curriculum, or in quality of teaching resources. Thus the project assumes the rethought course needs to be teachable by adjuncts and teaching assistants with little training in business.

The rethinking of business calculus with the availability of spreadsheets brings up an interconnected web of issues for the larger project. Changing the technology used in the course changes the topics that are accessible. In particular, it becomes possible to routinely include problems where the students are given data and need to choose a mathematical model and fit the data to the model. Spreadsheets also make it reasonable to address problems of rate of change and accumulation where the underlying function is not continuous. Changing to a spreadsheet should involve a reexamination of conventions used in mathematics classes. In the standard mathematics convention formulas should be compact and use single letter names. This convention aids symbol manipulation. Conventions for spreadsheets emphasize readability, so more descriptive variable names are used and more steps broken down. The use of data defined problems and the development of the programming skills in spreadsheets implies the exchange of electronic documents rather than allowing evaluation on paper and pencil tests. Since the teaching staff will include adjuncts, the material must address the training of instructors without a background in business. The author envisions the development of the underlying project will provide material for a series of SOTL projects.

In context, the underlying project fits in with calls from the professional organizations. In its report, *The Curriculum Foundations Project: Voices of the Partner Disciplines* (2004), the Curriculum Reform Across the First Two Years (CRAFTY) subcommittee of the Mathematical Association of America (MAA) made a number of suggestions for revisions of lower division mathematics courses aimed at partner disciplines. The report on business and management was the result of a conference at the University of Arizona in 2000. The report made a number of recommendations, including using spreadsheets rather than technology designed for the sciences, written reports with an emphasis on communication skills, and working with problems defined by data rather than formulas so that modeling is given greater emphasis.

In the 12 years since the conference in Arizona, the author is aware of only two projects to rethink the teaching of business calculus by reworking the course with the assumption that students will use spreadsheets as their primary computational tool for the course. The two projects are from Appalachian State University (Felkel and Richardson) and University of Arizona (Lamoureux and Thompson). The author has contacted the authors of both of those projects and in both cases wan told the authors neither made nor is aware of any SOTL studies connected with their projects.

The naïve background research question of the larger project is “Does the daily use of spreadsheets in a business calculus class lead to an increased success of the students in achieving the learning objectives of the course?” The problem with the naïve “What works?” question is that it assumes the objectives of the course are abstractly defined without any consideration of what is possible. At this preliminary stage a more appropriate question looks at what can be and compares it to what already is. The main research question at the center of the preliminary study is: “If Excel is available to students on a daily basis, how should the learning objectives of a one semester course in business calculus be reshaped to better meet the desires and expectations of the faculty of the business school?” The companion question to what can be asks about what is to give context. Thus, the study also looks at he current learning objectives, both those formally stated, and the de facto objectives deduced from looking at old exams.

A review of the literature failed to find any publications directly on point other than the CRAFTY report previously mentioned. There is some literature on the use of Excel in a business calculus class, but it all looks at using Excel as a supplemental tool where the text and syllabus was devised assuming no technology was available. (E.g., Liang and Martin, DuPont.)

For the preliminary study the author conducted a survey of the faculty of the business school. Respondents were asked to rank objectives on a scale from 0, not at all important, to 10, essential. Approximately 30% of the 60 faculty members responded. The survey listed 25 objectives broken into 4 groups. To establish a context, the author also looked at information from 2 other sources. The course has an official set of learning objectives published by the math department. To establish a de facto set of learning objectives the author grouped the official learning objective list into 9 clusters of objectives, then did an analysis of a random sample of 20 of the approximately 50 final exams for the course over the past 5 years to see how possible final exam points were distributed. (Table 1)

Table 1: De facto objectives taken from final exams from 20 sections

Cluster of learning objectives	Percentage of points
Algebra and pre-calculus skills	12.
Limits and continuity	11.4
The limit definition of derivative	3.6
Derivatives and graphs, finding max symbolically	16
Symbolic differentiation and finding tangent lines	22.6
Derivative word problems	10.4
Symbolic integration	16.0
Multivariable functions and partial derivatives	6.4
Miscellaneous	1.6

The results of the survey revealed some methodological problems with the construction of the survey. The author did not anticipate the extent to which the respondents would make differing use of the scale with one respondent marking 21 of 25 objectives as essential, while another respondent marked that none of the objectives were essential. To correct for this use of scale the author looked at which objectives each respondent put in the top and bottom half on in that respondent's ranking of the objectives.

Table 2: Results of faculty survey. Objectives and number of business faculty who ranked each objective as more essential than their mean response.

Math 132 is a calculus course that is typically the last math course taken by CSB students. For such a course, how important would you rank the following in terms of being a major objective of the course?	Times rated above mean (Of 18)
understanding the concepts of calculus	16
skill in proving theorems of calculus	2
skill with the mathematical techniques of calculus	12
skill in mathematical reasoning that comes from calculus	15
For general math course taken by CSB students, how important would you rank the following in terms of being an outcome of the course?	
skill with a graphing calculator	2
skill with using Excel for business problem-solving	13
skill in problem solving	18
skill with modeling data to formulas	18
skill with written presentations	10
skill with making logical arguments	15
skill with manipulating mathematical formulas	14
The proposed shift to Excel in MATH 132 allows some change in emphasis. How important are the following skills for CSB students taking this course?	
applying calculus to examples from business and finance	16
using mathematical reasoning in written presentations	2
working with functions that are only defined for whole number inputs (e.g., the amount of a payment as a function of the number of payments)	12
building and documenting templates that can be used for a type of problem	15

The current MATH 132, like most traditional calculus courses, has a heavy emphasis on symbolic manipulation skills. As we consider adjusting the content of MATH 132, how important are the following calculus skills for CSB students? (These are current student learning objectives of the course.)	
Computing with basic functions: lines, quadratics, etc.	12
Computing with exponential and logarithmic functions.	11
Demonstrate an understanding of limits and continuity.	2
Use the limit definition to compute derivatives.	3
Use differentiation rules to compute derivatives.	7
Use derivatives to sketch graphs of functions.	7
Word problems with derivatives, solve marginal analysis, related rates, and optimization problems.	13
Finding anti-derivatives and indefinite integrals	2
Solving word problems with integration and accumulation.	5
Understanding functions of several variables and computing partial derivatives.	7

The results of the survey showed some marked differences between the desires of the faculty in the client discipline and the de facto objectives determined by points possible on the final exams. A rough summary of the de facto objectives is that the course is modeled after the standard 3-semester calculus sequence designed to prepare for a major in physics, before graphing technology was readily available. To fit the course in a single semester, the emphasis is on symbolic manipulations skills and computational techniques. Virtually no problems required the use of any technology or the fitting of data to a mathematical model. Symbolic integration was considered a more important topic than functions of several variables.

The survey indicates business faculty members are looking for a substantially different course. Skill with modeling data to formulas was in the top half of the objectives for all respondents. Skill with using Excel for business problem solving was in the top half of the objectives for two thirds of the respondents, which had it more highly ranked than any of the computational objectives.

To a large degree the survey confirmed what the author had gathered from anecdotal evidence and the results of the CRAFY report. The student learning outcomes for business calculus desired by the faculty of the business school differ from the learning outcomes implicit in the current structure of the course. They are looking for a course that is more conceptual and less focused on computational technique. They are looking for a course with greater use of business technology, use of data, and applications. There was also an unexpected difference in ordering of importance of topics. The business faculty considered partial derivatives to be significantly more important that integration, while the final exams showed that the teachers placed much more emphasis on symbolic integration.

This study will help provide a baseline for determining objectives of student learning that should be measured in a further study. It will also lead to an adjustment to the syllabus to make both the standard and reformed sections more responsive to the needs of the client discipline.

Audience questions:

1) As the larger project moves forward, the author anticipates replicating this study with faculty from other business schools. What changes should be made to improve the study and gather better information.

2) For a period of at least two or three semesters, business calculus will be taught in both a traditional manner and in an Excel based approach. While the goals and syllabi will overlap, they will be distinct. What data should be gathered during the transitional period? (Possibilities include: responses to student evaluation questions about how much time the students put into the course and how relevant they see the course as being to their major, how student grade performance compares on an economic course that has business calculus as a prerequisite.)

3) Part of the larger project includes producing materials that allow the course to be effectively taught by adjunct faculty with a background in mathematics, but little training in business. What data should be gathered to evaluate the effectiveness of the teacher material?

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BRINGING THE FAMILIAR TO THE UNFAMILIAR: THE USE OF KNOWLEDGE FROM DIFFERENT DOMAINS IN THE PROVING PROCESS

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This report considers student proof construction in small groups within an inquiry-orientated abstract algebra classroom. During an initial analysis, several cases emerged where students used familiar knowledge from another mathematical domain to provide informal intuition. I will report on two episodes in order to illustrate how this intuition could potentially aid or hinder the construction of a valid proof.

Key words: [Abstract Algebra, Proofs]

A great deal of attention has been given to student construction of proof particularly in an abstract algebra setting (Weber & Larsen, 2008). Hazzan (2001) has observed that undergraduates' difficulties in group theory may partially be attributed to the abstraction level. Both Hazzan (2001) and Selden and Selden (1987) found that students often retreat to familiar number systems when working on algebra proofs.

As part of a larger project implementing a group theory curriculum based on guided reinvention of group and isomorphism concepts, this study considers student proof construction in small groups. Through an initial analysis, several cases emerged where students used familiar knowledge from another mathematical domain to provide informal intuition for an argument. This divergence from abstract algebra is consistent with the return to familiar contexts seen with Hazzan (2001) and Selden and Selden (1987) but unfolds in a different manner. In this proposal, I will present two such cases where students utilized prior knowledge in an unexpected way during the informal stage of their proof development.

Theoretical Background

Zandieh, Larsen and Nunley (2008) discussed the role of student intuition as students moved from informal notions to formal proofs. The researchers suggested, "As students search for key ideas and work to relate a key idea to the arguments needed to provide a rigorous proof, they often need to develop an intuitive sense of how the system in question works" (p. 122). The key idea connects intuition with a formal proof (Raman, 2003). Referencing Fischbein (1982), Zandieh, Larsen and Nunley (2008) categorized two types of intuition in proof: affirmatory and anticipatory. The former can be coercive in that "an individual with this intuition may not be able to consider other alternatives" (p. 126). Whereas, anticipatory intuition is associated with a feeling of certitude, but the "intuitions anticipate a further refinement" into formal proofs (p. 126).

I will consider two episodes where informal notions emerge when students bring prior mathematical knowledge into the abstract algebra context. The first example will illustrate affirmatory intuition whereas the second will illustrate anticipatory intuition.

Research Methodology and Problem Context

Four implementations of an inquiry-based Algebra course were videotaped over the course of two years. This course largely served as an introduction to group theory where many students were engaged in proving for the first time. Video data from these implementations were analyzed in this report. An initial search served to isolate the portions of the curriculum where students were prompted to prove in a small group setting. During this process, I noticed several occurrences of students referring to mathematical contexts outside of abstract algebra while in the early stages of constructing a proof. After consulting the literature, these videos were analyzed again with attempts to categorize the intuitions as affirmatory or anticipatory.

I will consider episodes from two classrooms implementing a curriculum where formal abstract algebra concepts are developed using students' informal knowledge. At this stage, the students have developed a familiarity with the symmetry group of an equilateral triangle. They have developed the symbols in terms of R and F to represent a 120 degree rotation and a vertical reflection respectively. Through their investigations they've developed a list of rules that describe the composition of symmetries (see Figure 1). Typically, the existence of inverses are not included. For more details of this curriculum see Larsen (2012).

<u>Rules</u>

The associative property: If A, B, and C are symmetries then $A(BC)=(AB)C=ABC$.

The identity property: Any symmetry combined with I is that symmetry. If X is a symmetry then $IX = X$.

$RRR=I$ and $FF = I$.

$RF = FRR$.

Figure 1. Student Rules for Triangle Symmetries.

While creating a table for the symmetries, students notice the pattern that each symmetry appears exactly once in each row. In order to motivate the idea of inverse elements, the prompt in Figure 2 was given. The students worked on proving both parts in small groups.

Conjecture: Each symmetry appears exactly once in each row of the table.

Can we prove this conjecture using our rules? If not, can we come up with another rule to add to the list that will allow us to prove the conjecture?

Hint: This conjecture can be broken down into two parts, work on one at a time.

1. Each symmetry appears at most once in each row.
2. Each symmetry appears at least once in each row.

Figure 2. Prompt

Results

The following cases illustrate students making connections to familiar mathematical contexts when beginning the informal process of proving the conjectures above. Each case comes from a different class where groups of four students worked together to attempt to prove the statements. The first case serves as an example where the intuition did not lead to the construction of a proof.

The second case illustrates how a student using prior knowledge provides the intuition for the key idea of inverses.

Case 1: Prime Numbers

In this first case, Bob began the discussion by introducing an analogy to prime numbers stating, “It's almost like prime numbers. You can have the composition of all of them but they trickle down to a single action. And it's 8 distinct actions, or I'm sorry not 8, 6 distinct positions [referring to the six symmetries of the triangle] in terms of orientation and location.” He continues his analogy, “Kind of like in the sense of a prime number, you can do all sorts of things to it to make it look different, when you factor it out into a prime you can't go any further. You can't split it any further without breaking it into a decimal.” Bob is attempting to connect his ideas of symmetries with his knowledge of prime number factorization.

Bob and his group-mates continue the conversation with reference to this simplification idea.

Roger: I see the spirit of what you are talking about and it's something in that uniqueness that you get that if you don't have ...if there are no repeats in the rows and columns that you are going to compose, then you can't get a repeat in that row or that column.

Bob: Because it always factors down to something that's already there.

At this point Roger and Bob are attempting to coordinate the connection to prime numbers with the uniqueness of each element in a row. This connection appears backwards since they were considering the multiplication of symmetries as opposed to factoring. They may have been confounding the idea of closure (that any combination of symmetries reduces to a known symmetry) with the property of appearing exactly once in each row. Alternatively, they may be thinking of one of the factors being held constant acting the same as a symmetry multiplied on a given row. This latter interpretation would align with the statements that follow. After the prime number discussion, Winston presented the following proof:

Winston: Since there are six unique functions and in each row and column these functions are composed with exactly one unique function that row or column must contain six unique functions because they all are equal to the same function.

Gayle: They are all unique so anything you do with them is going to be unique.

I would hypothesize that this prime number metaphor was serving an affirmatory role. All proof attempts resembled Winston's above where the uniqueness served as the reasoning. None of the group members considered justifying further. Instead, the intuition acted as a hindrance to the construction of a valid proof.

Case 2: Invertible Functions

In this second case, Logan contributed knowledge of invertible functions to make an argument for the conjecture:

Logan: Each of the operations is invertible because for any of the operations like R there is something you can compose with that to turn it into the identity. So if you have R and R squared and put them together, then you get the identity. And since it's invertible then it must be, um, one to one. I don't remember [inaudible.] Since it's invertible the same entry can never appear in the same row-

Henry: How does it being invertible prove that?

Logan: Because say you take FR and what's the inversion of that $(FR)^{-1}$. Actually, FR is the inversion of itself. So you have a bunch of starting points and you are mapping using a function FR and we know that if we do FR again, then it's going to map *them all back to the same place, so we know that we can't ever map two to the same one or we wouldn't be able to map it back. Because we know that FR FR equals I. So that means for any input rotation FR. For any two input rotations into FR, you can't get the same output rotation* (see Figure 3).

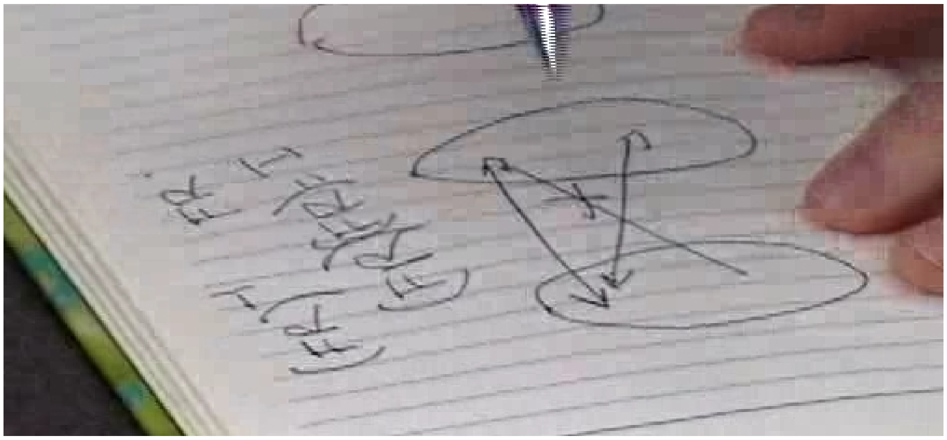

Figure 3. Logan's Function Diagram for the inverse of FR

Logan went on to conclude that “For each of them there is an inverse. Because they are invertible, they all have to be 1-1 and onto.” As evidenced by the diagram in Figure 3, Logan was able to build an intuition about the use of inverses. I would conjecture that this knowledge came from precalculus or other study of functions based on his use of a function diagram. Logan was identifying symmetries with functions mapping the group onto itself.

I would argue that this connection to invertible functions served in more of an anticipatory role. Logan used his function diagram and connected it to each symmetry having an inverse. The anticipatory intuition was further evidenced later in the conversation as the group attempted to formalize the idea culminating in Henry suggesting a proof by contradiction with direct reference to Logan's inverse suggestion.

Discussion

These episodes demonstrate two very different paths into a familiar domain. In the first case, the student group created an analogy to prime numbers that ultimately left them unable to prove the conjecture. In the second case, the introduction of invertible functions provided the foundation for the key idea of inverses. Further research into the types of connections made by novice provers could help inform instruction in order to encourage students to develop the intuition necessary to become successful in proof construction.

Questions for the Audience

1. What role can instructors play to encourage the formation of anticipatory intuition? Likewise, how can instructors help students move beyond affirmatory intuitions?
2. Have you noticed students appealing to prior mathematical knowledge during the intuition stage of proving?

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DEVELOPMENT AND ANALYSIS OF A BASIC PROOF SKILLS TEST

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Abstract: *We have developed a short (16 question) basic skills test for use in our institution's transition-to-proof course that assesses basic skills required to succeed in such a course. Using this test in our core introductory proof course, we have found that students are generally deficient in a number of skills assumed by instructors. In addition, using this test as a pre/post-test we have found that in this course students are learning some concepts well, but that learning gains on other concepts are much below desired levels. Finally, administration of the test to students in a higher level course has allowed us to assess retention of these skills. At this preliminary stage these skills appear to be retained into higher-level proof courses, but more data collection is needed, as well as a more extensive instrument to assess proof skills, rather than simply basic logic and comprehension.*

Keywords: Classroom research, Transition to proof, Retention, Basic skills

Introduction:

Our institution has a typical transition-to-proof course (Math 220) intended to bridge students from the computationally intensive calculus stream of courses to upper division proof-intensive courses. As is generally the case in such a course, students encounter a great deal of difficulty and instructors express a large degree of frustration with the poor learning outcomes. One source of this frustration appears to be a mismatch between students' incoming skills and those assumed by instructors. We have therefore endeavoured to identify these skills and assess them in the incoming population. An instrument we have developed to assess these skills is the “basic proof skills test” (BPST). This is a short (16 question) multiple choice test developed over several terms through a combination of instructor interviews, student interviews, and iterative development of test items, following the procedure outlined by Adams & Wieman (2011). The test is still under development, but at this stage it consists of 4 questions on algebra and graphing (hereafter the “Precalculus” content) and 12 questions on logic, mathematical quantifiers, and mathematical definitions (hereafter the “proof skills” content).

We developed this test to be used as a pre/post test in the core introductory proof course at our university with two main goals: (1) to assess students' skills at the start of the course, and hence to tailor instruction to address deficiencies and build on key skills and (2) to track learning gains for these skills over the term of the course. In addition, administration to students in a higher level course can allow us to assess retention of these skills. Over the past year we have administered the test to several sessions of the course, as well as in our introductory analysis course. Here we present the results of these and describe their implications on our teaching in these courses as well as on further improvement of the test for the future.

Methods:

We began with interviews with past instructors of Math 220, as well as with instructors of downstream courses (proof-intensive 300-level courses for which Math 220 was a pre-requisite). These were largely unstructured interviews where the discussion focused on (a) identifying what proof skills students were expected to have at the start of each of these courses, and (b) what common difficulties instructors have observed. Upon completion of these interviews, we constructed an initial version of the test that focused on the most basic of these expected skills. One of the authors was also an instructor for Math 220 for several years and so we relied heavily on his experience in this course in the development of the test. Test items were drawn from a number of sources. Several were adapted from the Field-Tested Learning Assessment Guide (FLAG) (Ridgway et al, 2001), and the remainder were drawn from local precalculus exams, Math 220 exams, or were newly created. Wherever possible, test items were first run in an open-ended form and the most common incorrect answers were used to create distractors in the final multiple-choice version. The initial test was created in Sept 2010 and since then it has been administered in Math 220 a total of 12 times, with 3 substantial revisions and many minor revisions. Table 1 shows the timeline of the development of the test.

Table 1. Timeline of the development of the BPST

Version	Changes	Date of Creation	Dates Administered
V1 – open-ended and multiple choice	Initial version	Sept 2010	Sept 2010 (pre-test), Dec 2010 (post-test), Jan 2011 (pre-test)
V2 - open-ended and multiple choice	Some questions removed, some new questions, and some questions converted to multiple choice	April 2011	April 2011 (post-test)
V3 – fully multiple choice	Some questions removed, some questions converted to multiple choice	May 2011	May 2011 (pre-test), July 2011 (post-test) Sept 2011 (pre-test), Dec 2011 (post-test) Jan 2012 (pre-test), April 2012 (post-test) May 2012 (pre-test), July 2012 (post-test)

In addition to the written answers on the test items, we used student validation interviews (in the form of think-aloud interviews) to assess the quality of the test, as well as we computed standard classical test theory statistics on each item of the test (item difficulty index, item discrimination index, item-to-total correlation, and item characteristic curves). Test items were removed or modified at each stage based on these interviews and statistics.

Results:

The pooled pre- and post-test scores for the current version (V3) of the test are shown in Table 2, as well as the normalized average learning gains (NALG). Note that only students who wrote both the pre- and post-test are included. NALG are computed using matched pre- and post-test scores for each student and are computed using the formula NALG = (<post-test> – <pre-test>)/(1-<pre-test>), where <pre-test> is the mean pre-test score (in %) and <post-test> is the mean post-test score (in %). The NALG therefore is the proportion of the total possible gain, given the particular pre-test mean score.

Table 2. Mean scores on the BPST and normalized average learning gains.

	Math 220 Pretest (N=192)	**Math 220 Post-test (N=192)**	**Normalized Average Learning Gains**	**T-test p-value for M220 pre- and post-test means**
Full test (out of 16)	9.63	11.83	34.58%	0.0006
Precalculus component (out of 4)	2.19	2.61	23.34%	0.1907
Proof skills component (out of 12)	7.44	9.22	39.04%	0.0005

As we can see from the NALG of ~35%, although students are clearly improving on some of the skills in the test, instructors are justified in their impression that students are not learning as much in this course as desired. Some of this is due to poor precalculus skills, but when we examine the test on a question-by-question basis, we see that there are a few proof concepts that students are consistently failing to improve on. As an example, we can consider problem 12 on the test, shown in Figure 1.

Figure 1. Problem 12 from the BPST

For each of the statements below, indicate whether each statement is
(a) always true: true for any choice of the variables
(b) sometimes true: true for some variable choices, but not for all choices, or
(c) never true: not true for any choices of the variables.

12. for real numbers x and y,

$$\sqrt{x^2 + y^2} < x$$

(a) always true
(b) sometimes true
(c) never true

This problem asks students to consider an open sentence and determine if the statement is true for all variable choices, true for only some variable choices or never true for any variable choices. Even at the end of term, only 71% of students correctly answered this problem. For another problem, given three “proofs” of the statement, students are asked to choose the proof that is correct and complete (there are 3 choices, one of which is incorrect, and one is incomplete). Even at the end of our transition-to-proof course, only 60% of students choose the proof that is correct and complete. In addition, the majority of wrong answers choose the incorrect proof that starts by explictly assuming the statement to be proved.

Discussion and Future Directions:

We have found that the BPST can be useful for identifying student difficulties at the start of term, as well as for tracking learning gains in our transition-to-proof course. Indeed, it has illuminated several persistent misconceptions to target in the course. However, further work is needed to establish the validity and reliability of the test, and also to determine if it would be useful to the broader transition-to-proof teaching community. Specifically, the test items need to be validated with a broad demographic of students to ensure they are being interpreted consistently and that wrong answers are chosen for the reasons that we assume. In addition, it is important to get feedback on the test from faculty at other institutions to ensure content validity.

Another possible use for this type of test in the future is to examine the retention of knowledge and skills to future courses. We have recently collected a small amount of data on this, by administering the BPST to students at the end of the Math 220 course, and then again at the start of the first analysis course, Math 320 – Real Variables I. We have a group of 27 students who have completed both of these tests. The mean scores on these tests are shown in the following table.

	Math 220 Pretest (N=27)	**Math 220 Post-test (N=27)**	**Math 320 start-of-term test (N=27)**
Full test (out of 16)	11.70	13.52	13.63
Precalculus component (out of 4)	2.59	2.93	3.00
Proof skills component (out of 12)	9.11	10.59	10.63

As we can see, the mean score in both groups is nearly identical, suggesting that there is very good retention of these skills to Math 320. In addition, these are not just skills that students already had coming into Math 220 because the mean pre-test score for this group is significantly lower ($p<0.001$). Of course, there are many skills beyond these basic ones that are not included in this test that play an important role in learning in a proof-intensive course. It would be another, bigger challenge to identify these and develop a similar instrument to measure them.

Discussion Questions:

- Are there key basic skills missing from our test? Is the set of basic skills different for other types of introductory proof courses?

- What further work is needed to improve the test and establish its validity and reliability? Also, how broadly useful could such a test be (i.e. specific to our institution and course, or more general)?

- This basic proof skills test has been helpful in identifying student difficulties and also in tracking learning, but we would also like to be able to track higher-level proof skills for large numbers (entire classes) in a systematic way. Is it possible to assess higher-level proof skills with a short test? If so, what format could be used for this?

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ASSESSING PRE-SERVICE TEACHERS' CONCEPTUAL UNDERSTANDING OF MATHEMATICS USING PRAXIS II DATA

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We summarize the preliminary results of a study of conceptual understanding of mathematics by pre-service secondary school math teachers. Our research involves the statistical analysis of data from an actual mathematics Praxis II licensure exam, which was administered nationwide. Through a quantitative, item by item analysis, using a classification of these test items by conceptual difficulty, we obtain insight into the conceptual issues that pre-service teachers have great difficulty with. Our preliminary results show a significant gap between computational and abstract mathematical processes. This in turn, affects the ability of pre-service teachers to be fluent in the domains of both subject and pedagogical content knowledge.

Key words: [Pre-service Secondary Teachers, Mathematics Praxis, Quantitative Analysis, Content Knowledge]

Introduction

The knowledge required for effectively teaching mathematics has been studied by many researchers in mathematics education. Ball, Thames, and Phelps(2008) have investigated this topic thoroughly for pre-service teachers for K-8. Also, Krauss et al.(2008) investigated the topic for secondary school mathematics teachers

In light of these investigations into the knowledge base for teaching mathematics, math teacher educators at the postsecondary level need to examine how well versed pre-service secondary math teachers are in their ability to move through different concepts and ideas in the mathematics they will teach. What are some ways we can obtain insight into students' mathematical understanding and flexibility? This is not an easy question to answer since the mathematical training of secondary teachers in the United States varies widely. However, the need for mathematical understanding and flexibility will be the same, regardless of where mathematics is being taught. In order to gain insight into pre-service teachers' abilities to conceptualize, we examine scores from an administration of a Praxis II Mathematics Content Knowledge Exam. Since virtually all states use the Praxis II as a licensure test, it is practically the only standard measure that cuts across all math teacher preparation programs in the country.

Literature Review

Thames and Ball (2010) indicate that solving every day "teaching problems" demands "mathematical understanding and flexibility". They have formulated a need for pedagogical content knowledge (PCK) as well as subject matter knowledge. Pedagogical content knowledge encompasses knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum

Krauss et al.(2008) investigated the topic of pedagogical content knowledge for secondary school mathematics teachers. In their work, they examine tasks that would assess pedagogical content knowledge in mathematics. These tasks involve moving beyond standard explanations for basic mathematical ideas.

Conceptually, the more difficult questions in the Praxis II exam also involve moving beyond the standard routines and procedures in mathematics. Thus, analyzing these questions can provide a window into the types of associations that pre-service teachers may have difficulty making. This can, in turn, increase awareness of concepts and ideas which need to be taught or emphasized in greater depth in the undergraduate training of math teachers. These concepts and ideas are necessary for *both* subject matter knowledge and pedagogical content knowledge. A teacher with a poor grasp of conceptual underpinnings of the mathematics they teach will be less able to facilitate students' understanding of those concepts.

Research Question and Methodology

Using the data on the Praxis II exams, we would like to ask the question, "what types of mathematical thinking do future teachers have the most difficulty with?". To help answer this question, we analyzed Praxis II exam results with a close examination of items that require connections across multiple mathematical domains and involve unpacking of the underlying knowledge. Our analysis is based on test records of 2299 examinees across the United States, who took the sixty-minute version of the Praxis II Mathematics: Content Knowledge test in November 2008. The data was collected by the Educational Testing Service (2008).

To answer our question, we follow the method of test item classification as proposed by Wainer, Sheehan, and Wang(2000). The results follow.

Results and Discussion

The Praxis II exam in mathematics consists of 50 questions, broken down into categories as follows. These categories are provided by the Educational Testing Service (2008).

Category	Topics covered	Number of questions
I	Algebra and Number Theory	8
II	Measurement, Geometry and Trigonometry	12
III	Functions and Calculus	14
IV	Data Analysis and Statistics and Probability	8
V	Matrix Algebra and Discrete Mathematics	8

Table 1

The percentage of the 2299 respondents who answered each item correctly was given by the Educational Testing Service(2008). The side-by-side boxplots in Figure 1 show the percentage of respondents choosing the correct answers for questions in the various categories.

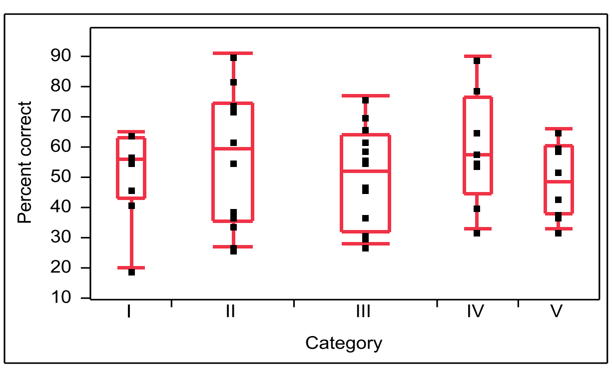

Figure 1

Test items in categories II and IV had a maximum of 90% correct, while Categories I and V had the lowest maximum, at 65% and 66% correct, respectively. Categories I, III, and V represent the more abstract and conceptual portions of the test. Categories II and IV involve more numerical computations.

Wainer, Sheehan, and Wang(2000) also propose a different classification scheme for a more meaningful analysis of test results. In their work, they analyze Praxis I test scores for Education in the Elementary School Assessment according to hierarchically ordered skill levels. Gitomer(2010) analyzes results of Praxis II mathematics exams in terms of major trends using the total scores. We analyze some of the Praxis II mathematics exam results with a close examination of items that require connections across multiple mathematical domains and involve unpacking of the underlying knowledge. To do so, we modified the classification system proposed by Wainer et al.(2000) to reflect the mathematical complexity and processes involved in the Praxis II questions. The classification is as follows, with each of the fifty questions assigned only to a single category.

Category	Processes involved	Number of questions
NC	primarily involves straightforward numerical computation	15
GR	primarily graphical or geometric reasoning	16
AC	involves computation, but uses abstract reasoning, and perhaps links multiple concepts	10
AR	involves abstract reasoning, but no numerical answer is produced	9

Using descriptive data analysis, we found that the students fared significantly worse in the categories of AC (abstract reasoning with computation) and AR (abstract reasoning without numerical answer). Side-by-side boxplots for the percentage of respondents choosing the correct answers for questions in the different categories are shown in Figure 2. The data clearly indicates a need for explicit emphasis on connections of concepts in mathematics.

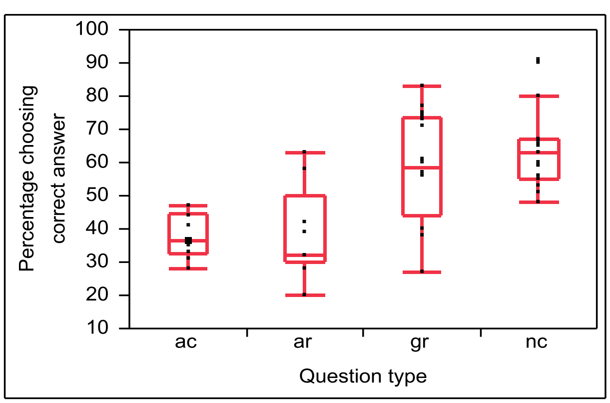

Figure 2

Our work thus far suggests large gaps between computational and abstract understanding of mathematics by pre-service teachers. We expect to further analyze the data with more refined methodologies in testing and measurement. To this end, we plan to modify the research methodology discussed in Wainer et al.(2000) and Sheehan and Mislevy(1990).

We also plan to use this data to investigate implications for the undergraduate teaching of mathematics, and how it can better serve the needs of pre-service mathematics teachers.

Questions

1. Is the proposed classification robust for the type of questions that appear on the Praxis II exam?
2. What are the implications of these results for the teaching of undergraduate mathematics courses, both at the lower and upper level?
3. Should further avenues for exploration expand on the measurement details or on the implications for pedagogy in undergraduate mathematics?

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SCALING UP REINVENTION: DEVELOPING A FRAMEWORK FOR INSTRUCTOR ROLES IN THE CLASSROOM

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Studies have shown that students have difficulty with the concept of limit, especially when reasoning about formal limit definitions. We conducted a five-day teaching experiment (TE) in a second semester calculus classroom in which students were asked to reinvent a formal sequence convergence definition. Oehrtman et al. (2011) detailed how pairs of students reinvented sequence convergence definitions but did not attempt the same instructional heuristic in the classroom. Our analysis focused on the instructor prompts and the TE students' subsequent group discussion through their use of key words and visuals in revising their definition. An interview with the instructor was conducted to investigate his intention of using specific prompts and his thinking about the TE group's choice of words and visuals. In our preliminary analysis, we found that the roles of the instructor were extended beyond those roles previously reported as roles for facilitators with pairs of students.

Keywords: Guided reinvention, Sequence, Limit, Calculus, Classroom Discourse, Instructor Role

Much of the previous literature on limits has focused on students' misconceptions within their informal understanding of limits (e.g., Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Williams, 1991). Students given a formal limit definition can have great difficulty making sense of the intricacies of that definition (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). Recent studies have used empirical results to detail how students come to understand formal limit definitions (Cottrill et al., 1996; Martin et al., 2011; Oehrtman et al., 2011; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012). This paper attempts to add to this body of research by exploring the nature of instructors' role as facilitators of students' discussion in a classroom where students were asked to reinvent a formal definition of sequence convergence by focusing on instructor prompts. We ask, what are the roles of the instructor as a facilitator in a classroom where the instructor is implementing guided reinvention for a formal definition of a limit concept?

Literature Review

Within the theory of Realistic Mathematics Education (RME; Freudenthal, 1973), guided reinvention is an instructional heuristic that aims to position students in experientially real contexts to support the emergence of formal mathematics through students progressively constructing the mathematics for themselves (Gravemeijer, 1998, 1999). Gravemeijer (1999) stated that the "idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 158). Concerning the reinvention of formal limit definitions, recent studies have detailed challenges students face and the importance of engaging those challenges (Martin, Oehrtman, et al., 2012; Oehrtman et al., 2011; Swinyard, 2011; Swinyard & Larsen, 2012), the role of quantitative reasoning (Martin, Cory, et al., 2012), and how computer generated dynamic graphs might support decisions made during reinvention (Cory et al., 2012). In particular, Oehrtman et al. (2011) has described the reinvention process as an iterative refinement process (IRP). Within this process, students write a definition, evaluate their definition against a rich collection of examples and non-examples, acknowledge problems with their definition, discuss potential solutions, and revise their definition, thereby initiating another iteration.

The previous studies with pairs of students described three main roles of a facilitator as nudging students forward to the next logical phase of the IRP (Steering the Ship), producing conflict (Conflict Producer), and providing timely solutions (Solution Provider) (Oehrtman et al., 2011). When students persist in overlooking a problematic issue with their evolving definition, facilitators can act as *conflict producers* by asking students to interpret their definition applied to a particular graphical example that their definition does not appropriately capture. After students wrestle with a problem for a significant time and have sufficient understanding of solution elements but remain unable to come to a satisfactory resolution, facilitators might act as *solution providers* while preserving the students' intellectual ownership of the process.

These previous guided reinvention studies were conducted with pairs of students while acknowledging that whole class implementation of such activities needed further research. Therefore, we implemented a guided reinvention approach to support students in constructing a formal limit definition as a part of a series of lessons implemented within a second semester calculus classroom. Video data revealed differences between the interview and classroom settings in terms of the role of the facilitator (instructor in the classroom), and the nature of interaction between students and facilitator. To address these differences in a systemic way, we are starting from these three facilitator roles described above, and have identified subcategories focusing on instructor prompts for the whole class and small group (Table 1).

Theoretical Framework & Method

We conducted a five-day TE in a second semester calculus classroom with 11 students at a medium-sized public university. Over the past three years, the instructor, who is the second author, has been both a researcher and facilitator with three guided reinvention teaching experiments with pairs of students. A *developmental research* approach "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, and formal knowledge, while (c) complying with the basic principle of intellectual autonomy" was adopted (Gravemeijer, 1998, p. 279). The students were first asked to create examples and non-examples of graphs of sequences converging to 5. Students were then asked create a formal definition for sequence convergence by completing the statement, "A sequence converges to 5 provided..." The instructor then guided the class through the IRP. There were two groups of four students and one group of three students. The activity of one group of four students was video and audio recorded (referred to as the TE group) and a second video camera captured whole class interactions with the instructor.

In our preliminary analysis of instructor's roles while conducting a guided reinvention in the classroom, we are developing an analytical framework for instructor prompts through an open coding of classroom videos (Strauss & Corbin, 1990). We especially focused on instructors' use of words and visuals, and his choice of students' words and visuals (Sfard, 2008).

Emerging Results

During our preliminary analysis, we found new categories of instructor roles in addition to providing more detail to previously identified facilitator roles. The current framework is shown in Table 1. Examples for subcategories were direct quotes from classroom videos.

Table 1
Categories of Instructor's Role

Category	Subcategory	Description	Example
Framing the Task	Inclusion (IN)	Explaining what needs to be discussed in IRP	"Your definition is supposed to be precise and concise, and capture all the things it should, all the examples, and exclude all the non-examples" (Day 2).
	Exclusion (EX)	Explaining what is not going to be discussed	"We don't question ... the convergence of these graphs" (Day 1).
Steering the Ship	Procedure (PC)	Telling students what to do next	"I want you to take another stab at your definition" (Day 1).
	Intra-Group Activity (IntraG)	Leveraging multiple- person activity within the group: Asking students to move from individual activity to group activity	"Exchange what each of you did… discuss and then make another attempt at a group definition" (Merging 4 individual definition into 1) (Day 2).
	Inter-Group Activity (InterG)	Leveraging multi-group activity in the whole class: Giving students an opportunity to see what other group was doing	"You're going to see everybody else's [definition], some problems they wrestled with and how they may have resolved some. If you see something that you think, oh, that might be a good thing to adopt, or oh, we don't want to" (Day 3).
	Focus on Visuals/Words (FI)	Directing student focus toward specific visuals or words provided by instructor (e.g., a graph, particular attributes of a graph)	"How would you describe the end behavior of [Graph]?… How many dots above and below each time [on Graph G]"? (Day 1)
	Focus on Definition (FD)	Directing student focus toward their use of specific visuals or words in their definition	"Where did the 'ultimately' [included in a TE's definition] occur like on Graph B?" (Day 2)
	Presenting (PR)	Directing students how to talk/present	"I want you to present your problem, not necessarily your definition" (Day 3).
Conflict Producer	Selecting Examples (SE)	Ask students to apply the definition to selected graphs	"How does your definition work for [Graph] A?" (Day 3)
	Third Person (TP)	Ask students to responds to the third person's question	"The student can look at this, and say 'I'm not sure exactly what this 'approach' is. I'm not sure if it captures this B graph?'" (Day 1)

Table 1 (continued)
Categories of Instructor's Role

Category	Subcategory	Description	Example
Conflict Producer (continued)	Other Group (OG)	Let student in one group talk to another group	A student in one group to another group "It [Graph B] does not always approach 5." (Day 1)
	Highlighting Process (HP)	Noting how students applied their definition in ways not captured by their definition	With hands spread over graphs, "What did you always draw first?... Do you feel like... your definition captures that?" (Day 4)
Solution Provider	Direct Solution (DS)	Correcting/clarifying meaning	"You can have error bounds without necessarily having over and under [estimates]. Error bounds just means, bounding the error" (Day 2).
	Student Source (SS)	Selecting one students' work among TE students	"I heard, [student's name], you saying something, but I don't see it incorporated here." (Day 2)
	Group Source (GS)	Selecting a group's work among other groups	While addressing whole class, "So you decided error bound. [Student's name], could I get you to quickly illustrate? I mean a lot you've done here [on group's board]." (Day 3)

The first main category includes instances where the instructor explicitly explained what will or will not be part of the IRP. The second category, steering the ship, includes cases when the instructor moved students forward in the IRP by explaining the procedure or asking students to focus on specific words and visuals in his examples, or in students' own definitions. The third category included the instances where the instructor attempted to produce conflict for students by selecting examples to which a current student definition did not apply, choosing words in their current definition that may cause conflict, using a third person who disagrees with their definition, asking a student in other group to speak who had a different view about a problem that a group was trying to resolve, and highlighting a process being used to apply their definition that was not captured in their definition. Codes SE and HP in Conflict Producer involve various instructor prompts for students to see the consistency/inconsistency between their written definition and their application in the examples/non-examples. Usually, the cases for SE identified when the instructor asked students to apply their written definition on the graphs, and the cases for HP identified when instructor pointed out the inconsistency between the students' written definition and their graphical explanation after the students applied their definition to the graphs. In the last category, solution provider, the instructor either provided an explicit definition of words for TE students to use or promoted one student's or another group's work to help resolve a problem. We note that the instructor could use non-TE groups' work either to produce a conflict (OG) and provide solution (GS). Because the difference between these two codes based on the video data only were subtle, such instances were complemented by the interview with the instructor about his intention on using other groups' work.

As this study built on previous research with pairs of students (e.g., Oehrtman et al., 2011) to groups of four students within a whole class setting, new detail has emerged in describing instructor roles. The subcategories indicate that having four students in the TE group and other groups in the classroom enabled the instructor to use other student work to produce conflict or provide solution. For example, we note that as a conflict producer, the instructor used the work of *another group* for TE students to realize problems in their definition. As a solution provider, instead of providing a crucial element of a formal definition, the instructor let *other groups* presents their definitions and problems to the whole class and provided TE students opportunities to adopt or reject other students' idea.

Discussion

Our preliminary analysis revealed various aspects of instructor and student discourse, which were different from interview settings. In particular, instructor's promoting an individual student's work or the work of another group made the nature of instructor-student and student-student interaction different from those previously reported with pairs of students. We anticipate that applying Table 1 as an analytical framework to our data will help us to better detail the nature of classroom discourse. In the future, we will attempt to apply this framework to address the following questions:

1) What are characteristics of the instructor's discourse during instructor-student interaction while complying with the principles of guided reinvention?
2) What are relations between the instructor's prompts and students' process of reinventing a formal definition?

To address the first question, we will investigate various aspects of instructor discourse using our categories of instructor's role, including the frequency of each subcategory and changes in the frequencies over time. We will also explore the nature of instructor-student and student-student discussion after a certain instructor prompt to identify types of instructor prompts that support productive student activity in the sense of progressing through the IRP. In our presentation, we will seek feedback on our categorization of instructor's roles and the systematic characterization of instructor-student classroom interaction. Because our primary goal was implementing a guided reinvention in a classroom setting, we will also solicit suggestions on how we can provide better resources for instructors who plan to use this approach in their classroom.

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STUDENT RESPONSES TO TEAM-BASED LEARNING IN TERTIARY MATHEMATICS COURSES.

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Abstract

Starting in 2009 we have implemented a Team Based Learning (TBL) model of delivery in two mathematics courses and one mathematics education course involving a total of 295 students. Qualitative data from evaluations, observations and interviews is used to begin to answer four questions raised by the Seldens (2001) regarding teaching mathematics at tertiary levels. Our analysis indicates that students say that TBL creates an environment in which they are active, have productive arguments and discussions and benefit from immediate feedback. There is scant evidence of any group being disadvantaged by this model of delivery.

Key words
Team-based Learning, Tertiary Mathematics Teaching, Group Work

Introduction

In their 1999 report, "Tertiary Mathematics Education Research and its Future," later included in an ICMI study, the Seldens raised important questions about co-operative learning: "How might one change the classroom culture so students came to view mathematics, not as passively received knowledge, but as actively constructed knowledge? What are the effects of various cooperative learning strategies on student learning? What kinds of interactions are most productive? Are some students advantaged while others are disadvantaged by the introduction of cooperative learning?"(Selden & Selden, 2001, p. 207). In this paper we will use students' responses to the introduction of a mode of delivery called Team-Based Learning (TBL) into two mathematics courses and a mathematics education course at Auckland University, New Zealand to begin to answer these questions.

Team-Based Learning is a particular cooperative learning strategy. It is widely used in the health sciences and medical education (Haidet, O'Malley & Richards, 2002; Michaelsen, Knight & Fink, 2002; Searle, Haidet, Kelly, Schneider, Seidel & Richards, 2003) and in business education (Michaelsen, Foml & Knight, 1997). In a study of the responses of 304 medical and dental students in an introductory course in medical science education in the United Arab Emirates 91% agreed with the statement "TBL increased the extent of my usual classroom involvement" and 33% of their open responses were categorised as "Small group learning: TBL is a great learning experience, motivating, enjoyable, different from the traditional lectures" (Abdelkhalek, Hussein, Gibbs, & Hamdy, 2010). Despite this very few mathematics lecturers use the approach.

We have previously argued, based on lecturer response as evidenced in task development, that there are sound theoretical reasons why TBL is particularly effective in promoting and developing a mathematical disposition and mathematical thinking and that it provides learning opportunities that are aligned with with social-constructivist learning theories and provokes mathematical thinking (Paterson & Sneddon, 2011). In this paper our focus is on student responses.

What is Team-Based Learning (TBL)?

TBL is a pedagogical model that shifts responsibility for learning to the students. The constitution of the teams is one of the key characteristics of this model. Teams usually have 5 - 7 students, and team membership is fixed for the duration of the course and work together on all team tests and tasks. Teams are not friendship groupings, but resemble the types of teams businesses construct to maximise productivity. Teams are constructed by the lecturer to distribute as fairly as possible the skills, knowledge and attributes needed to solve problems in the context of the course.

Students receive grades for course work in two different ways. Firstly, they prepare for each section of the course by reading carefully selected pre-readings and take a multiple choice test on this reading twice, once individually and once in their fixed teams. These are the readiness assurance tests or RAPs. When doing the test for the second time the students receive immediate feedback on their answers through the use of the Immediate Feedback Assessment Technique (IF-AT) These group tests are administered using IF-AT cards, on which four options are shown with the correct answer indicated by a star when covering is scratched off. If they are not correct first time they return to their discussion to gain part marks for being correct the second or even third time they 'scratch and win'. This immediate feedback means students always know the correct answers by the end of the RAPs. The power of immediate, goal-directed feedback has been identified in large-scale meta-studies by Hattie and Timperley (2007). Secondly they complete a team task that involves applying the ideas, concepts and skills learnt in the section. The structure of these tasks requires that all teams do the same task at the same time and that they submit one solution per team.

In 2009 and 2010 our focus was on course development and teaching of the Mathematics Education and Combinatorial Computing courses and the collection of data was largely opportunistic. We kept records of lecturer observations of class interactions, discussions between developers, and student evaluations. These evaluations were both Likert scale responses and open responses to the prompt " What helped or hindered your learning in this course?" In 2011 data was drawn from a parallel research project on Student Identity in mathematics involving the students in the Combinatorial Computing course. They were asked, amongst other things, about how they felt about TBL as a model of delivery. These interviews add an important dimension since they were conducted by impartial researchers not involved in teaching the TBL courses. In 2012 the data was augmented by audio recordings made in three situations: of two teams in Combinatorial Computing as they worked on tasks in class; two teams in the Dynamical systems course reflecting back on how they worked on the tasks in response to questions from an interviewer; and unstructured interviews with students from Combinatorial Computing that focussed on their experience of learning in a TBL environment.

We recognise that the fact that we are all enthusiastic implementers of the model may skew the outcomes and that this is a comparatively small study. We also acknowledge that we have, at times, pushed the envelope of the TBL structure to fit our mathematical needs.

Discussion

In the following discussion of Selden and Selden's (2001) questions, the data from all these sources are combined. While the learning experiences, content and lecturers in the three courses was different, we found the students' responses sufficient similarity to allow us to categorise them to begin to answer the questions. We note that students in the mathematics education course have completed at least three undergraduate mathematics courses and the students in the post graduate course will al have done a number of both pure and applied mathematics courses in their first degree. Unless otherwise indicated the direct quotes are from students in the undergraduate Combinatorial Computing cohorts, either from their open responses to evaluations or from the interviews. Quotes from students in the mathematics education course are coded ME and the Dynamical systems students are DS.

Question One: How might one change the classroom culture so students came to view mathematics, not as passively received knowledge, but as actively constructed knowledge?

It is our contention that TBL changes the classroom culture so students begin to view mathematics, not as passively received knowledge, but as actively constructed knowledge. What is happening as students work in established teams on a task or a problem that they have to work on together to succeed? The students talk about '*pooling their expertise'* and '*drawing on different people's ways of working*' and of the importance of learning to listen as others '*explain their thought processes (ME)'* and of the role that argumentation and defence of conjectures and ideas pay in reaching correct solutions. These behaviours align with the mathematical thinking described by Mason and co-authors (Mason, Burton and Stacey, 1982; Mason, 2008).).

The students say they find ideas are more accessible: "*I've certainly found it easier than my other level three papers. That could just be from the teaching strategy, the way it's been taught."* They valued the group discussion and the new arguments and perspectives they developed from working with their peers: "*It gave me alternative perspectives, ... coming from someone on my level rather than from a lecturer or from a text book"* and *"Working in a group you get to bounce ideas off of each other (ME)"* and *"it's just the whole environment where you can learn where you've gone wrong and you can correct it."* Not all the students felt they pushed themselves as much as they might have on their own: "*I found that I probably was lazier with the tasks than I otherwise would have been, had I had to do it myself."*

The responses to the pre-readings and RAPs were almost universally positive indicating that, when compelled, students value being better prepared to engage mathematically: "*I really liked the RAPs, sort of back to the primary school pre-test style of learning, it was really good." "I enjoy the idea of having to learn the definitions first and then learn the topic. I think that in particular is something that could be applied to almost all the math courses. That I really enjoy."* The frequent in-class assessment meant they needed to '*stay on the ball the whole time, no cruising till the end (ME).*" We suggest that these students showing an awareness of the need to be able to bring a knowledge of definitions to work on proving theorems and solving problems.

They also spoke about the feeing of accomplishment when the team was able to solve a problem no-one had been able to do alone: "*I remember at least one instance of our team when nobody got the right answer on the individual IRAP and then we all got it for the team RAP.*" They refer to students bringing ideas from other parts of mathematics and that when they worked together the focus was on thinking: "*When we got together .. **thinking** (*her emphasis*) about it - it was better.*" An awareness developed of team member's strengths and that communication improved over time: "*We understood each other – we knew who's strong in what areas (DS)," "I can remember one instance when we were doing the tree traversal algorithms where I brought, I remember presenting them with a problem which they hadn't thought of, which I had from experience previously."* There are strong indications of their extending their personal example spaces (Watson & Mason, 2002a, 2002b) The following is typical of a number of responses in all three courses:

> *Someone else always has a different view on the mathematical searches than you have … it really makes you understand what you're talking about because sometimes you're so limited to your own way of thinking that if you look at it from another way, the solution is really easy but if you focus from your way then you get stuck at some point, … if you do it with four or five people solving exercises it's quite easy because a lot of knowledge and a lot of different views on the same thing and it really helps you to expand your way of thinking.*

*Question Two: What are the effects of (*various co-operative*) TBL learning strategies on student learning?*

> *This is like working with a team in the real world where we very seldom work on our own - co-operation is vital to any career… This idea of getting a good mark because you understand the material and not solely because you were aiming for a good mark is what all assessments should aim for. (ME)*

We can begin to answer this question from data collected when the interviewer asked each of the twelve students: "So thinking about the style of the course, it was a Team Based Learning approach, how did that fit in with your style of learning?" Of the 12 students interviewed ten responded positively, two had mixed feelings and none were negative. One of the two was the person who was concerned about being lazy and the other felt the team went too quickly and was dominated by a strong mathematician. The most interesting responses were from students who in the pre-interview were negative but shifted their perspective: "*I actually thought I was going to hate it. … I hate team work. … But I actually did like it. And I did actually enjoy it in the end. So maybe it is my style of learning, even though I didn't know it was. I liked it because actually you could talk about stuff*"

Team Based Learning encourages students to ask themselves questions. In the discussions about drawing a phase portrait the students in the Dynamical Systems class were heard saying a number of things the lecturer regarded as very useful, things she hoped they would be asking and observing: *"This direction doesn't match with that direction" "Choose a point and see where it goes" "How do you know that?" "Direction does matter" "Have you considered the eigenvectors?" "Does that make sense?" "What's the stable manifold doing? Is it just floating around?"* During the next lecture she gave the whole class this list.
There were however teams that functioned less effectively, particularly if students did not always come to class. One student commented: "*When we were actually doing the tasks we would have had perhaps four people really engaging with it. But then they weren't turning up to class for other things as well, the hangers on, so they were kind of special cases. Anomalies. I wouldn't put too much attention on them. Certainly other teams didn't have that kind of thing happening. It was just that we had this particular weird little set."* The mathematical language he uses to describe this is fascinating.

Question Three: What kinds of interactions are most productive?

> *From a constructivist point of view the instant feedback makes sense - to learn from our mistakes and adapt to our environment this kind of testing and working together is far more beneficial than a number out of 10 you receive a week later. (ME)*

A number of students commented on the value of the immediate feedback: "*It was great knowing immediately you were right"* and on the way that working with others either confirmed or disproved their ideas.

Arguments of various types played an important role in this. The usual sort of mathematical argument was mentioned frequently: "*If I can convince the other one my way is correct then I am sure my idea is correct*." Students who said they usually work alone found the opportunity to interact rewarding: "*Normally I avoid asking for help and talking to others. I* *try and get everything done on my own, researching online or studying the notes or the text book or trying to figure it out myself. And that normally works well for me, but it does take longer. Whereas this one, anything I didn't understand I felt my team mates were able to explain it adequately, and then it was good being able to argue about it, and discuss it and to attack a problem from different points of view. And to be able to see where the other person was coming from.*" A number of students referred to the usefulness of hearing someone explaining how they approached a problem.

There were more heated arguments: "*Though one member of my team and I did not quite get along as smoothly as we might have liked. And there may have been one or two very sort of heated team tasks. But a little aggressive stand offs may have been helpful in the mathematical learning ... I was working very, very hard to do everything in my power to prove him wrong all the time. That sounded like a nasty thing to do. But it did make me more focussed, albeit for the wrong reasons."*

There were also instances when a team member who had not been part of a discussion needed more time than was available to be '*brought up to speed*" and in the end "*we had to convince the guy again and it took like five or ten minute extra to convince him ... that point I was like okay, doesn't matter if you don't understand, just be quiet and work with us*" This raises the question of the role of the person who was seen as the team leader. If they were a strong mathematician whose focus was on gaining the maximum grade the interactions were less productive than when they collaborated willingly with the team. The recognition of "*playing to different people's strengths (DS)*" appears to be an important aspect in the creation of effective teams.

Some of the students who did not identify themselves as 'mathematicians' spoke about not always following the discussion but they said it was useful to be a '*listener*' and to hear the ideas being discussed by a number of people and not just by the lecturer: *"Certainly one of the tasks especially helped my understanding where one of the team members sort of went about the problem and something just clicked for me, to see someone do it that way and that really helped, counting symmetries."*

Question Four: Are some students advantaged while others are disadvantaged by the introduction of TBL (**cooperative learning**)?

Some of the more talented students expressed concern at being held back but the consensus was that "*Although at first it seemed that the groups might hold back students working at higher levels, they appear to have worked by encouraging these students to study the material in more depth - developing the breadth and depth of their knowledge rather than accelerating them through the curriculum.(ME)"* In fact the researchers in the Identity project from which we have drawn data observed that "Those who most strongly identified as mathematicians were most open to the team-based learning format." (Barton, Ell, Kensington-Miller & Thomas, 2012, p 4)

This is in contrast to a study of student performance in a medical gross anatomy and embryology course that found that the students who benefited most from TBL were the academically at-risk students "who are forced to study more consistently, are provided regular feedback on their preparedness and given the opportunity to develop higher reasoning skills" (Nieder, Parmelee, Stolfi &Hudes, 2005 p 56) This is supported by statements made by a music and mathematics major student who is dyslexic and has found mathematics very challenging. She said: "*I like to listen when others explain how they see the ideas and then I try to see the patterns as we solve the problem, I did better in this course than my other stage*

3 courses."

In another group there was a student who was finding the work hard and one of her team mates talks about her and the problem of helping someone when the interaction is being recorded: "*I think I was concerned about someone in my group because I think she wasn't as confident as say the rest of us in just putting our ideas out there. And I probably could have managed this better in terms of helping her and stuff, ... especially with the camera trained on you, you don't really want it to be recorded."* She helped her more at other times. Despite the negative feedback on the mode of data capture this does underline the sensitivity for one another's feelings evidenced.

Conclusion and Questions

The data presented largely supports our contention that implementing a Team Based Learning approach to delivering mathematics and mathematics education lectures can allow, and even prompt, the lecturer to create an environment in which students play a more active role in the construction of their mathematical knowledge and there is evidence that TBL serves both ends of the academic spectrum well.

TBL promises an effective alternative to the traditional mode of course delivery in higher mathematics. We would welcome further research that explores the use of TBL in large first year classes and quantitative studies on the impact of the model. We welcome the reviewers' questions and input and I will address them in the presentation and in the longer paper.

How do you create a classroom culture in which students came to view mathematics, not as passively received knowledge, but as actively constructed knowledge? What mathematical behaviours do you encourage in lectures? How do you do this?

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ASSESSMENT OF STUDENTS' UNDERSTANDING OF RELATED RATES PROBLEMS

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This study started with a thorough analysis of student work on problems involving related rates of change in a first-year differential calculus course at a large, research-focused university. In two sections of the course, students' written solutions to geometric related rates problems were coded and analyzed, and students' learning was tracked throughout the term. Three months after the end of term, "think-aloud" interviews were conducted with some of the students who completed the course. The interviews and some of the written assessments were structured based on the classification of key steps in solving related rates proposed by Martin (2000). Our preliminary findings revealed a widespread, persistent use of algorithmic procedures to generate a solution, observed in both the treatment of the physical and geometric problem, and the approach to the differentiation, and raised the question of whether traditional exam questions are a true measure of students' understanding of related rates.

Key words: Related rates, Calculus assessment, Misconceptions

Introduction and Research Questions

In many traditional differential calculus courses in North American universities, after learning about rates of change and various techniques of differentiation, students learn to apply these ideas to solve related rates problems, that is, problems that require the evaluation of "the rate of change (with respect to time) of some variables based on its relationship [often geometric in nature] to other variables whose rates of change are known" (Dick & Patton, 1992). Existing research on students' difficulties with these problems indicates that students lack conceptual understanding of variable and have trouble in distinguishing between variables and constants (White & Mitchelmore, 1996, Martin, 2000), as well as trouble in engaging in covariational reasoning (Engelke, 2004). A classification of the main steps in solving geometric related rates problems was proposed by Martin (2000), who discusses the results of assessing students on the specific steps, reporting greater correlations between procedural knowledge and success at solving related rates problems, while Engelke (2007) discussed a possible framework to describe how a mental model for a related rates problem is developed during the solution process.

Being a classic course topic, related rates problems were chosen as the setting for a classroom experiment that took place in 2011 in two sections of a large calculus course (Code et al. 2012). As part of that project, test items similar to traditional exam questions were developed to assess students' skills at solving related rates problems. The detailed analysis of student work performed in that study brought to light specific limitations of these assessment tools and questioned the effectiveness of traditional exam questions as an accurate measure of understanding of related rates. Motivated by these findings, we conducted a follow-up study aimed at deepening our understanding of students' difficulties with related rates. Using a similar framework to that presented by Martin (2000), we assessed students' mastery of specific steps in solving related rates problems, extending her methodology with the use of student interviews. The main goal of this study is to investigate the following questions:

What are the sources of common misconceptions observed in students' solutions to related rates problems on written exams?

Do traditional exam questions involving related rates accurately assess students' understanding of such topic?

Methodology

Written solutions of geometric related rates problems from four different assessments were collected for N = 300 students enrolled in a large Calculus 1 course at a research-focused university. The course is primarily aimed at Business and Economics majors with some prior knowledge of calculus (high school calculus), but it shares most core material with the science-oriented Calculus 1 offered at the same institution (about a third of its student population are in fact science majors). Our sample represents about 25% of the total course enrolment, and was selected from two of the 11 course sections. Student work was collected at four different stages during the term: on a short diagnostic test at the beginning of the term, a quiz at the end of the week of instruction on related rates problems, a midterm exam two weeks later, and a final exam at the end of the course. Both the midterm and the final exams accounted for a portion of the final grade, while the diagnostic and the quiz were part of a number of in-class activities that were worth a small fraction of the final grade (1%), awarded based on participation. About three months after the end of the course, "think aloud" interviews were conducted with 11 students randomly selected from the original sample.

Preliminary Results

From the analysis of students' written work and the tracking of performance over the term, we observed significant improvements of key skills in solving related rates as a result of both instruction and feedback from tests. After targeted instruction and homework involving related rates problems, the majority of students showed improved ability in performing the early steps of a solution compared to their incoming skills at the beginning of term. Differentiation, however, appeared to be one of the major stumbling blocks for students. Despite several weeks of review and practice of the basic concepts and rules of differentiation, when students start to work with related rates they had not yet developed the sufficient skills to carry out sophisticated calculations such as the derivative (with respect to time) of a functional expression containing more than one time-dependent variable, like for example the function representing the volume of a growing cone. Skills improved over the course of the term, but these difficulties were not fully resolved by the end of the course, and in some cases persisted beyond the end of the course, as confirmed by the student interviews. A preliminary analysis of student thinking observed in the interviews would suggest that the source of these difficulties stems from lack of a deep understanding of the differentiation process, rather than some misunderstanding of the specific physical problem at hand. Interestingly, to bypass the challenge posed by these complicated functional expressions, instructors and textbooks often teach students to reduce the number of variables by performing an appropriate substitution before taking the derivative. While this strategy simplifies the problem significantly for students, the data we collected suggest that proficiency in implementing this solution strategy is likely an indication of procedural knowledge rather than conceptual understanding, raising the question of whether testing the students on how proficient they are in providing written solutions for these problems is a true measure of their understanding of related rates.

Discussion Questions

Do students really possess the technical skills to handle the mathematical sophistication that related rates problems present?

Are traditional questions testing the ability to generate a full, correct solution a true measure of students' understanding of related rates?

What assessment strategies can be developed to effectively measure understanding of related rates?

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WAIT A MINUTE…IS THAT ENOUGH TO MAKE A DIFFERENCE?

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The "one-minute paper" (Stead, 2005) is a technique for facilitating communication between students and the teacher and promoting reflection. In this paper we focus on the types of questions students ask and how they may be related to success. We present preliminary results from an introductory university-level calculus course, indicating that the nature of questions asked by more successful and less successful students are different, suggesting that the types of reflections that students engage in may have a significant impact on the efficacy of such an intervention.

Key Words: Calculus, Classroom research, Formative assessment, Self-regulation

Introduction

Formative assessment is a process of *evoking* information about learning and *using* it to modify teaching and learning activities (Black, Harrison, & Lee, 2003). Practicing formative assessment in large lecture courses presents a challenge, both due to the large number of students and the teacher-centered practice that such a classroom tends to promote. Accordingly, the "one-minute paper" is a technique that has been used to improve communication between students and the teacher (Stead, 2005). Students spend "one minute," near the end of class, to write a reflection on what they learned and what questions they have about the content from that day. In this paper we explore the types of questions that students ask and how they may be related to success.

Effective learners are self-regulating and engage in many important learning activities, such as: setting powerful goals, monitoring their performance toward those goals, and adapting their future activities based on the results of their performance (Zimmerman, 2002). In order to promote self-regulation in truly productive ways, simply asking students to reflect is insufficient; students must be taught to ask the *right* types of questions to monitor their progress, for instance. Because problem solving and understanding in mathematics are domain-specific (cf. Schoenfeld, 1985), it is evident that students must be taught mathematics-specific reflection skills. In order to evoke a shift in how students engage in mathematics and monitor their engagement, it is important to change for students what it means to know and do mathematics (Boaler & Greeno, 2000). If students perceive mathematics as a game of recalling facts and procedures (cf. Schoenfeld, 1988), they will not develop the appropriate reflective thinking skills necessary for success in mathematics.

Method

This paper reports on a portion of a larger ongoing study focused on promoting explanation and reflection in a first-semester university-level calculus course (Reinholz, 2013). The intervention promotes metacognition by integrating three "key" questions as a regular part of classroom discourse: (a) Why would you...?; (b) Why can you...?; and (c) What does it mean that...?. These particular questions are exemplars of *categories* of questions, representing three different viewpoints from which students can self-assess their understanding of a given problem or concept. These questions are meant to push students away from thinking about math as memorizing facts and procedures. Additionally, students ended each day by answering 3 reflection questions:

1. On a scale from 0 to 100%, how well did you understand today's lecture?
2. What questions do you have? (What was unclear? How does today's lesson relate to other math concepts?; **write at least 2 questions.**)
3. Tell me something else you think I should know.

Daily reflections gave students an opportunity to incorporate the key questions used by their instructor into their regular reflective practice. Additionally, these questions played an important role in shaping future classroom practice (cf. Black & Wiliam, 2009).

Results and Analysis

As an ongoing study, this paper reports on results from the beginning of the semester, including performance on the first midterm exam, which has been shown to predict success in the course with 80% accuracy (using logistic regression; Reinholz, 2009). To investigate the relationship between the types of reflection questions asked by students and student performance, two groups of 5 students each were randomly constructed based on exam performance. "Successful" students scored 90% or above on the exam (the A cutoff), while "unsuccessful" students scored below a 60% (the C cutoff). The daily reflection questions asked throughout the semester (approximately 16 per student) were analyzed using a grounded theory approach (cf. Glaser & Strauss, 1967). In addition, the impact on instruction will be discussed.

Through the analysis of student responses, 7 major categories of questions emerged. The results are presented in Table 1.

Question Type	"Successful" Students (39 reflections/78 questions analyzed)	"Unsuccessful" Students (38 reflections/76 questions analyzed)
Future Topics	13 (16.7%)	2 (2.6%)
Connecting Concepts	4 (5%)	9 (11.8%)
Specific Concepts	29 (37.2%)	22 (28.9%)
Real-life Connections	4 (5%)	3 (3.9%)
Scoring/Logistics	20 (25.6%)	4 (5.3%)
"How-To" / Procedures	5 (6.4%)	21 (27.6%)
Other/Non-Mathematical	3 (3.85%)	15 (19.7%)

Table 1: Types of reflection questions asked by students

For the purposes of this brief report we focus on a few significant aspects of the coding and results. Successful students asked many more questions focusing on future topics and the logistics of exams, such as how to present a complete solution (see Table 2 for sample responses). It seems that successful students saw the course more holistically, rather than focusing only on a single lesson, considering what topics would be on the exam and what they would need to do to demonstrate mastery of the material.

Question Type	Sample Student Response
Future Topics	When will we learn to find f' without using limits?
Connecting Concepts	How does the definition of tangent relate to secant?
Specific Concepts	When does the derivative not exist?
Real-life Connections	What is a real application of limit?
Scoring/Logistics	Do we need to keep our answers in exact form?
"How-To" / Procedures	How do I "long divide" equations involving x?
Other/Non-Mathematical	Still unsure about limits

Table 2: Sample student responses for various question types

In contrast, unsuccessful students were much more focused on "how-to" solve problems and use procedures, and asked many more irrelevant or non-specific questions. In particular, questions in the "other" category often expressed confusion but did not construct a concrete question that could be answered to resolve the confusion.

There was some evidence that students asked the 3 key questions modeled by the instructor in class. These responses belonged to the categories of connecting concepts and specific concepts. However, there was no significant difference between the usages of these two questions across groups. The most likely explanation is that the usage of these questions increased for both groups as a result of classroom practice, but the increases in usage were similar between groups.

Impacts on Instruction

In addition to promoting reflective thinking, the one-minute paper also had a direct impact on instruction. The first question: On a scale from 0 to 100%, how well did you understand today's lecture?, gave the instructor immediate quantifiable feedback about how the students felt about the day's lesson. Regardless of the absolute accuracy of student judgments, when a number of students marked low percentages, it indicated that some of the content might need to be re-addressed.

Re-addressing content, however, was not always an easy task. Fortunately, the second question from the one-minute paper helped address this: What questions do you have? (What was unclear? How does today's lesson relate to other math concepts?; **write at least 2 questions.**). Student responses to this question provided further detail on what ideas and concepts the students were struggling with. This enabled the instructor to adapt future lessons according to student needs.

Discussion and Conclusions

The one-minute paper is a tool for promoting communication between an instructor and her students, as well as for promoting reflective thinking in students. Crucially, however, is not just that students ask questions, but that they ask the *right* types of questions. Preliminary analyses indicate that successful students were better able to see the course as a whole. In contrast, less-successful students focused more on "how-to" and expressed general confusion, both of which are harder to use to monitor performance and understanding. In addition, the one-minute paper had an impact on instruction, allowing the instructor to address the needs of the students on a daily basis.

Open Questions

1. How might we design an intervention to be more effective at influencing the types of questions students ask?
2. How might we use such an intervention to study the types of questions students ask causally rather than correlationally?
3. In what other ways might we classify the types of questions students ask?

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AN INVESTIGATION OF PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' DEVELOPMENT AND PARTICIPATION IN ARGUMENTATION

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Abstract: This study investigates how two professors and pre-service secondary mathematics teachers engage in argumentation and proof in two courses. One course under investigation is a geometry course; the second is a methods of teaching mathematics course. The research also studies the how professors and pre-service teachers construct arguments and proofs. Examining the classroom discourse to understand how it may impact argumentation practices is another aspect of the research. Case study and grounded theory approaches are used to guide the data collection and analysis. Some data collected include interviews with the two professors and pre-service teachers and observations of the two courses and the pre-service teachers' classrooms during their student teaching. Data analysis so far indicates the geometry professor engages students in argumentation and proof in multiple ways.

Key words: Pre-service teachers, Mathematics, Proof, Argumentation

Introduction and Literature Review

According to Forman, Larreamendy-Joerns, Stein, and Brown, "students must also learn about the nature of mathematical argumentation as they attempt to use it to gain an understanding of the mathematical objects under discussion" (1998, p. 529-530). In other words, an aspect of learning mathematics is learning the domain's argumentation practices. Walshaw and Anthony claim argumentation practices in mathematical discourse are "a defining feature of quality classroom experience" (2008, p. 516). Thus, the teaching of proof is valuable for learning mathematics, but we have little research about it (Stylianides, 2007). Bass questions if proof is an "endangered species," which is a form of argumentation in mathematics (2011, p. 98). Uhlig (2002) claims students have little experiences with proof before college so it is plausible that teachers place little emphasis on proof when teaching. Cross (2009) believes engaging in argumentation can foster and help disseminate mathematical ideas; in particular, practices such as conjecturing are those classroom discourse can support. Yet Cross (2009) claims teachers need to model and help students understand the argumentation practices they want students to adopt.

The Common Core State Standards (2010) acknowledge proof and argumentation as a valuable part of students' mathematical learning. These standards outline opportunities students should have related to argumentation in their standard "Construct viable arguments and critique the reasoning of others"; this includes giving students opportunities to conjecture. Some elements of the standard of mathematical practices, besides conjecturing, are to provide time for students to "Constructing arguments", "recognize and use counterexamples, and "make plausible arguments". Evidence of the value they place on argumentation is that in every grade listed practices related to argumentation are cited. Teachers need the ability to formulate strong mathematical arguments and proofs because they must respond to students' mathematical claims or explanations. Walshaw and Anthony claim, "effective pedagogy is inclusive and demands careful attention to students' articulation of ideas" (2008, p. 527). Thus, the experiences teachers' design for mathematical argumentation has pedagogical importance. In fact, Krummheuer (2007) considers mathematical argumentation an everyday activity in the mathematics classroom. Teachers are expected to teach proof and argumentation and engage students in mathematical argumentation. Thus, teaching and fostering argumentation and proof supports recently adopted standards (see Common Core State Standards, 2010).

The focus of the research is to study the mathematical argumentation of two professors and pre-service secondary mathematics teachers enrolled in two courses- a methods of teaching mathematics course and a geometry course. The research pursued here is guided by the following research questions: How do the students and teacher engage in argumentation in mathematics? How are arguments constructed? How do argumentation practices develop over time? How does the classroom discourse around mathematics influence argumentation? Some "students" in the geometry course will be identified as "teachers" once they begin their student teaching.

Methodology

Data collection began in the fall semester of 2012 and will continue through May 2012 and so I will continue to collect data after submission of the proposal. Thus, the analysis presented here is based on data collected up to this time. Observations of the courses and suggestions from the professor will be taken into account for selecting students to participate in interviews and classroom observations. Open and in-vivo coding will be conducted for initial coding stages of the interview data of the professor and students, which are at present fully transcribed. Triangulation will be used in data analysis, which according to Merriam "remains a principle strategy to ensure for validity and reliability" (2009, p. 216). Also, member checks, and efforts to establish researcher reflexivity (e.g., Cho & Trent, 2006; Merriam, 2009) will be used.

The research study's participants include two professors in education at a university in the Rocky Mountain region and undergraduate pre-service secondary mathematics teachers. One professor is the instructor of a geometry course which the pre-service teachers (henceforth called students) are enrolled for Fall 2012. This course was chosen for several reasons, but one is that it is an advanced level mathematics course and so forms of argumentation such as proof and counterexamples are likely to be encountered. Many of the students are concurrently enrolled in the methods and geometry course. Thus, the possibility to observe how teaching proof and argumentation is addressed was a reason for choosing the methods course.

Data consist of field notes from observing the two courses, interviews with the professors and selected students, written work collected from students, and observations of classroom visits to see the students teach in their own classrooms. Interviews throughout the semester with the professors are based on observations of the course and written work produced by students. Questions posed during interviews with professor are meant to draw out information regarding how they plan to and did engage students in argumentation and proof, forms of argumentation they saw students using, and how students constructed proofs and mathematical arguments. Students selected for interviews and observations have not been chosen yet, but will be chosen based on the forms of argumentation they may have employed, questioned, or in the way in which they responded to a given mathematical argument. Interviews with students will focus on forms of argumentation and proof, how they engaged in proof and argumentation, and explore how they constructed them.

Grounded theory and case study approaches are employed. Because a small number of students will be selected (possibly two or three) to participate in interviews and observations of their teaching practice, the phenomenon under investigation is considered "intrinsically bounded", a criteria for case study (Merriam, 2009, p. 41). Stylianou, Blanton, and Knuth (2011) claim there is little research on how proof is taught in schools. Thus, observing pre-service teachers during their student-teaching experiences can help address this gap in the research. Each professor and student will be considered as separate cases. The interactions between professors and students will provide valuable data concerning the engagement, teaching, use, and development of mathematical arguments and proofs. This context represents one described by Grbich as "interactions between persons or among individuals and specific environments", which justifies grounded theory is a suitable approach (2007, p. 70). Also, because little is known how teachers develop arguments in mathematics, it is another reason why grounded theory is an appropriate approach (Grbich, 2007). Toulmin's model of argumentation has been used by numerous researchers (e.g., Giannakoulias, Mastorides, Potari, & Zachariades, 2010; Krummheuer, 2007; Pedemonte & Reid, 2011), to analyze the structure or proofs and arguments in mathematics and will be used for this research.

Preliminary Findings and Discussion

My current data set consists of field notes of course observations, documents from the course (e.g., syllabi, assignments), and interview data (transcripts). Preliminary analysis of observations indicates there is a variety of ways the geometry professor engages students in argumentation practices. One approach he has used often is to make historical references to proof. These reference highlight mathematical claims individuals have attempted to prove throughout time and the changing emphasis of proof in schools. Also, he related the structure of mathematics to proof. An instance of this is one of the discussions of non-Euclidean geometries such as Lobachevski, based on a restatement of Euclid's fifth postulate. An unexpected change in the acceptable form of proof took place after the second quiz. The instructor noticed many students struggled with constructing proofs on the quiz; all students wrote proofs in a narrative form, according to the professor, but many with circular reasoning or an absence of justifications for assertions. Thus, from that point on the professor required students to write proofs on quizzes and exams in the two-column format until he felt confident they could construct narrative proofs that provided assertions followed by justifications. His description of expectations for two-column proofs is consistent with Weber and Alcock's notion of proof, which "must be based on accepted axioms and definitions" (2011, p. 323). To analyze students' arguments and proofs, Toulmin's model may be used.

Questions for the audience:

1. What might be important concepts that lend themselves well to studying proof and argumentation?
2. The use of Toulmin's model of argumentation has been used by many other researchers. Are there other models or frameworks better fitting to this research?

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AN EXAMINATION OF PROVING USING A PROBLEM-SOLVING FRAMEWORK

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A link between proving and problem solving has been well established in the literature (Furinghetti & Morselli, 2009; Weber, 2005). *In this paper, I discuss similarities and differences between proving and problem solving by using the Multidimensional Problem-Solving Framework created by Carlson and Bloom* (2005) *on Livescribe pen data from a study of proving* (Savic, 2012). *I focus on two participants' proving processes: Dr. G, a topologist, and L, a mathematics graduate student. Many similarities were revealed by using the Carlson and Bloom framework, but also some differences distinguish the proving process from the problem-solving process. In addition, there were noticeable differences between the proving of the mathematician and the graduate student. This study may influence a proving-process framework that can encompass both the problem-solving aspect of proving and the differences found.*

Key words: Proof, Proving, Proof construction, Problem solving

Proof and proving are central to advanced undergraduate and graduate mathematics courses, yet there is little discussion in these courses of the proving process behind the proofs presented. Since there is an overlap between proving and problem solving (Furinghetti & Morselli, 2009, Weber, 2005), one might look at the problem-solving literature in order to describe some of the aspects of the proving process. I used the Multidimensional Problem-Solving Framework created by Carlson and Bloom (2005) in order to examine the proving processes of a topologist, Dr. G, and a mathematics graduate student, L. I discuss the adequacies and limitations of their framework for describing the proving processes.

Background Literature

Selden, McKee, and Selden (2010) stated that the proving process "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (p. 128). In addition, professors teaching upper-division undergraduate mathematics courses often seem to ask students to produce original proofs to assess their understanding.

In the mathematics education literature, there are several analytical tools about proof production or the proving process, including "proof schemes" (students' ways of "ascertain[ing] for themselves or persuad[ing] others of the truth of a mathematical observation") (Harel & Sowder, 1998, p. 243), affect and behavioral schemas (habits of mind that further proof production) (Furinghetti & Morselli, 2009; Douek, 1999; Selden, McKee, & Selden, 2010) and semantic or syntactic proof production (Weber & Alcock, 2004). Both aspiring and current mathematicians seem to need flexibility in their proving styles in order to be successful in mathematics (Weber, 2004). Yet, there seems to be no overall proving-process theoretical framework that encompasses most of the above ideas about the proving process.

Past research has indicated connections between proving and problem solving, usually citing proving as a subset of problem solving. In Furinghetti and Morselli (2009), the authors stated that "proof is considered as a special case of problem solving" (p. 71). In Weber's (2005) paper, he considered "proof from an alternative perspective, viewing proof construction as a problem-solving task" (p. 351). This connection influenced my research questions: Can Carlson and

Bloom's (2005) Multidimensional Problem-solving Framework be used to describe the proving process? If changes or additions are called for, what might they be?

Carlson and Bloom's Multidimensional Problem-Solving Framework

The Multidimensional Problem-Solving Framework described by Carlson and Bloom (2005) has four phases: orienting, planning, executing, and checking. Each of these phases can be associated with four attributes: resources, heuristics, affect, and monitoring.

The orienting phase includes "the predominant behaviors of sense-making, organizing and constructing" (p. 62). The planning phase is when a participant "appeared to contemplate various solution approaches by imaging the playing-out of each approach, while considering the use of various strategies and tools" (pp. 62-63). In addition, during the planning phase Carlson and Bloom often observed an additional sub-cycle consisting of (a) *conjecturing* a solution, (b) *imagining* what would happen using the conjectured solution, and (c) *evaluating* that solution. The executing phase involved "mathematicians predominantly engaged in behaviors that involved making constructions and carrying out computations" (p. 63). Finally, the checking phase was observed when the participants verified their solutions.

Carlson and Bloom (2005) stated that "the mathematicians rarely solved a problem by working through it in linear fashion. These experienced problem solvers typically cycled through the plan-execute-check cycle multiple times when attempting one problem" (p. 63). They also stated that this cycle had an explicit execution, usually in writing, and formal checking that used computations and calculations that were also in writing.

Research Setting

One topologist, Dr. G, and one graduate student, L, were given a set of notes on semigroups and a Livescribe pen and paper, capable of capturing both audio and real-time writing using a small camera the near end of the ballpoint pen. These were two participants of a larger study of nine mathematicians and five graduate students (Savic, 2012), who were asked to answer two questions, provide seven examples, and prove thirteen theorems in the notes. Dr. G used the Livescribe equipment for proving or answering for a total of five hours and 31 minutes, while L used the equipment for a total of three days, 22 hours, and 11 minutes. I selected Dr. G's data because he spoke a significant amount of the time while proving and also encountered impasses when proving this theorem. I chose L's data because he was one of only two graduate students who attempted a proof of this theorem and I hoped that his transcript would be amenable to analysis using the Carlson and Bloom framework. I focused the coding of the proving processes on the theorem, "*Theorem 20: A commutative semigroup with no proper ideals is a group.*" The audio/video recordings were transcribed so that the audio and actions on the paper corresponded. Once the sessions were transcribed, coding was done with the Carlson and Bloom (2005) framework. A sample of the transcription, with coding, can be seen in Table 1.

Table 1: Coding the proving process of Dr. G

Time	Writing/Speaking	Coding
7:02 AM	Th 20: A comm semigp w/ no proper ideals is a gp.	Orienting (Resources)
7:03 AM	Hmm . . . I'm taking a break, breakfast, etc. Back to this <u>later</u>. <u>Must think on this.</u>	Orienting (Resources)
BREAK 7:04 AM - 8:07 AM		
8:07 AM	Ok, I thought about this while on a cold walk in the fog.	Planning (Heuristics)

	Pf: Given $g \in S$, a semigp., consider the ideal g. . . . (Then he stops and puts "comm." between "a" and "semigp.") . . . $gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ \ni gg^{-1} =$ (32 second pause, then he strikes through the whole proof)	Executing (Resources) Checking (Monitoring) Executing (Resources) Checking (Monitoring)
8:09 AM	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.	Checking (Resources), Planning (Resources) Cycling back (Back to planning, re-approaching to answer why there is no identity)

A Description of the Coding of the Sample

At 7:02 AM Dr. G started the proving period of Theorem 20; he had proved the theorems in rest of the notes in the two hours prior to this first attempt. This was his first proving period on Theorem 20. He wrote the theorem on paper, probably orienting himself to what he needed to prove. There was a one-minute pause, which I infer that he was planning or orienting himself to the theorem. Since he had proved Theorems 13-19 quite quickly, his decision to take a walk at 7:04 AM might have been because he had not quickly seen how to attempt this proof. I assume that during the break he might have been planning how to prove Theorem 20, probably using the conjecture-imagine-evaluate cycle mentioned above. At 8:07 AM, he started executing the idea that he had generated during the walk. He corrected his work by inserting "comm." to be precise, something that I coded as "checking" and "monitoring" for correctness. Then Dr. G went back to executing his idea, using an element, g, in the semigroup and multiplying it by the whole semigroup to create an ideal. There was a 32-second pause, and then he crossed out the entire proof that he had just written. This was coded as checking. In fact, at 8:09 AM, he wrote why he crossed out this proof attempt: he needed an identity, which had not been given. I coded this as "checking (resources)," because Dr. G apparently used what he knew about groups to verify this attempt. He then cycled back to planning, because there was a 95-minute gap before he wrote something else, beginning with a different idea, and eventually re-orienting himself.

Similarities and Differences Found when Using the Problem-solving Framework

For most parts of the transcripts, the Multidimensional Problem-Solving Framework could be used to adequately describe the proving process. For example, there were multiple situations in both the transcripts of Dr. G and L that involved both the conjecture sub-cycle (conjecturing, imagining, evaluating) and the larger cycle of planning, executing, and checking. For example, in Table 2, L can be seen using the planning-executing-checking cycle.

Table 2: Coding the proving process of L

10:19 AM	First we want to show S has an identity 1.	Planning (Heuristics)

	(pauses for 45 sec)	Planning (Cycling)
10:20 AM	(pauses for 20 sec) If possible. Suppose S has no identity. Then for every $a \in S, ab \neq a$ for all	Planning (Monitoring) Executing (Heuristics) Executing (Resources)
10:21 AM	(pauses for 25 sec, then crosses out "Then for every $a \in S, ab \neq a$ for all") Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$.	Checking (Monitoring), Planning (Resources) Executing (Resources)

However, one can also see in L's, and especially in Dr. G's, transcripts five aspects of proving that are not covered in the problem-solving framework and perhaps should be included in a proving framework: (1) *Re-orienting* is not a part of the problem-solving framework, but appears to sometimes occur in proving. At 9:44 AM Dr. G wrote he was "suspicious that [the theorem] was [not] true." He thought about finding examples to show this, which seems different from the original orientation of preparing to write a proof. (2) Dr. G also experienced an *impasse*, a time during which he saw he was not progressing and had no more new ideas. This led to a period of *incubation* during which he did not pursue the proof and later to an *insight* leading to the proof. The larger study of the nine mathematicians also suggests impasses and incubation are important in proving. (3) *Planning and checking can be an integral part of executing*. For example, in Table 2, L hesitates for 25 seconds, crosses out his previous attempt (which can be taken as a sign of checking), and immediately proceeds with another approach. The writing of the new idea can be seen as executing, but any planning, such as formulating the new idea, must have occurred simultaneously with the earlier checking or with the later executing. Also, when L finished the proof of the theorem, he immediately went on to prove the next theorem, thus apparently not completing the full planning-executing-checking cycle. (4) Planning for Dr. G and L seems to be of two kinds, *global* and *local*. By *global planning* I mean planning the overall structure of a proof, for example, L's deciding to prove a theorem by contradiction. By *local planning* I mean planning how to proceed on a small part of a proof. In some cases, this can consist of "exploring," that is, collecting new information without knowing how, or whether, it will be useful. In Theorem 20, one can find that $ax = b$ is solvable for x without immediately knowing how, or whether, this might be useful. (5) Checking can also be *local*, for example, checking whether a definition has been properly applied, or *global*, that is, checking whether one's entire proof is sound.

Discussion and Limitations

Carlson and Bloom's framework describes the process of problem solving and some aspects of the proving process well. When posed a problem like those in the Carlson and Bloom study, mathematicians can rather easily and quickly get conversant with the constraints (orienting) and then go about solving the problem (planning-executing-checking). In my study, however, the responsibility was on the participants to figure out newly introduced concepts while proving theorems that they did not always consider to be true. Carlson and Bloom were also in the room when the interview took place and were able to take notes on their participants' behavior, so observations of affect and heuristics were easily witnessed. My participants, on the other hand, were allowed to take the notes home and work on the proofs whenever they pleased, which gave them a naturalistic setting but did not allow me to view these attributes personally. Finally, Dr. G took long breaks, which may have included incubation (Savic, 2012), in his proving process. Incubation was not observed in the Carlson and Bloom (2005) paper perhaps due to the time constraints of an in-person interview, but was extremely helpful in the proving process.

Using Carlson and Bloom's (2005) framework, I was also able to distinguish some differences between the proving processes of Dr. G and L. For example, Dr. G asked more questions (both in writing and verbally) about the structure of and the constraints of the theorem. L never went back to orienting himself after the first few minutes. One advantage to using notes, like those on semigroups, is that they are easily accessible to novice, intermediate, and advanced provers, while providing challenging theorems to prove. Could a proving-process framework be created to include the Carlson and Bloom framework and also the differences noted? Also, is it beneficial that professors have such a framework to diagnose problems with a student's proving process?

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A CODING SCHEME FOR ANALYZING GRAPHICAL REASONING ON SECOND SEMESTER CALCULUS TASKS

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Abstract: As a first step in studying students' spatial reasoning ability, preference, and their impact on performance in second semester calculus, I ran a pilot study to develop interview tasks and a coding scheme for analyzing the interviews. Four videotaped interviews were conducted with each of the five participants and the video was coded for graphical reasoning. I will discuss my coding scheme and share some preliminary results. I hypothesize that the coding scheme may help identify a student's preference and ability for spatial reasoning.

Keywords: Second Semester Calculus; Graphical Reasoning; Coding; Interview

Introduction and research questions

Second semester calculus is widely regarded by undergraduates as one of the most difficult or most failed math classes. Underlying the primary concepts in a second semester calculus course is the limit concept, which research has shown to be difficult for many students (e.g., Tall & Vinner, 1981). Furthermore, difficulties with the concepts of differentiation and integration are often related to deficiencies in a student's understanding of limits (Orton, 1983; Tall, 1992). Combining this with the abstract nature of the material, techniques of integration and infinite series, creates a rich environment for studying student learning.

In addition to the complexities of understanding the content, I am also interested in how students' spatial reasoning impacts their success in second semester calculus. Comacho and González-Martín (2002) investigated how students fare at performing non-routine tasks using improper integrals, as well as students' use of the graphic register versus the algebraic register in completing the tasks. In particular, several of their survey questions asked students to use graphical thinking to answer and interpret a given question. They found that students preferred to use the algebraic register, even when specifically asked to use or create a graph, and that, "generally speaking, they are unable to articulate information between these two registers" (Comacho & González-Martín, 2002, p. 9). This finding highlights the importance of the psychological perspective—considering students' spatial and symbolic abilities—when instructing them on limits. A recent study of Haciomeroglu and Chicken (2011) considered several measures of cognitive abilities and performance of high school students on three measures of mathematical performance (AP Calculus AB exam scores and scores on both the Mathematical Processing Instrument and the Mathematical Processing Instrument for Calculus). They performed a correlational analysis on their data and determined that spatial orientation ability seems unrelated to calculus, although "visualizing mathematical objects from different perspectives is crucial to understanding calculus" (Haciomeroglu & Chicken, 2011, p. 68).

The research questions my larger study seeks to answer are: (1) How is the limit concept understood by second semester calculus students across the contexts of limits of functions, definite integral approximation and error, improper integrals, and infinite sequences and series? (2) Do individual differences in students' visual and symbolic reasoning skills impact students' ability to understand the limit in calculus or their performance on tasks? The purpose of this preliminary study was to develop tasks and a schema for analyzing the interviews. These research questions will hopefully be addressed in the larger follow-up study currently underway.

As an initial step to answering these research questions, I ran a pilot study during Spring 2012 that I will discuss here. I had two primary goals for this study: (1) develop and test tasks for interviews and (2) develop a coding scheme for analyzing the interview videos. I hope to get feedback on the research goals as well as the coding scheme.

Methods/subjects

The five participants in the study were chosen from one instructor's sections of a second semester calculus course at a small Midwestern University. I will refer to them by the pseudonyms Daniel, Jon, Laura, Sarah, and Travis. The participants were chosen in part because they had no previous second semester calculus experience, including no AP Calculus BC experience. The participants were all freshman, with an average age of 19 years old, with STEM majors. They volunteered in exchange for extra credit and met with the researcher for approximately one hour per week for a total of six weeks. I will focus on four weeks of videotaped interviews here (a total of 20 interviews). In each of these interviews, students worked individually on specific mathematical tasks: see Figure 1 for examples. Students were instructed to "think aloud" while solving the tasks. If needed, students were asked to clarify their thinking.

1. Find $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

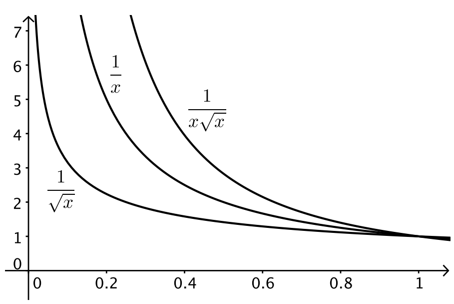
2. Find $\lim_{x\to\infty}\frac{x^5}{2^x}$.

3. Do the following integrals converge or diverge?

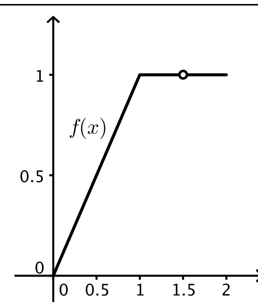
a. $\int_0^1 \frac{1}{\sqrt{x}}dx$

b. $\int_0^1 \frac{1}{x\sqrt{x}}dx$

c. $\int_0^1 \frac{1}{x}dx$



4. Find $\int_0^2 f(x)dx$.



5. Does $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ converge or diverge? Can you draw me a picture of what that series represents? Is there any connection to $\int_1^\infty \frac{1}{x}dx$?

(Note: we do not teach students the harmonic series in second semester calculus)

Figure 1: Examples of tasks

The resultant video from the interviews was coded for use of graphics when solving tasks. Graphs arose in four primary ways: a graph was given as part of the task; the participant created a graph with prompting; the participant created a graph without prompting; or the participant reused a previously created graph. See Figure 2 below for the frequency of each type. Note that the number of graphs given as part of the task is not consistent: depending on how previous participants fared on an individual problem, the presentation of the problem was sometimes adjusted. In addition, some participants frequently created graphs without prompting, while others required prompting. In each instance where a graph occurred, the graph was labeled in three separate categories, as appropriate: graph creation, reasoning from graph, and connection to symbolic reasoning (see Figure 3 below). Frequently more than one label from a category would be applied to an instance. For example, when using a graph, a participant would initially reason incorrectly with the graph. After consideration, correct and helpful reasoning would occur. Such an instance would be labeled for both "Incorrect reasoning with graph" and "Correct and helpful reasoning from graph."

In addition, I am particularly interested in those instances where a participant solves a problem symbolically, solves the same problem graphically, gets two different answers, and then tries to resolve the two answers. Several tasks were developed to generate this conflict. For example, Daniel solved the problem $\lim_{x\to 1}\frac{x^2-1}{x-1}$ and was able to correctly do the symbolic manipulation to get the limit of 2. Then, after prompting, Daniel created an incorrect graph (see Figure 4 below). In using this incorrect graph, Daniel had both consistent and inconsistent reasoning: the limit does not exist because it is different on either side; the limit is x because it is a slant asymptote. In either case, his reasoning contradicted his symbolic reasoning and he was unable to resolve this conflict. This instance, where the graph was created with prompting, was labeled as "Student created incorrect graph," "Consistent reasoning with incorrect graph," "Inconsistent reasoning with incorrect graph," and "Reasoning contradicts symbolic reasoning."

	Daniel	Jon	Laura	Sarah	Travis
Graph given as part of task	14	18	16	15	15
Graph created with prompting	7	9	15	5	14
Graph created without prompting	7	5	4	14	5
Reused previously created graph	3	6	4	4	3

Figure 2: Types of occurrences and preliminary results

	Daniel	Jon	Laura	Sarah	Travis
Graph creation:					
Easily produced correct graph	9	11	10	3	15
Able to produce correct graph after prompting or errors	0	3	6	4	5
Unable to create graph after help	1	0	1	0	0
Interviewer generated graph after unsuccessful attempt	0	0	1	0	0
Created incorrect graph and used it to complete task	4	0	0	9	0
Connection to symbolic reasoning:					
Reasoning contradicts symbolic reasoning	4	1	4	6	3
Reasoning supports symbolic reasoning	7	5	9	4	5
Reasoning from graph:					
Correct and helpful reasoning from graph	12	24	21	13	20
Correct but unhelpful reasoning from graph	3	2	5	1	2

Incorrect reasoning with graph	12	15	11	15	16
Did not use graph for reasoning	3	0	1	2	1
Unable to reason with graph	2	1	0	4	3
Consistent reasoning with incorrect graph	4	0	0	8	0
Inconsistent reasoning with incorrect graph	1	0	0	1	0
Total graphs considered	*31*	*38*	*39*	*38*	*37*

Figure 3: Labels for each occurrence and preliminary results

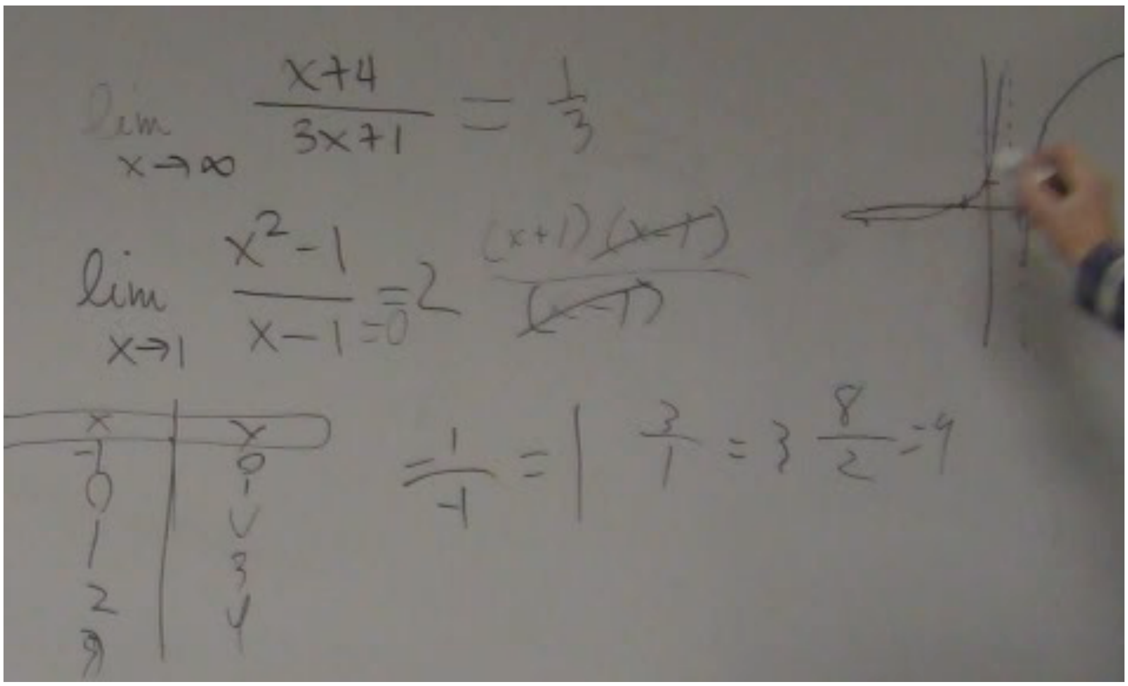


Figure 4: Daniel's solution to a limit problem

Results and future directions

From the preliminary data in Figure 3 above, we can hypothesize that Daniel and Sarah have less graphical reasoning ability than the other participants because they created more incorrect graphs (4 and 9, respectively) and fewer correct graphs (9 and 3, respectively) than the other participants. In addition, both Daniel and Sarah have less "Correct and helpful reasoning from graph" than other participants (12 and 13, respectively), supporting the hypothesis that they have less graphical reasoning ability. Sarah's inability to create correct graphs contrasts her strong willingness to create graphs without prompting: Sarah created 14 graphs without prompting. From this, I hypothesize that although Sarah may have less graphical reasoning ability, Sarah may prefer to use graphical reasoning when solving calculus problems. Finally, Daniel used fewer graphs than the other participants, which may indicate Daniel's preference for using symbolic reasoning when solving calculus problems.

During 2012-2013, I will be running a study using the piloted tasks as well as several psychological measures of spatial ability and preference, including Haciomeroglu and Chicken's (2011) Mathematical Processing Instrument for Calculus, a shortened version of Suwarsono's Mathematical Processing Instrument (1982), as well as the Form Board Test, Paper Folding Test, Card Rotations, Cube Comparisons, Diagramming Relationships and Nonsense Syllogisms Test from the Kit of Factor-Referenced Cognitive Tests (Ekstrom et al, 1976). I hope to be able to connect the interview data to the measures of spatial ability and preference. In addition, I will be collecting participants' midterm and final exams and thus will be able to track their performance in their second semester calculus course.

Questions for the audience

1. Would it be better to keep track of each occurrence of correct or incorrect reasoning with a graph, instead of just labeling each use of a graph as having correct or incorrect reasoning? Often a student would have multiple occurrences of correct or incorrect graphical reasoning within a task and in the current scheme, these would only be counted once per graph.
2. What other information might be useful to track from these interviews?
3. What other changes or additions should I make in the next round of coding?

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An Analysis of First Semester Calculus Students' Use of Verbal and Written Language When Describing the Intermediate Value Theorem

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This preliminary report describes the second stage of data collection and analysis in a larger study that examines students' written and verbal language when studying basic theorems in a first-semester calculus course. We examine students' difficulties with understanding and using mathematical language and notation in both formal written work and informal verbal descriptions. Not surprisingly, the students in our study rarely use formal mathematical language without being prompted to do so. One surprising result was that while many students do understand the mathematical notation in the theorems, and can illustrate this graphically when prompted, they still do not use this notation when providing their own written (or verbal) description of a theorem. Preliminary results suggest that our biggest obstacle as teachers is not in getting our students to understand the notation, but instead lies in convincing our students of the power that comes from this notation in describing a concept, thus encouraging our students to use this notation in their own written work.

Keywords: Calculus, Mathematical language, Mathematical notation, Emergent perspective, Constructivism

Introduction and Literature Review

The precision of mathematical language directly contrasts with the imprecise language used daily (both written and spoken) by the average undergraduate student. Additionally, human language is naturally ambiguous and variable, two qualities which only increase students' imprecision. This contrast can become the source of difficulties when these students are expected to read, write, understand and graphically interpret mathematical language and be able to move between these methods of communication freely in an undergraduate mathematics classroom. Our research is motivated by the need to develop a greater awareness of our students' specific understanding and difficulties with mathematical language.

We are particularly interested in how our students understand, interpret and express mathematical theorems given a certain level of conceptual understanding of the content. A significant portion of the recent literature in mathematics education focuses on developing a conceptual understanding of a mathematical entity in the minds of our students. However, in our pilot study, we found that even after students developed a conceptual understanding of a theorem, they were still unable to successfully describe that theorem in a written form. We believe that at least one portion of a solution to this disconnect involves helping students to increase their metalinguistic awareness of their language. The idea of metalinguistic awareness is a term from language and linguistics research and involves developing an ability to objectify and analyze language. It means that students will gain the ability to see the structures of language and manipulate them to achieve the targeted genre of language, specifically mathematical language and notation. Metalinguistic awareness has been widely recognized as an important area of attention for many educational endeavors throughout a student's development, from kindergarten on (Cazden, 1974; González & González, 2000), and we believe that this work will help us to understand how our students come to learn and use the language of mathematics.

The Intermediate Value Theorem (IVT) is often introduced in a first-semester calculus course in the beginning of the semester. As such, it is a convenient theorem in which to begin to understand students' use of mathematical language and notation. In Stewart's *Essential Calculus*, this theorem is introduced in Section 1.5, which discusses an informal notion of continuity. Recall that the IVT states that if a function f is continuous on the closed interval $[a,b]$ and N is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number c in (a,b) such that $f(c) = N$. (See Figure 1.)

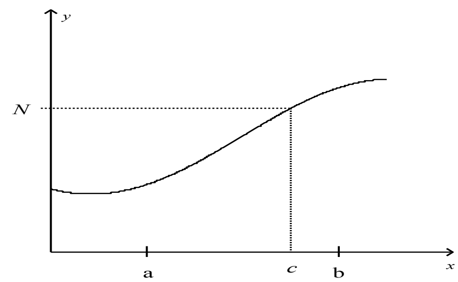

Figure 1: Illustration of the Intermediate Value Theorem

While student understanding of calculus concepts has been investigated enough to result in a relatively large research base (work on limits, functions, derivatives, etc.), the work on theorems is more limited. Abramovitz et al.(2007, 2009) developed a process for *learning* theorems (the self-learning method) to help students better understand the hypotheses and conclusions of the Mean Value Theorem and Rolle's Theorem, though language use was not a focus of this project. There has also been work conducted in the area of how students (and experts) construct and evaluate proofs of theorems in undergraduate mathematics (e.g. Weber, 2001; Weber & Alcock, 2004), though proof of mathematical theorems was not our focus in the current project. Instead, we focus on students' use of mathematical language and their ability to *express* theorems verbally and in writing.

To date, language-related issues in the mathematics education community, including classroom discourse and multi-lingual classrooms, have generated great interest (Brown, 1997; Sfard, 2000; Adler, 2001). However, the intersection of applied linguistics and mathematics education has emerged more recently. The mathematical register is the set of terms and grammatical constructions that are most appropriate for communicating mathematical ideas clearly and with the maximum amount of meaning. Barwell (2005) notes that there has been little attempt to "relate…the acquisition of the mathematical register with the acquisition of mathematical concepts" (p. 97). This work focuses on the development of mathematical language through interacting with mathematics and attention to metalinguistic awareness.

Theoretical Perspective

The theoretical perspective used in this research relies on both on Piaget's structuralism (1970, 1975) as well as on the emergent perspective described by Cobb and Yackel (1996). First, we believe that students must construct for themselves an understanding of the mathematical concepts used in each theorem. As such, the first phase in our classroom teaching of each theorem involves time for the students to explore the concepts, often in the context of an activity designed to elicit specific ideas about each theorem.

Next, we turn to Cobb and Yackel's (1996) emergent perspective to describe the teacher's role in helping the students to understand and use traditional mathematical notation and language. As teachers, we want our students to become comfortable using this formal notation and to see the power this notation provides. Cobb and Yackel (1996) describe the teacher's role as "proactively supporting both students' individual constructions and the evolution of classroom mathematical practices so that students increasingly become able to participate effectively in the mathematical practices of the wider society" (p. 186). Our current research examines which aspects of mathematical language students have appropriated into their own language as well as which aspects they understand, but have not yet taken as part of their own vocabulary.

Phase I: Data & Results

All participants in the study were first-semester calculus students at a large, public research university. The first round of data collection occurred in the Fall 2011 semester in two sections of Calculus I courses, both taught by one of the authors. Two groups of students from each section were videotaped while working on an activity that was designed to guide students to construct an understanding of the hypothesis and conclusion of the Intermediate Value Theorem. This activity was given before the students were formally introduced to the IVT by the instructor. Students were asked to draw a series of functions that satisfied some of the conditions given in the IVT. Two class periods after completing the activity, all students (n = 54) were given a pop quiz which asked students to state the Intermediate Value Theorem in their own words.

Written responses were collected and first analyzed using Corbin and Strauss' (2008) open and axial coding. Preliminary results from this phase were discussed at the 2012 Conference on RUME (Sealey, Deshler, & Toth, 2012). Our leading observation was that students used a series of unconnected indicative clauses to describe some aspects of mathematical theorems without fully understanding all logical possibilities. The data suggested that even though most students were unable to correctly write the theorem, the majority of these students did, in fact, understand the concepts behind the IVT and could express the theorem verbally. The data from the written work certainly showed that students struggled with writing the theorem correctly, but unfortunately, we did not have adequate data to show that these same students could verbally describe the IVT.

Phase II: Data & Results

Phase II of the study was designed to support the hypothesis from Phase I, namely that students are able to verbalize the IVT but not able to accurately express it in written form. Data collection for this phase occurred in the Fall 2012 semester, again in the same author's classroom as the previous year. For this phase, students worked through the same worksheet from Phase I to develop an understanding of the concept of the IVT. During class, the instructor/researcher wrote the theorem on the board and spent a significant portion of class time discussing the mathematical notation used in the theorem. We note that the instruction during this phase was more likely to have addressed many of the issues that students in Phase I were shown to have. Thus, we are not attempting to compare the students from Phase I and Phase II.

Over the next two days, a small group of self-selected students volunteered to participate in out-of-class interviews during which they were asked to describe the IVT in their own words, while being videotaped. After providing a verbal description, they were asked to provide a written description, then watch their previous verbal description (via video-recording) and to compare their written and verbal responses. Finally, the students were asked to draw a graph that illustrated the IVT and discuss how the graph related to their written response.

Video data is currently in the preliminary stages of analysis, though we have preliminary results which contradict our previous hypothesis. Namely, we are not seeing students on video who *can* express the IVT and show an understanding of the concepts behind the IVT, but are unable to express it in written form. Individual interviews seem to indicate that students possess similar written and verbal abilities with respect to being able to describe a mathematical concept. Further analysis is being conducted on the previously collected written work (pop quiz, n=54), which was the basis of Phase I.

Another interesting finding from the preliminary analysis is the discrepancies in what constitutes a "good" statement of a theorem, depending on the mode of language. Specifically, verbal descriptions were initially given higher ratings by the researchers than written descriptions that contained the same mathematical content. While this may not be surprising, it is certainly important to be aware of the discrepancies when evaluating both written and verbal responses from students.

Questions for the Audience

1. Even though our students were knowledgeable about some of the specific mathematical terms used in the formal description of the IVT, they did not provide that information during the task of describing the IVT in their own words. How might we elicit *all* the knowledge the students possess about the mathematical theorem at hand to get a better sense of their full understanding when given a written task, without providing them with prompts to use certain notation?
2. How does this work which appears to show there is not as great a disconnect between the student understanding of a mathematical concept and their ability to express it (either verbally or in written form) as previously thought by researchers fit into the larger mathematics education research knowledge?
3. How do we move forward to understand why there seemed to be a disconnect between written and verbal descriptions in Phase I, but no disconnect in Phase II? We think this could be due to the selection of students (self-selected in Phase II), a result of the instruction (since the instructor/researcher was aware of many issues from Phase I), or possibly simply that talking about the theorem first enabled students to express it in writing in a more coherent way.

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TALKING MATHEMATICS: AN ABSTRACT ALGEBRA PROFESSOR'S TEACHING DIARIES

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The world of a mathematician, with all its creativity and precision is fascinating to most people. This study is an account of collaboration between mathematicians and mathematics educators. In order to examine a mathematician's daily activities, we have primarily employed Schoenfeld's goal-orientated decision making theory to identify his Resources, Orientations and Goals (ROGs) in teaching an abstract algebra class. Our preliminary results report on a healthy and positive atmosphere where all involved freely express their views on mathematics and pedagogy.

Keywords: Abstract algebra, Mathematician's thought processes, Resources, Orientations, Goals

Literature

Mathematics has been a passion for mathematicians for centuries. Thurston (1994) poses the question: "How do mathematicians advance human understanding of mathematics?" In his view, "what we are doing is finding ways for people to understand and think about mathematics" (p.162). While this statement is encouraging, in reality the communication between the mathematicians and those outside of the community is very limited. According to Byers (2007):

> People want to talk about mathematics but they don't. They don't know how. Perhaps they don't have the language, perhaps there are other reasons. Many mathematicians usually don't talk about mathematics because talking is not their thing-their thing is "doing" of mathematics. Educators talk about teaching mathematics but rarely about mathematics itself. Some educators, like scientists, engineers, and many other professionals who use mathematics don't talk about mathematics because they feel that they don't possess the expertise that would be required to speak intelligently about mathematics (p.7).

Research in teaching and learning at the university level is fairly new. A study by Speer, Smith, and Horvath (2010) showed that research conducted has hardly examined the daily teaching practice of mathematicians and gave the following possible reasons for this lack of research:

> First, lecture can be taken as a description of teaching practice, rather than a common instructional activity within which teaching takes place. Second, the professional culture of mathematics may obscure differences in teaching and forestall discussions of teaching within the set of shared norms. Strong content knowledge and the ability to structure it for students may be taken as sufficient for good teaching. Third, collegiate mathematics teachers have limited exposure to and knowledge of pre-college research, where aspects of practice have been productively analysed (p.111).

On the other hand, some recent research activities are as a result of a partnership between research mathematicians and mathematics educators. In Nardi's (2007) view this is a 'fragile' but never the less a 'crucial' relationship between the two parties. To close this gap, Hodgson (2012), in his plenary lecture at ICME 12, raised the point about the need for a community and forum where mathematicians and mathematics educators can work as closely as possible on teaching and learning mathematics. In recent years various institutes and individuals are more willing to examine and reflect on their own teaching styles and the trend is slowly changing. For example, a study by Paterson, Thomas and Taylor (2011) described a supportive and positive association of two groups of mathematicians and mathematics educators from the same university which allowed the "cross-fertilization of ideas" (*ibid*, p. 359). The group met on a regular basis and discussed teaching strategies while watching small clips of each other's videos during a teaching episode. Similarly, the research conducted by Hannah, Stewart and Thomas (2011) indicated a case in which a mathematician took careful diaries of his actions and thoughts during his linear algebra lectures and reflected on them with two mathematics education researchers.

In describing the role of a mathematician Thurston (1994) clearly declares that: "We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics" (p. 163). Although, Mason (2002) maintains similar views, he adds that:

> .... This does not mean that it is effective to walk in and solve a lot of problems, formulate definitions and prove theorems in front of them, mindless of their presence. On the other hand, neither is it effective to give a truncated and stylised presentation which supports the impression that mathematics is completely cut, dried and salted away, that it is something that one can either pick up easily or not at all(p. 4).

The aim of this study is to investigate how mathematicians build mathematical knowledge, so in return, help students to think like mathematicians and encourage them to become independent researchers. This study is not about imposing mathematicians what to do or how to teach, rather support and share ideas with one another in order to reflect on the teaching process.

Method

The study described here is a case study that took place at the mathematics department, University of Oklahoma in Fall 2012. Two mathematicians and two mathematics educators formed a community of enquiry to examine how mathematicians think and create mathematical knowledge. The data for this research comes from the research mathematicians' day-to-day reflection of his teaching an abstract algebra course which after each class which he posted it on his website and made it available immediately to the group; the mathematics educator's observation of the classes and her written notes; weekly discussion meetings of the whole group after reading each of these reflections and the recordings of each meeting which was later transcribed for further discussion.

Theoretical framework

We have employed Schoenfeld's (2011) ROGs model to describe the professor's Resources, Orientation and Goals. By resources Schoenfeld focuses mainly on knowledge, which he defines "as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (p. 25). Goals are defined simply as what the individual wants to achieve. The term orientations refer to a group of terms such as "dispositions, beliefs, values, tastes, and preferences" (p. 29). We are also interested in mathematicians' decision making moments as part of this study and would like to examine them in detail.

Preliminary Results

Many of the emerging results deal with the professor's orientations. For example, he decided to cover ring and field theory prior to group theory (and chose the course textbook accordingly). His rationale for this approach included that he himself had been taught rings before groups, and that he believes that fields are more intuitive and familiar to students than groups. Another emerging theme in the data is both the tension and collaboration between his orientation as a mathematician and his orientation as a teacher. (This was noticed in studies by: Hannah, et al., 2011 and Paterson, et al., 2011.) For example, in one of our weekly discussions, he discussed how he was disappointed with a particular homework question on which they had spent significant time in class: calculating the degree of a field extension. He mentioned that, while most of the students seemed to understand the basic idea, they were sloppy in their execution. From a mathematician's standpoint, it is not only correctness that matters but also elegance. From a teacher's perspective, however, he resolved to be patient and decided to take the next class period to review this topic. Going on with new material, he believed, would have left a scar for the rest of the course (orientations). This compromise proved to be quite effective: the students seemed relieved. This example, in addition to providing information about the professor's orientations, also lays the groundwork for a more in-depth examination of the relationship between these two orientations as the course continues.

On another occasion the professor stressed the power of visualization of abstract concepts. This is in line with a study by Sfard (1994) in which she interviewed mathematicians in regards to the power of their visualization. In her view, "they stressed that pictures, whether mental or in the form of drawings, are only a part of the story. They support thinking, but they do not reflect it in all dimensions" (p. 48).

So far the effect of this collaboration has been positive in the sense that everyone in the group are not only focuses on the research mathematician's teaching strategies and thinking processes, but also their own teaching and decision making on day-to-day basis. Moreover, it has provided a platform allowing mathematicians to talk about mathematics freely and share their thought processes with each other.

We would like to ask the audience the following questions:

What are some themes that we could explore? What areas of study we should concentrate on that we haven't thought about? What future data we should plan to seek from? Is Nardi's "fragile" relationship between mathematicians and mathematics educators still an issue stopping the community to work closely together?

We hope to see more collaboration to produce tomorrow's mathematicians who are not only capable in doing high quality research, but also have a passion for teaching undergraduate courses.

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ANALYZING CALCULUS CONCEPT INVENTORY GAINS IN INTRODUCTORY CALCULUS

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Research in science education, particularly physics education, indicates that students in Interactively-Engaged classrooms are more successful on tests of basic conceptual knowledge. Despite this, undergraduate mathematics courses are dominated by lectures in which students take a passive role. Given the value of such tests in assessing students' conceptual knowledge, the method for measuring such change is largely unexplored. In our study, students were given one such inventory, the Calculus Concept Inventory, in introductory Calculus classes as a pretest and posttest. We address issues of how gains might be measured on this instrument using two techniques, and the implications of using each of these measures.

Keywords: Calculus, Measurement, Concept Inventories

A recent report in the MAA Notices stated that almost two thirds of college and university instructors surveyed agreed with the statement that "Calculus students learn best from lectures, provided they are clear and well-prepared" (Bressoud, 2011, p. 1). Strong support for traditional lecture as the primary means of undergraduate mathematics instruction remains in spite of a growing body of research supporting the claim that students learn best when they are interactively and cognitively engaged with subject matter in mathematics and other sciences (Epstein, 2007; Hake, 1998a; Prather, Rudolph, Brissenden, & Schlingman, 2009; Rhea, n.d.; Smith et al., 2005). These studies often use instruments called concept inventories to measure conceptual knowledge gains by giving the instrument as a pretest and posttest. We discuss two types of measures of gain on one such instrument, the Calculus Concept Inventory, given to introductory calculus students at a large southwestern university, and the differences observed by using each of the measures of gain.

Background

Conceptual understanding may be measured through instruments called concept inventories. The first concept inventory, the Force Concept Inventory (FCI), is a test in introductory mechanics which paved the way for analyzing student conceptual understanding of the basic ideas in a subject area (Hake, 1998a, 2007; Hestenes & Wells, 1992; Hestenes, Wells, & Swackhamer, 1992). Since then, many concept inventories have been written in various subject areas (Allen, 2006; Anderson, Fisher, & Norman, 2002; Carlson, Madison, & West, 2010; Carlson, Oehrtman, & Engelke, 2010; Garvin-Doxas, Klymkowsky, & Elrod, 2007; Libarkin, 2008; Marbach-Ad et al., 2009, 2010; Mulford & Robinson, 2002; Prather et al., 2009; Rhoads & Roedel, 1999). There has been active discussion about how to interpret the results of the FCI (Heller & Huffman, 1995; Henderson, 2002; Hestenes & Halloun, 1995; Huffman & Heller, 1995). The concept inventory discussed in this paper is the Calculus Concept Inventory (CCI), developed by Epstein (2007).

One of the most well-known uses of the FCI was Hake's (1998a) comparison of classrooms which utilized Interactive-Engagement (IE) methods with those which were described as "traditional lecture" (TL). In his study, Hake defined IE teaching as a collection of teaching methods which are “designed at least in part to promote conceptual understanding through interactive engagement of students in heads-on (always) and hands-on (usually) activities which yield immediate feedback through discussion with peers and/or instructors” and found differences between the two types of classes of almost two standard deviations (Hake, 1998a, p. 65). The concept of an IE classroom has been further explored in physics by Hake (1998b) and in mathematics by Epstein (2007). IE teaching styles share features with Peer Instruction (Mazur, 1997) including ConcepTests (Pilzer, 2001), and pure discovery learning (Paris & Paris, 2001).

Normalized Gain

Normalized gain is a measure first used by Hake (1998) in his study of Interactively-Engaged teaching styles with the Force Concept Inventory to measure how much material has been learned by students during a course. This measure is almost always used in concept inventory studies for measuring gains. In particular, this is the gain score that Epstein (2007) and Rhea (n.d.) used to report their findings on the Calculus Concept Inventory. The normalized gain score is defined as

$$\langle g \rangle = \frac{Final - Initial}{Total - Initial}$$

and measures the fraction of unknown material learned throughout the course. For example, if a student correctly answered 50% of the questions on the pretest, and 75% on the posttest, the normalized gain would be <g> = 0.5, meaning that student correctly answered half of the 50% of the material they did not know at the beginning of the class. Normalized gain is often calculated using section averages of pretest and posttest scores, so each section of a course will be assigned a single normalized gain score. Many studies compare the effects of instructional practices on student learning, so the effect of interest is at the section level: normalized gains calculated at the section level allow one to analyze the effect of instructional practices on the entire class. One can also create an individual normalized gain score by using the pretest and posttest score for each student. The effect of computing individual normalized gains for each student has been investigated and compared to using section-level normalized gain scores (Bao, 2006; Coletta & Phillips, 2005). The two methods produce close, though not numerically identical, results. The advantage of considering individual-level normalized gains is that class-level variables can be considered along with student-level variables such as demographics or previous mathematics courses.

Item Response Theory

Item Response Theory (IRT) is a modern approach to analyzing instruments like tests or surveys (Embretson & Reise, 2000). IRT is based on the idea that an instrument measures a single latent trait or ability, such as conceptual knowledge of calculus. While this trait cannot be directly observed, the effects can be observed through answers to questions. In an email to Hake, Mislevy (n.d.) pointed out that IRT has some benefits over the use of normalized gains such as handling floor and ceiling effects (e.g., students who obtain a perfect score obtain a normalized gain of 1 regardless of initial ability levels). IRT also provides the opportunity to analyze individual questions as opposed to a single test score for each individual.

IRT is a methodology for predicting ability levels based on an instrument and can be leveraged to determine gain scores by measuring ability levels on the pretest and on the posttest. The difference between these two ability levels is the change over the course, and so measures gain (Wallace & Bailey, 2010). Despite the benefits of using IRT to create gains scores instead of using normalized gains, as Mislevy suggested to Hake for analyzing the FCI, IRT analysis is very rarely used in science and mathematics education research (Wallace & Bailey, 2010). The

FCI has been analyzed using IRT (Wang & Bao, 2010), and comparisons of IRT and normalized gain methods have been made in astronomy (Wallace & Bailey, 2010). An IRT analysis of the CCI has not been published, nor have multiple measures of gain been studied for this instrument. Our study contributes to the existing literature in both of these areas.

Purpose of Study

Our study builds on previous studies by considering the implications of using both normalized gain and IRT to measure gains on CCI. Previous studies have investigated the effects of instructional practices on student learning by using concept inventory pretests and posttests. In order for the connections between instructional choices and learning gains to be studied, a method for measuring learning gains needs to be established. It is also important that these gain scores accurately reflect learning since they may be used for practical decisions, such as administrative decisions involving the careers of teachers. If a teacher's career is affected by how much their students' scores improve on an exam, it is worth considering that there are multiple ways to measuring these gains which may produce different results.

Methods & Analysis

The subjects in this study were Calculus I students at a large southwestern university. All Fall 2010 Calculus I students at the university were required to take the Calculus Concept Inventory as a pretest and posttest as part of the course, and consenting students had their scores collected along with demographic information. The pretest was graded for course credit on completion, and the posttest scores were factored into students' final grades. Of the 880 students who took the pretest and 668 students who took the posttest, 507 students took both tests and consented to participate in the study. Of these students, 482 had non-zero scores on the CCI pretest, CCI posttest, and the final exam, meaning they did not miss any of the tests. There were 26 sections, with a maximum capacity of 35 in each section. On average, 18.5 students per section participated in the study, ranging from 10 to 26.

Analysis was done using a combination of the software tools BILOG-MG and R. These are commonly used software tools for conducting IRT analysis (BILOG-MG) and general purpose statistical analysis (R).

Results and Implications

Normalized Gain Scores

In his 1998a study, Hake grouped sections by their normalized gain, <g>, scores: “low-g” sections were defined as those with <g> values less than 0.3, “medium-g” as those between 0.3 and 0.7, and “high-g” as those above 0.7 (p. 65). In his study, Epstein (2007) gave the CCI to 1100 students at 12 American universities and 1 university in Finland. He found <g> values largely clustered between 0.15 and 0.23, similar to the scores in traditional lecture physics classes on the FCI. A large midwestern research university with a department-wide IE focused teaching style reported an average <g> score of 0.35 among their 51 sections, with a range of 0.21 to 0.44 (Rhea, n.d.). Ten of the sections had <g> scores above 0.40.

The mean normalized gain for the entire participant group at the large, southwestern university where our study was conducted was 0.25, meaning that 25% of the previously unknown material was learned during the course. Normalized gain scores ranged from 0.14 to 0.36. By Hake's definition, 4 of the 26 sections had medium-g scores, and the remaining 22 had low-g scores.

Individual normalized gain scores were then computed so that a comparison with IRT gains could be made, since IRT gains are computed at the individual level. A histogram is given in

Figure 1. The difference between using individual normalized gains averaged by class and class average normalized gains makes almost no difference in this data set, as the mean was slightly higher, but still 0.25 when rounded to two decimal places.

IRT Gains

Item Response Theory allows for a variety of models to be created. The Rasch model is the simplest of these, estimating a single parameter for each item. In the Rasch model, the probability of an individual, *i,* correctly answering question *p* is given by

$$P(X_{pi}=1|\theta_p,b_i)=\frac{\exp[\theta_p-b_i]}{1+\exp[\theta_p-b_i]}$$

which results in a logistic curve for each item (Embretson & Reise, 2000). For the CCI pretest under the Rasch model, the plot of curves for each item is given in Figure 2. The interpretation is that for any level of conceptual knowledge of calculus (θ), there is a corresponding item-specific probability of correctly answering that item. The difficulty of item i, b_i, is the ability which is required for a 50% chance of correctly answering the question. For the Rasch model, the difficulty of each item uniquely determines this curve, called the item characteristic curve.

One can introduce a different model by introducing a new parameter, α, called the discrimination. This model is given by the formula

$$P(X_{pi}=1|\alpha,\theta_p,b_i)=\frac{\exp[\alpha(\theta_p-b_i)]}{1+\exp[\alpha(\theta_p-b_i)]}$$

The effect of the discrimination parameter is to change the slopes of all the item characteristic curves. The larger the value of α, the steeper the slope of the curve, and the more discriminating the item. This model is known as the one parameter logistic model (1PL). A plot of the item characteristic curves for the CCI pretest is given in Figure 3.

The two parameter logistic model (2PL) relaxes the condition that the discrimination parameter must be the same for all items, resulting in the following model.

$$P(X_{pi}=1|\alpha_i,\theta_p,b_i)=\frac{\exp[\alpha_i(\theta_p-b_i)]}{1+\exp[\alpha_i(\theta_p-b_i)]}$$

The graphs of item characteristic curves for the CCI under the 2PL model is given in Figure 4.

Looking at Figure 4, it is apparent that the behavior of the last item on the test is counter to what one would expect for a test measuring a single construct: the item characteristic curve for item 22 is decreasing, indicating that as one's conceptual knowledge of calculus increases, the likelihood of answering the item correctly decreases. The same item was also a poor fit on the posttest, where the questions were reordered. This is seen in Figure 5, where item 20 does not fit the pattern of the other items well. Since no explanation for this behavior was apparent, the item was removed from future analysis. Using the method proposed by Bejar (1980), it was determined that the remaining items were assessing the same construct.

To ensure that the pretest and posttest scores were comparable, item parameters were estimated from the pretest, and these item parameters were used to estimate individual ability levels for both pretest and posttest students, following the method used by Wallace and Bailey (2010). Therefore, a score of 0 was a common score, interpretable as the ability of an average student taking the pretest. A student who is estimated to have an ability level of 0 on the posttest would then have an ability comparable to an average student at the beginning of the course.

Comparison of the Two Gain Models

A comparison of the section average normalized gains and IRT gains averaged by section is displayed in Figure 6. The two measures of gain are strongly correlated, $r(21) = 0.92$, $p < 0.01$, so 86% of the variation in one measure is explained by the other. This is low if the two quantities are actually measuring the same trait – the knowledge gained over the course. At the student level, the correlation between the normalized gains and IRT gains is similarly correlated, $r(480) = 0.92$, $p < 0.01$, suggesting that 15% of variation of one measure is still unaccounted for by the other measure. This relationship is shown in Figure 7. If one were to interpret one of the two gain measures (IRT or normalized gains) as "correct," then using the other measure would introduce error. This suggests that, unless one measure can be objectively preferred to the other, the two measures provide different information, and answer different questions.

Practically, one of the fundamental differences between normalized gain scores and IRT gain scores is the dependence of the normalized gain score on the pretest score. Given two classes with the same difference between pretest and posttest scores, the normalized gain will be larger for the class with the higher pretest score (Wallace & Bailey, 2010). Consider a concept inventory with 22 questions. Suppose a class with a pretest average of 18 points achieves a posttest average of 20 points, and another class with a pretest average of 11 points achieves a posttest average of 13 points. The normalized gain score would be higher for the first class more than the second. Those 2 points were likely more difficult to achieve than the 2 points achieved by the second class since the pool of available questions to improve upon is smaller for the first class. Additionally, if the difficulties of the questions are distributed roughly normally, a class with a medium pretest is likely to encounter questions which are not far beyond their ability, while a high achieving class may encounter questions which are farther from their (current) ability. In this way, the normalized gains are rewarding classes in a reasonable way.

From a theoretical point of view, the normalized gain is a test-specific measure of gain. The normalized gain is the total learned out of the total that could be learned, implying a maximum knowledge level. Once a student has correctly answered all of the questions on the instrument, the total knowledge has been achieved for that construct. IRT does not have this type of frame of reference of total knowledge. In this way, normalized gains are a measure much more closely tied to a particular instrument than IRT gains. Consider the following example. Suppose version 1 of a concept inventory has 2 questions which are correctly answered by everyone who takes the test, and version 2 of the concept inventory replaces those 2 items with 2 items which are answered incorrectly by everyone on the test. In an IRT analysis, this change would not make any difference since these 2 items provide no information that allows individuals to be compared. In a normalized gain analysis of the concept inventories, however, the normalized gains will be different. If the concept inventory had 22 questions, a change from 14 to 16 ($\langle g \rangle$=.25) on version 1 would become a change from 12 to 14 ($\langle g \rangle$=0.2) on version 2. These scores are still worth considering and can reveal a great deal about gains during a course, but the dependence on the instrument itself should not be ignored. This is also noteworthy when comparing normalized gains on the FCI to normalized gains on the CCI. These are two completely different instruments, and so comparing normalized gains on one with the other may not be reasonable. In particular, useful cutoffs for high-, medium-, and low-$\langle g \rangle$ scores may not transfer from one test to the other.

Conclusions

Measuring gains using a pretest and posttest appears like a simple task, but the two measures investigated here produce quite different results. Both IRT gains and normalized gains aim to determine the amount of learning that has taken place during a course, but they are measuring this quantity in different ways, producing different results. A priori, there is no objective way to chose one measure as preferred to the other, as each have advantages. IRT produces measures which are test and population independent, but are more difficult to interpret than normalized gain scores. IRT's test independence is an advantage. With IRT, if a different test were created which measures the same construct as the CCI and were given to the same population, the individual ability estimates would remain unchanged. This is not the case for a normalized gain analysis. The maximum is achieved when the student has successfully mastered the material that is on the test. While much can be learned from studies of other concept inventories, he CCI is measuring mathematics-specific ability, and so needs to be studied further. The difference between normalized gains and IRT gains demonstrates that we can assess gains on the CCI in different ways and achieve different results, highlighting the need for attention to the type of gain score used.

Plans for Future Research

The Calculus Concept Inventory plays an important role in our study as an externally created and validated measure of student understanding of the concepts of introductory calculus. To build upon the analysis of gains presented here, hierarchical linear models will be created to incorporate student-level variables such as demographic information and mathematics background, and final exam scores will be used to consider the relationship with potentially different types of knowledge.

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Figures

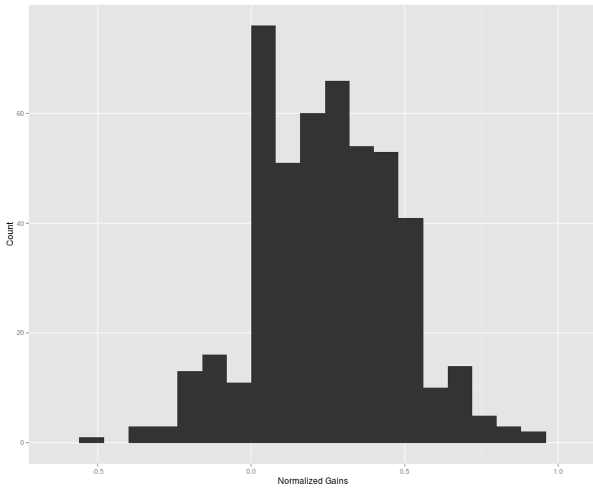

Figure 1: Histogram of Normalized Gains

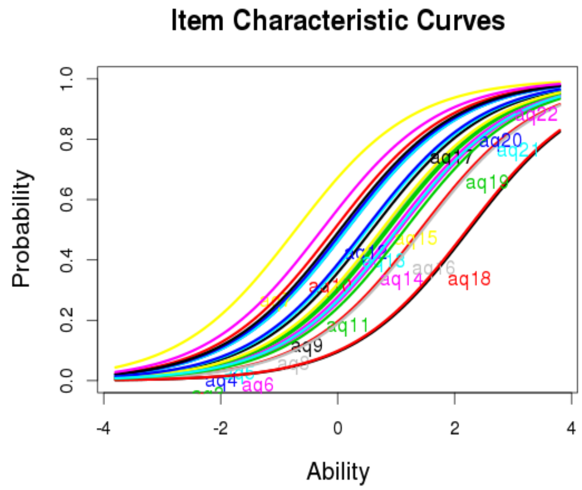

Figure 2: Rasch Model for CCI Pretest

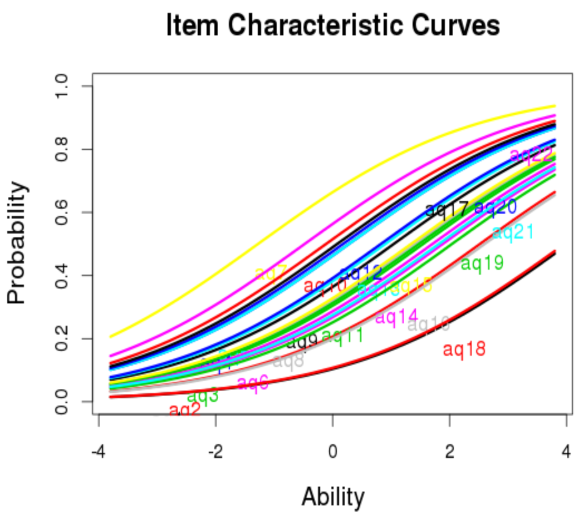

Figure 3: 1PL Model for CCI Pretest

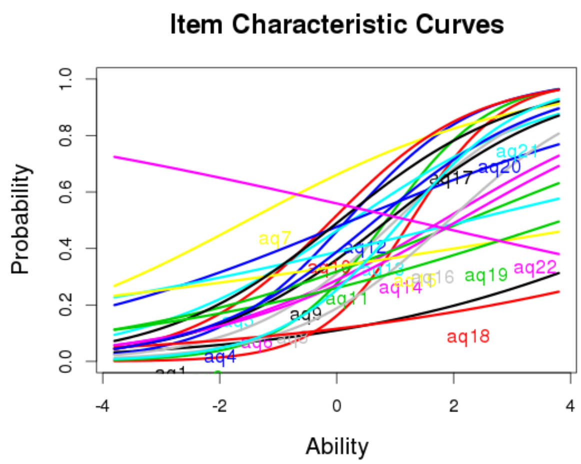

Figure 4: 2PL Model for CCI Pretest

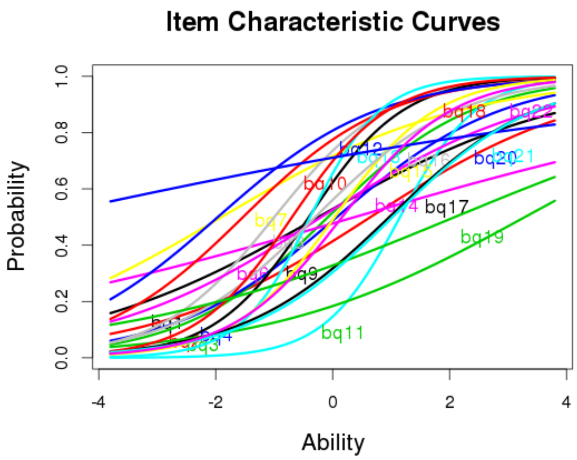

Figure 5: 2PL Model for CCI Posttest

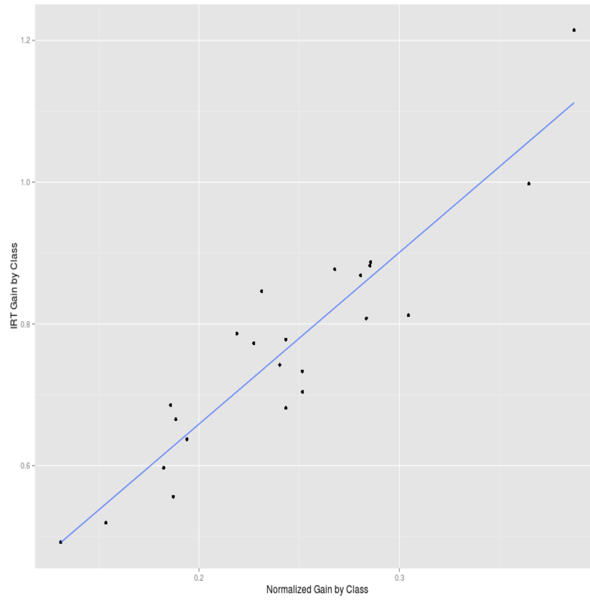

Figure 6: Normalized Gain vs. IRT Gain by Section

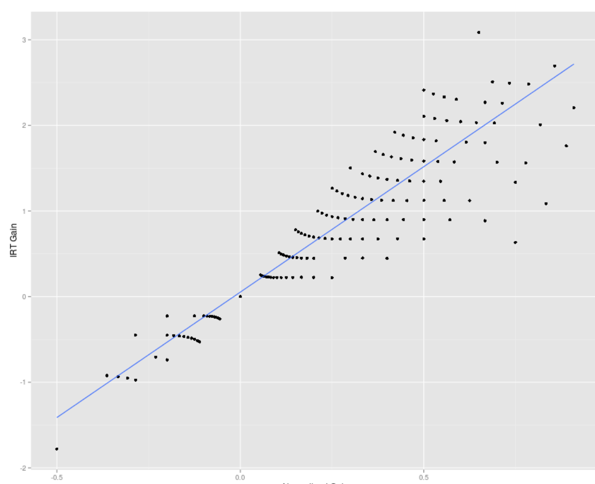

Figure 7: Individual Normalized Gains vs. IRT Gains

EXPERT PERFORMANCE ON ROUTINE AND NOVEL INTEGRAL APPLICATION (VOLUME) PROBLEMS

Preliminary Research Report

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Past research has shown that students struggle when applying the definite integral concept, and these difficulties stem from incomplete understanding of the integral's underlying structure. This study aims to provide insight into the construction of effective mental structures for integrals by examining experts' solutions to volume problems. Seven mathematics faculty members from a large, public university solved three volume problems (two routine, one novel) in videotaped interview sessions. Preliminary analysis shows that the experts have a rich understanding of definite integrals, and the few instances of errors seemed to be a result of inattention as opposed to a deficit in understanding. Their problem-solving process was highly structured and detailed. The experts' visual representations varied from sparse and static to fully 3-dimensional and dynamic. We hope to use this and past student data to construct a framework for analyzing student understanding of integral volume problems.

Keywords: Calculus, definite integral, volume, visualization, expert

Introduction and Literature. Determining volumes of solids is an application of the definite integral that is routinely covered in a second-semester calculus course, but very few studies have been conducted with the aim of understanding how students conceptualize these problems. Previous research has found that students have a very weak intuitive understanding of the definite integral and its underlying approximation and accumulation structure (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Thompson & Silverman, 2008; Sealey, 2008; Huang, 2010). In an early study on student understanding of integration (Orton, 1983), students were asked to give detailed explanations of their reasoning when solving integration problems. Orton observed that students had very little idea of the dissecting, summing, and limiting processes involved in integration. Huang (2010) observed students focusing on "calculating correctly, while ignoring the true meaning of the concepts behind the calculations."

A key component in successfully solving volume problems is visualization of the solid. Stylianou and Silver (2004) compared frequency in and nature of the use of visual representations by experts and novices while solving arithmetic word problems. They found that, even though the construction of a diagram or picture is helpful, it is the quality of the picture that is most important. Experts more frequently recognized and highlighted critical features of the visual image, which helped to focus their attention and guide the rest of the problem-solving process. For volume problems, visualization of the solid and its constituent parts guides and dictates the construction of the corresponding volume integral. Students must first translate the information given in the statement of the problem into a visual representation of the physical situation, usually by sketching. Then, they must extract specific information from their sketch and represent the information symbolically in the form of a definite integral. Successful completion of the second transfer of information (from pictorial to symbolic) requires that the student know what type of information to obtain from the sketch as well as what to do with it.

Research Question. The current work expands on the above literature in an attempt to understand how visualization ties into students' ability to set up a definite integral for tasks involving solids of revolution. The goal of this study is to more deeply explore student understanding of applications of the definite integral. The first phase involved identification and classification of common mistakes students make when setting up and solving volume problems. The second phase (discussed here) involved interviews with mathematics faculty members (which we consider to be experts) during which they solved integral volume problems of varying levels of difficulty. Our goals in interviewing experts were to observe their techniques and strategies for solving routine volume problems, and to examine how they approach novel volume problems. We are primarily interested in learning (1) if and how experts use visualization techniques like gesturing and diagrams, and (2) how each of the pieces of the underlying structure of the definite integral contribute to their problem-solving processes. Specifically for the second research question, we will compare the important elements of the structure of the definite integral (product, summation, and limit) the experts employ when solving volume problems to those employed by the subjects in Sealey's (2008) study involving other applications of the definite integral. Based on the analysis of the expert data, we aim to create a preliminary framework for analyzing student understanding of definite integral volume problems.

Theoretical Perspective. Our research is built on the foundation of the constructivist learning theory (Piaget, 1970). We certainly acknowledge that the experts in our study do not need to construct an understanding of the definite integral, but we do believe that the actions of the experts can give insight into their mental structures. The conceptual framework guiding data analysis is taken from the work done by Zazkis et. al (1996) on student understanding of the dihedral group D_4. The Visualization/Analysis (VA) model deals with both visual and analytic thinking and how these two types of cognition are used together in problem solving. Application problems in calculus almost always require a transfer of information from one form to another (written, pictorial, symbolic, numerical, etc.) and volume problems in particular involve the transfer of information from visual representation into symbolic form. Using the VA model, we will be able to more systematically examine and categorize understanding of integral application problems by focusing on the instances of information transfer during the problem-solving process.

Research Methodology. Interviews with experts (mathematics faculty members) were conducted during the Fall 2012 semester at a large, public, research university. The seven participants were faculty members of a mathematics department who responded to a faculty-wide e-mail sent out by the primary investigator. The sample included four full professors, two assistant professors, and one teaching assistant professor. Two of the professors are female and five are male. The sample also varied in how recently the professors had taught a course that included discussion of integral volume problems – four professors had taught integral volume problems within the past five years, and three professors had either never taught the topic or had taught it more than five years ago. While we acknowledge that their ability to recall information about volume problems may be weak, we still consider all of the participants to be experts in mathematics.

During the interviews, the participants were asked to complete three problems concerning volumes of solids of revolution; the first two problems were considered "easy" or routine problems that are typically solved by students in a second semester calculus course, and the third was more complex than would be expected in such a course (see Appendix A). As they worked through each problem, the participants were asked to think out loud and elaborate on what they had written. The experts were asked to first approach each problem from an expert point-of-view and then discuss how they would teach the concept in class. Each interview was videotaped and transcribed for analysis.

Preliminary Results. In the first phase of this study, students exhibited errors in nearly every component of the problem: variable of integration, bounds of integration, integrand formula, accuracy of sketch, and relationship between sketch and integral set-up. The experts' mistakes coincided with students' mistakes, although the experts' mistakes were less numerous, and no difficulties with sketching or determining the variable of integration were observed.

As expected, the experts had very few difficulties with the first two problems, although only two of the seven were able to complete both routine problems error-free. More errors occurred in the first problem than the second problem despite the fact that the second problem is generally considered to be more difficult. We suspect that if we had switched the order of the problems, the experts would have more errors in whichever problem they attempted first, and we do not see the errors as being a conceptual misunderstanding with these experts, but instead an inattention to details. Interestingly, a mistake that would be considered evidence of a student's complete misunderstanding of integral volume problems was made by two different experts on the first problem. Experts' visual representations varied greatly with respect to aspects like level of detail, use of dynamic indicators like rotation arrows, and 3-dimensional details.

The third problem was indeed novel to all seven participants despite the fact that second-semester calculus knowledge is sufficient to solve it. Our aim in presenting the experts with a novel problem from a relatively simple area of mathematics was to get them out of their comfort zones and to see how their methods for solving the routine and novel problems compared. Even though a majority of the experts approached the third problem by first rotating the coordinate axes – a technique that may not be accessible to most second-semester calculus students – we believe that analysis of this data has the potential to produce useful insight into effective thinking. At this time, it seems as though the ways in which the experts attended to and explained the "*dx*" in the first two problems are indicators of how their solution strategies and performance on the novel problem. As of the submission of this proposal, data analysis is still in the preliminary stages, so we look forward to being able to share more of our findings at the conference.

Future Research/Teaching Implications. The research discussed here is one phase of a larger study on student understanding of the definite integral when applied to finding volumes of solids. We plan to use the expert interview findings in conjunction with previous research on integration to produce a framework for analyzing student understanding of the definite integral. Volume problems can be considered as somewhat atypical compared to most single-variable calculus application problems due to their highly visual nature. The vast majority of physical situations encountered in first- and second-semester calculus application problems are 2-dimensional, and there are many physical situations discussed that cannot be accurately represented by a picture (e.g., work). We believe that this aspect of volume problems makes them a powerful tool in improving students' mathematical maturity and strengthening their ability to transfer knowledge between different representational domains.

Questions for Audience.
1. How does analysis of expert problem-solving techniques contribute in a meaningful way to advancing student understanding?
2. Are there any other aspects of expert data to which we should be attending?
3. How do we learn from our experts' mistakes?
4. Are there any examples of "real-world" problems or situations in which integration is used in determining the volume of a 3-dimensional object (e.g., geology)?

Appendix A. *Interview Problems*

For each problem, set up the integral that gives the volume of the solid.

1. The solid obtained by rotating the region bounded by the curves $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

2. The solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{x}{2}$ about the line $y = 3$.

3. The solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = x$ about the line $y = x$.

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COMPARING A "FLIPPED" INSTRUCTIONAL MODEL IN AN UNDERGRADUATE CALCULUS III COURSE

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In this small comparative study, we explore the impact of "flipping" the instructional delivery of content in an undergraduate Calculus III course. Two instructors collaborated to determine daily content and lecture notes; one instructor altered the instructional delivery of the content (not the content itself), utilizing videos to communicate procedural course content to students out-of-class, with time in-class spent on conceptual activities and homework problems. Student performance on tests for both classes will be compared to determine any significant differences in achievement related to "flipping" the instructional delivery.

Key words: Flipped Classroom, Technology, Calculus III, Comparative Study

The educational landscape has been transformed in the past 25 years due to the accessibility and integration of technology in the classroom. One of the latest technological trends attempts to completely re-conceptualize the in-class and out-of-class experience: "flipping" the classroom. The premise for this instructional model is that student and teacher interactions during class time can be maximized by offloading course content onto videos or screencasts to be watched from home (Bergman & Sams, 2008). In this study, we explore the impact on student performance of "flipping" an undergraduate Calculus III course.

Literature

At the turn of the 21st century, educators began discussing the potential benefits of "inverting" the classroom model. Lage, Platt, and Treglia (2000) described the inverted classroom model as "events that typically take place *inside* the classroom now take place *outside* the classroom and vice versa" (p. 32); they presented anecdotal evidence through student and faculty perceptions of an undergraduate economics course to claim the model had potential to help create an inclusive learning environment for diverse students. In the past 10 years, increased accessibility to free video servers (e.g., YouTube), podcasts (e.g., iTunes U), and tablets with screencasting software have made this approach less cost-prohibitive and more feasible on a large scale (e.g., Khan Academy).

Two high school science teachers, Bergman and Sams (2008), popularized an analogous approach that has come to be known as the "flipped" classroom. In the past few years, technology conferences have included panels and had numerous sessions about the "flipped" model (e.g, ISTE Conference); there is also an annual conference (The Flipped Class Conference), an online professional learning community (Flipped Learning Network), and large funding for "flipped" initiatives in education (e.g., Gates Foundation funding the Khan Academy). However, while demand and interest are high and the rationale for the approach is compelling, the majority of research is either anecdotal or contains minimally convincing, non-comparative data collected within a single course (Bergman & Sams, 2008; Gannod, Burge, & Helmick, 2008). Perhaps the most rigorous study, which demonstrated statistically significant gains in a physics class from a comparative study (Deslauriers, Schelew, & Wieman, 2011), is only partially related to the "flipped" model. While there is optimism about its potential, some benefits of "flipped" instruction may not translate to mass implementation (Hertz, 2012). There is little information about how the "flipped" model impacts learning in various subject areas or age groups. We aim to add research to the existing literature about the "flipped" classroom approach that is: 1) specific to undergraduate mathematics instruction; and 2) has a comparative research design, to explore the causal impact of utilizing different instructional delivery models for identical course content.

Methodology

Two mathematics professors and a mathematics educator at a mid-size private university collaborated to study the impact of "flipping" an undergraduate Calculus III course. Two sections of Calculus III (each professor taught one) were both taught three days a week (50-minute class periods) during Fall 2012. The research questions addressed were:

Does "flipping" the instructional delivery in an undergraduate Calculus III course:

1. Impact students' overall performance, or their performance on procedural or conceptual mathematics problems?
2. Impact students' opinions and perceptions about the course regarding in-class and out-of-class interactions with the content and the professor?

For this research, problems on homework assignments and exams were categorized as primarily *procedural* or *conceptual*. The purpose in doing so was to add an additional layer of analysis into the study, aiming to report about whether the "flipped" instructional model had more impact on students' performance in either problem type. Adapted from the National Research Council (2001), we defined: *procedural* questions as those that primarily require carrying out a standard mathematical procedure or algorithm (e.g., calculate the partial derivative of a function); and *conceptual* problems as ones that primarily require explanation/generalization of mathematical concepts or application of procedures in non-standard settings (e.g., interpret the partial derivative of T=f(x, y, z) with respect to z).

To answer the first research question, significant efforts were made to make the content in two Calculus III courses as identical as possible in order to isolate the impact of instructional delivery. In addition, the labor of "flipping" the classroom was made into as simple and manageable of a process as possible. Data collected includes: student demographic information, attendance, homework completion, student exam scores, with sub-scores for procedural and conceptual problems, field notes from both classes, and a student perception survey. For this study, we characterize each model of instruction by the following:

Traditional instructional model: In-class, the professor primarily lectures by writing notes and examples on the board. Students mainly take notes, with minimal student-to-instructor dialogue and no student-to-student interaction. Out-of-class, homework problems are assigned for students to complete on their own.

"Flipped" instructional model: Out-of-class, students watch short videos (~20 minutes) prior to the class of a lecture prepared by the professor on primarily procedural course content. In addition, students complete one or two procedural homework problems based on the video. In-class, the professor facilitates whole-class and small-group discussions based on more conceptual course content, using additional problems from the lecture notes or homework problems. Part of the class period will be spent having students work on homework problems, turned into activities for learning more conceptual content.

During the Fall 2012 semester, two mathematics professors collaborated on lecture notes, homework assignments, and assessments for the course. The course was split into three units, with an exam at the end of each unit. For the first one-third of the course, both professors followed the same lecture notes and taught according to a traditional model of instruction. The purpose was to substantiate that students would perform similarly when the professors taught similarly. For the second two-thirds of the course, one professor continued to teach in a traditional manner whereas the other switched to a "flipped" instructional model. The mathematics education researcher helped establish a system to "flip" the instruction, from selecting video content to altering homework problems for meaningful in-class discussion.

To guarantee that students in both sections received nearly identical content under the two instructional models, the following precautions were taken: 1) *Both professors followed the lecture notes agreed upon for each class.* Professor A (traditional) gave a lecture in-class. Professor B ("flipped") selected the lecture notes that were primarily procedural and created a video for students to watch out-of-class; the remainder of the lecture notes was turned into discussion-based activities for in-class. 2) *Both professors assigned and collected the same homework problems.* Students of Professor A turned them in the following class period. Students of Professor B turned in one or two of the procedural problems after watching the video, and had the opportunity to work on and complete the remainder of the homework problems in-class, turning them in the following class period. This helped provide similar out-of-class time demands for both sections. 3) *Both professors gave identical exams*. The professors collaborated to write and select the exam questions for each unit. And 4) *Both professors graded exams.* To make sure there was no bias in grading, the professors split the grading so that each professor graded the same exam questions for both sections.

To answer the second research question, additional data will be gathered from a student survey at the end of the course and a group interview with a representative sample of students from each course. The researchers will write the survey instrument and interview protocol, taking into account instruments used in other research studies on the "flipped" model; both the quantitative and qualitative data should help answer the second research question.

Preliminary Findings

Nearly all students in both classes agreed to participate in the research study: 41 out of 45 in the traditional section, and 39 out of 44 in the "flipped" section. Based on an analysis of participating students' demographic information, there are no statistically significant differences ($p<.05$) in the student makeup of each course regarding: gender, age, ethnicity, major, class (e.g., freshman, sophomore), Calculus II grades, or SAT Math scores.

During the first one-third of the course, the mathematics educator researcher took field notes during two different class periods, which helped characterize the instruction for this period. While both professors had slightly different ways of explaining ideas and interacting with students, they covered the same content and their instruction was similar. Based on field notes and students' performance on the first unit exam, there was no statistically significant difference between the two sections in the distribution of scores ($p=.342$), indicating that student performance was similar when both professors used a traditional instructional model.

We are still collecting data from the rest of the semester (complete by December 2012). While we currently do not have data on the impact of the "flipped" approach, we have evidence that the participating students and professors are similar, which provides a good foundation for the comparative analysis. Once data has been gathered, statistical analysis for comparing the populations will be coupled with visual trends, as the size of the study may limit statistical conclusions. Drawing decisive conclusions about the effectiveness of different instructional models by comparing only two undergraduate classrooms is inadequate; however, the results from this study will add to the growing body of literature on the "flipped" model. In particular, the comparative design and efforts to make content as identical as possible to isolate the instructional delivery should give meaningful conclusions.

During the presentation, we will: detail our process for transforming lecture notes and the homework assignment into a "flipped" model with identical content; present data about the impact on student performance of the "flipped" instructional model; and discuss results regarding students' perceptions about the two courses. The questions we will discuss during the presentation include: What limitations do you see in our process of removing variables to isolate the content delivery, as characterized by the two instructional approaches? How manageable was the process for "flipping" lecture notes and homework assignments?

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TWO STUDENTS' INTERPRETATION OF RATE OF CHANGE IN SPACE

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This paper describes a model of the understandings of two first-semester calculus students, Brian and Neil, as they participated in a teaching experiment focused on exploring ways of thinking about rate of change of two-variable functions. I describe the students' construction of directional derivative as they attempted to generalize their understanding of one-variable rate of change functions, and characterize the importance of quantitative and covariational reasoning in this generalization.

Key words: Rate of change, covariation, functions, graphs, quantitative reasoning.

Background and Research Question

Mathematics and science are concerned with characterizing the behavior of complicated systems. The transition students make as they shift from thinking about systems with two quantities varying to systems with three or more quantities varying has not been fully explored. In response to the need to understand how students model change in complicated systems, this study sought to gain insight into,

> *What ways of thinking do students reveal in a teaching experiment that is focused on the meaning and measurement of rate of change in space?*

Understanding rate of change is foundational to ways of thinking about ideas in calculus, yet many students possess difficulties reasoning about rate (Carlson et al, 2001; Carlson et al., 2003; Monk, 1987; Rasmussen, 2000; Thompson & Silverman, 2008). Students' difficulties understanding rate of change range from problems interpreting the derivative on a graph (Asiala et al., 1997) and focusing on cosmetic features of a graph (Ellis, 2009). Thompson (1994) found that the difficulties student's displayed in understanding the fundamental theorem arose from impoverished concepts of rate of change and incoherent images of functional covariation. Thompson described a coherent way of thinking about average rate of change of a quantity as, "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred". The characterizations of difficulties students have in thinking about rate and Thompson's scheme for thinking about rate were the foundation for a conceptual analysis. The following conceptual analysis uses Thompson's scheme of meanings and extends it to rate in space.

Theoretical Framework and Conceptual Analysis

I based the study on quantitative and covariational reasoning, which served as the theoretical lens through which I constructed the tasks. The following conceptual analysis represents a plausible way for a student to coherently understand rate of change in space, and served as the basis for the construction of tasks.

Instantaneous rate of change can be thought of as an average rate of change over an infinitesimal interval. The average rate of change of a quantity C ($f(x,y)$) with respect to quantities A (x) and quantity B (y) in a given direction in space can be thought of as the constant rate at which another quantity D would need to change with respect to quantities A and B to produce the same change as quantity C in the same direction that (x,y) changed. Then quantity D accrues in a constant proportional relationship with quantity A, and simultaneously accrues in a proportional relationship with quantity B.

The average rate of change between two points in space is the constant rate at which another function g(x,y) would need to change with respect to x and y over the intervals $[x_0,x_1]$ and $[y_0,y_1]$ to produce the same change as f(x,y) over those intervals. The function g(x,y) must change at a constant rate with respect to x and a constant rate with respect to y and those constant rates must remain in an invariant proportion. An "exact" rate of change is an average rate of change of f(x,y) over an infinitesimal interval of $[x_0,x_1]$ and$[y_0,y_1]$.

Thinking about the rate of change of f(x,y) as above supports thinking that any accrual d of either x and y must be made in constant proportion b/a. This proportion a/b actually specifies the direction of change. Thus, rate of change of f(x,y) with respect to x can be reformulated as $f_u'(x,y)=\lim_{d\to 0}[f(x+h,y+k)-f(x,y)]/d$, where $ad = h$ and $bd = k$ so h must be a/b'ths of k and k must be b/a'ths of h. Then, d can be thought of in the same way as h in the one-variable case, where the derivative is an average rate of change of a function over infinitesimal intervals and the proportional correspondence between h and k means they have a linear relationship resulting in approaching the point (x_0,y_0) along a line. This conceptual analysis served as the basis for the construction of tasks, and interpretation of student responses during the teaching experiment.

Method

This study used Steffe and Thompson's (2000) account of a teaching experiment to build models of students' ways of thinking about mathematical ideas by focusing on the mathematics of students, which refers to ways of thinking that, were a student to have them, would make the student's words and actions sensible for the student. I generated a set of hypotheses about the students' ways of thinking, and used the tasks to test these hypotheses using grounded theory and open and axial coding. The cycle of task hypotheses testing and generation is depicted below (Figure 1).

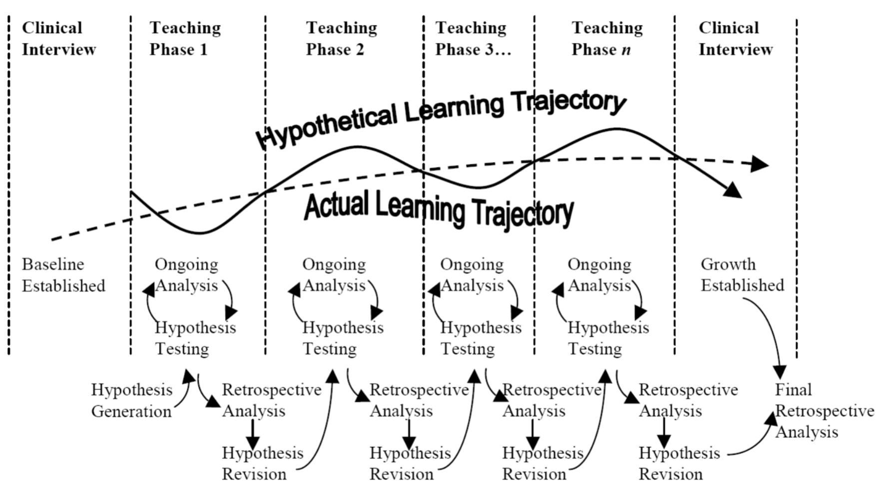


Figure 1. Characterzation of a teaching experiment methodology.

Results & Discussion

The following excerpt centered on how Neil and Brian would interpret the meaning of rate of change at a point in space.

Excerpt 1

EW: How would you think about rate of change at a point in space?

Brian: My first though is that it has multiple rates of change, kind of like sitting on a hill, depending where you look, the steepness, slope at that point can be different.

Neil: I agree with that, I thought about the kind of example too, or just sitting on the surface we swept out, maybe we can use the z-x and z-y rates of change?

Brian: Yeah! Umm, let's see though, if we want to make a rate of change function and then graph it in space, we need to figure out a way to program it, and do a sweeping out.

Neil: What about just plugging in *x*, *y* and the rate of change? Oh, I guess we don't know the rate of change yet, so my point thing wouldn't work.

Brian: Alright, for z-x, it is kind of like, okay let's back up, let's say we are at a point (a,b,c) in space. Then for z-x, we fix *y* at *b*, then do the normal rate of change except it has to be two-variables.

Neil: Yeah, that makes sense, so we need an *h*, maybe like *f*(x+h,b)-*f*(x,y), then divided by *h*, and then for z-y, we just say y+h and fix *x* at *a*?

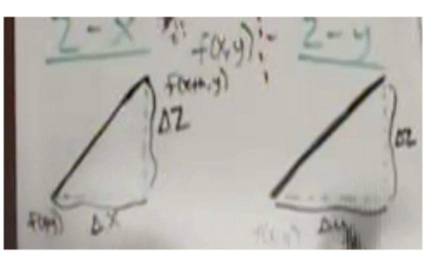
Brian: Yeah, let's go with that.

EW: Okay, so where do you want to go next then, what is your plan?

Brian: We need the two rates of change to make an overall rate of change function, then we can graph it by doing the sweeping out I think.

Neil: I'd rather just draw the two calculus triangles that I am imagining each in a perspective.

Brian's description of rate of change (Excerpt 1, lines 2-3), indicated he was thinking about rate of change in a direction "at" a point on the surface of the function's graph. Brian's suggestion of considering multiple rates of change from a perspective (Excerpt 1, lines 14-16) led to Neil's sketching of perspective dependent calculus triangles (Figure 2). Their descriptions of multiple rates of change, as well as specific rate of change functions for z-x and z-y perspectives, indicated they were imagining rate of change occurring in at least two directions. I intended to understand if they imagined the rates of change occurring simultaneously.



$$r_{f_x}(x,y) = \frac{f(x+h_1,y)-f(x,y)}{h_1} \quad (Z\text{-}x)$$

$$r_{f_y}(x,y) = \frac{f(x,y+h_2)-f(x,y)}{h_2} \quad (z\text{-}y)$$

Figure 2. Calculus triangles from the z-x and z-y perspectives.

Brian described the z-x perspective rate of change as the average rate of change of *f* with respect to *x* while holding *y* constant, and z-y as the average rate of change of *f* with respect to *y* while holding *x* constant. In the following excerpt, I asked Neil and Brian to expand on their description of their perspective dependent calculus triangles, in particular their use of h_1 and h_2 in the denominators of the open form rate of change functions. I anticipated that thinking about the relationship of h_1 and h_2 would be critical to their creating a need for considering rate of change in a direction.

Excerpt 2

EW: So I noticed that you guys constructed your calculus triangles in each perspective and labeled the *h*'s as different. Can you say more?

Neil: Yeah, well basically the *h*'s are independent, so they don't have to be the same, but I guess they could be.

Brian: I was thinking about this more last night, and had a sort of moment. We talked earlier about being on a hill, or walking on a function, for example, and the rate of change depended on the direction you were facing, I think the same thing applies here.

EW: What do you mean by a direction?

Brian: Okay, so let's say we head from a point, or we went directly Northeast, and imagine we are doing this from overhead, it was looking at the x-y perspective that made me think of this.

EW: Okay, so if you were heading Northeast, what's the significance of that?

Brian: Then we know how the two *h*'s are related to each other, because most of the time you don't go in a direction of just north, west, east, whatever, you head in some combination.

Neil: Okay, not sure I am following, but basically if we head Northeast, the change in *x* and change in *y* would be equal, a different direction, change in *x* could be twice as big as change in *y*, which are like the *h*'s right?

Brian: Yeah, let me make an illustration here. Then we can just call the numerator a change in *z*, $f(\text{x}+h_1, \text{y}+h_2) - \text{f(x,y)}$ either *h* value that we want to.

Brian introduced direction as a way to account for all possible rates of change in space (Excerpt 2, lines 10-12). I believed Brian had an image of the relationship between h_1 and h_2 to define a direction (Excerpt 2, lines 15-17). Brian's key insight was that *any* direction was more general than considering only z-x and z-y perspectives.

Neil and Brian continued to work on developing a two-variable open form rate of change function, and they agreed that the numerator represented a change in the output, represented by $f(\text{x}+h_1, \text{y}+h_2) - f(\text{x,y})$, where either h_1 or h_2 was written in terms of the other *h*-value. However, both Neil and Brian questioned how they have a single denominator that represented a change in *x* and a change in *y*. Even though they saw that h_1 and h_2 depended on each other, that dependence did not immediately resolve their issue of what change to represent in the denominator. I believed that this was because they were focused on trying to represent a single change in the denominator while they understood that there were changes in both *x* and *y*.

Excerpt 3

EW: So, what is your denominator in the function, your conjecture?
Brian: I was thinking either *h*-value, whichever you have in the numerator.
Neil: But doesn't that kind of just delete it, it goes away?
Brian: No, you just define it in terms of h_1, if you are talking about h_2.
EW: So if one *h* is in the denominator, and we make that value small, what happens to the other *h* value??
Brian: Well, oh yeah, it becomes small as well, it's not like the *h*'s have the same value, but they can end up getting so small it doesn't matter.
Neil: Ah, I see what you mean. So like in class we talk about making changes so small, because we have an equation to relate the two *h*'s, then when one gets really small, the other one has to get smaller too? I was thinking multiply in the denominator I guess, but that doesn't need to happen, you just need one of them to become small because they are related, the *h*'s I mean, so then both do.
Brian: So, I guess that's our overall rate function for two variables. Now we can sort of program the points to do the graph.

Brian's insight that the changes in *x* and *y* became smaller in tandem allowed him to conjecture that using a single parameter in the denominator (Excerpt 3, lines 7-8) was acceptable (Excerpt 3, line 2). Neil appeared to focus on deleting one of the parameters, but Brian's insight allowed him to think about the equation they had specified between h_1 and h_2. By imagining progressively smaller values for h_1, he found that h_2 became smaller as well given the proportional relationship specified by choosing a direction in space (see Figure 3). These insights allowed them to construct an average rate of change function (Figure 3).

$$r_f(x,y) = f(x+h_1, y+h_1 d) - f(x,y) \,/\, h_1$$

direction = d d= fixed $\frac{\Delta y}{\Delta x}$ $h_1 = \Delta x$ $h_2 = \Delta y$

Figure 3. Brian and Neil's two variable average rate of change function.

Conclusions

Brian and Neil problematized how to interpret and calculate a rate of change at a point in space because they were concerned with defining rate of change at a point in space. Neil's construction of "simultaneous calculus triangles" and Brian's reflection on them as a way to represent partial rates of change indicated to me that they were imagining change occurring in different directions in space. The x-y perspective allowed them to think about relating the *simultaneous* changes in *x* and *y* by considering a direction, and supported their construction of an open form rate of change function. By attempting to generate the graph of the rate of change function, Brian, and then Neil, were able to problematize the existence of rate of change at a point in space. This problematization occurred without formal focus on directional derivative, and was based off of the students' understanding of a one-variable rate of change function.

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DEVELOPMENT OF STUDENTS' WAYS OF THINKING IN VECTOR CALCULUS

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In this talk, we describe the development of the ways of thinking of 25 vector calculus students over the course of one term. In particular, we characterize the generalizations that students made within and across interviews. We focus on the construction of the semi-structured pre and post interviews, trace the construction of explanatory constructs about student thinking that emerged from those interviews, and describe how those constructs fit within the broader literature on student thinking in advanced calculus. We conclude by exploring implications for future research and practical applications for educators.

Key words: *Calculus, quantitative reasoning, rate of change, covariational reasoning, generalization.*

Background Literature and Research Question

The transitions that students make as they progress through the calculus sequence is important for both students' immediate success and their meaningful application of those concepts to other fields. The most important transitions that students make are in their understanding of function and rate of change that allow them to represent, predict and explain relationships between quantities in a system and in the generalization of those ways of thinking to systems with many quantities. However, research that explores those transitions is limited (Martinez-Planell & Trigueros, 2012; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Researchers have suggested that quantitative and covariational reasoning are foundational to students' coherent understanding of functions and rate of change (Carlson, Oehrtman, & Thompson, 2008; Saldanha & Thompson, 1998; Thompson, 1994; Weber, 2012). Studies have also shown the importance of quantitative reasoning to generalization (Ellis, 2007a, 2007b). This research led us to hypothesize that to study the development of students' ways of thinking about function and rate of change across the calculus sequence we would need to a) Focus on the transition between single and multivariable calculus, b) Characterize the development of the students' quantitative and covariational reasoning, and c) Understand the generalizations students make as they progress through the transition from single to multivariable calculus.

Given our hypothesis, we constructed a series of semi-structured interviews of students as they participated in vector calculus at a mid-size Northwestern institution. We used these interviews to gain insight into the major research question:

> *How do vector calculus students' ways of thinking about function and rate of change develop as they participate in a typical vector calculus course?*

This major research question led naturally to gaining insight into the role of quantitative and covariational reasoning as well as generalization in the development of those ways of thinking.

Theoretical Framework

Quantitative reasoning and generalization were the foundation for the construction of the semi-structured interviews and interpretation of the data gathered from them. Quantitative reasoning refers to a way of thinking that emphasizes a student's cognitive development of conceptual objects with which they think about mathematical situations (Smith III & Thompson, 2008; Thompson, 1989). Quantitative reasoning focuses on the mental actions of a student conceiving of that situation, constructing quantities in that situation, and relating, manipulating, and using those quantities to make a problem viable. Thompson (2011) argued that for a student to imagine that a function of two variables is a representation of the invariant relationship among three quantities, the student must construct those quantities, whether it is from an applied context or an abstract mathematical expression. As an example, if a student is to think about a complicated situation with three quantities and construct quantities and the invariant relationship between those quantities, a dynamic mental image of how those quantities are related is critical. That image positions a student to think about how quantity 1 varies with quantity 2, how quantity 2 varies with quantity 3, and how quantity 1 varies with quantity 3. Understanding these individual quantitative relationships allows a student to construct a function that expresses an invariant relationship of quantity three as a function of quantities one and two, so that its value is determined by the values of the other two. This example is a powerful characterization of how students might reason about situations with functions of many variables, and Ellis' (2007) generalizing actions framework provides a framework for how this extension might occur for a student. Her characterization of three generalizing actions: relating (Type 1), searching (Type 2), and extending (Type 3), served as a tool with which to construct the semi-structured interviews with a focus on quantitative reasoning, and to explore and categorize the generalizing actions students exhibited in those interviews. Together, Thompson's characterization of quantitative reasoning and Ellis' generalizing framework provided a foundation for exploring how students made generalizations over the course of their vector calculus course.

Method

We selected 25 students enrolled in vector calculus at a mid-size, Northwestern University. We chose the vector calculus course at this university because the course was the students' first exposure to multivariable functions in mathematics, which allowed us to observe the students' initial fits and starts with systems with many quantities. The students were randomly selected from all vector calculus students enrolled during that term, and were asked to participate in the study. They were compensated for their participation in the interviews.

Each student participated in a pre and post interview, and completed a number of problems during the vector calculus course. The pre and post interviews consisted of questions designed to gain insight into the students' ways of thinking about function and rate of change. The pre-interview focused on single-variable functions and rates, as the students had not yet been exposed to these ideas in their course. The pre-interview questions were identical across students. The problems they completed during the term were based on the progression of the vector calculus course and were identical for each student. These problems were open ended and designed to gain insight into the generalizations that students were making. The post-interview took place at the end of the term, and consisted of four common items, and four items based on the students' responses on the pre-interview and problems completed during the term.

Analysis of the data was multi-phased. We used the pre-interviews to characterize students' quantitative and covariational reasoning in explaining their ways of thinking about function and rate of change. We used open and axial coding to identify common behaviors and responses across interviews in constructing a number of ways of thinking that we hypothesized students had. As the students progressed through the course, we used the initial ways of thinking we identified in the pre-interview to understand how those ways of thinking played a role in the generalizations students made as they learned about multivariable functions and derivatives. We documented the development of the students' generalizations over the course of the term, and gained further insight into these generalization actions with the post-interviews. As a result, we were able to create a model for each student's way of thinking about function and rate of change and compare those models across students. This process of constructing inferences about student thinking allowed us to explain the role of quantitative reasoning in the generalizations students made as they transitioned from single to multivariable calculus.

Results & Discussion

Analysis of the data is ongoing, but our preliminary interpretations suggest that many of the students in vector calculus likely possess "flawed" understandings of functions and rate of change. By flawed, we mean that the students make inconsistent, and often incorrect statements about rate of change and interpretations of functions' graphs. We believe the students struggles are explained by their lack of ability to reason quantitatively. These findings are not necessarily surprising given the prevalence of literature that has documented students' difficulties with functions and rates, but we believe that their struggles are related to the type of generalizing actions that they exhibited during the course of the study.

We believe that the students' difficulties can be partially explained by their attention to calculational reasoning. By calculational reasoning, we mean that the students are concerned primarily with arriving at an answer without attention to the meaning of that answer in a particular context. The calculations reasoning that students exhibited constrained the type of generalizing actions they were able to perform. For example, their Type I and Type II (see Figure 1) almost entirely focused on the numbers and answers in a given problem. Rarely was the meaning of a particular mathematical idea the driving force behind a student's generalizing action. In the presentation, we will provide an overview of the students' actions in the context of calculational reasoning. We believe these insights support the categories defined by Ellis (2007), but also present an opportunity to expand on the generalizing actions framework.

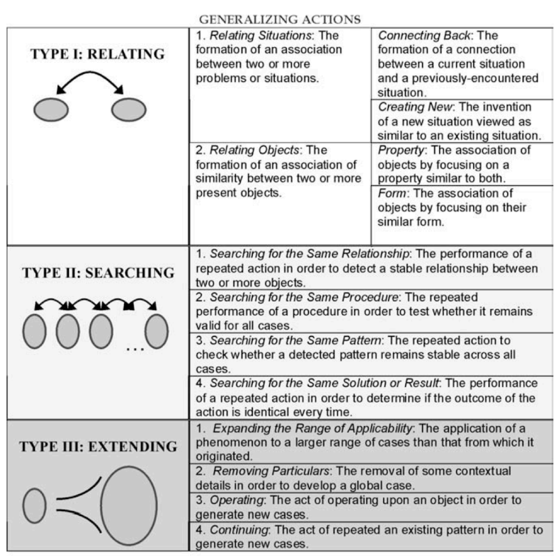

Figure 1. Ellis' (2007) framework for generalizing actions.

Questions for Audience

1) Based on the study's research question, in what ways could the research design be improved for future iterations of use with vector calculus students?

2) Based on the data presented, what were the strengths and weaknesses of the constructs we identified?

3) What are some essential questions you think must be answered to effectively characterize students' transitions from single to multivariable calculus?

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Students' Sense-Making in Mathematics Lectures

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It is important for us to develop an understanding of how students make sense of lectures, particularly in upper-level mathematics classes, where the focus is on definitions, proofs and examples. To do this, we observed abstract algebra lectures, collected students' notes, and interviewed students about their experiences while re-reading their notes and re-watching the lectures. Our preliminary findings indicate that students attempt to make sense of components of the lecture on multiple levels, which influences their understanding of the mathematical aspects of the lecture.

Key Words: Lectures, Abstract Algebra

Despite efforts to make college classrooms more inquiry-based, typical college students still spend approximately 80% of their time in class listening to lectures (Armbruster, 2000). Lecture listening is a difficult cognitive task for college students (Ryan, 2001), but it is important, since what students take away from lectures is closely linked to what they learn (e.g., Titsworth & Kiewra, 1998).

The goal of this research project is to develop and use a framework to describe the ways students make sense of mathematics lectures and the various factors that influence and constrain this sense-making process. In the work reported here, we focus on the research questions:

1. In what ways do students make sense of upper-level mathematics lectures?
2. What aspects of the lecture influence the ways students make sense?
3. What aspects of the student—their beliefs, habits, and knowledge—influence the ways they make sense of lectures?

Background

Most prior research on students' participation in lectures has focused on their note-taking habits. Students' notes can be viewed as a "symbolic mediator between the content taught by the teacher and the knowledge constructed by students" (Castello & Monero, 2005, p. 268), and Ryan (2001) generated metaphors that students might use to describe and guide their note-taking practices and lecture-observing habits. However, this—and much of the prior research (e.g. Van Meter, Yokoi, & Pressley, 1994)—has focused solely on students' self-reported habits rather than empirical evidence and did not describe the ways that these various habits might affect students' learning. Most other research has focused on factual recording and recall rather than deeper understanding of the content (e.g. Kiewra, 1991; King, 1992). In addition, there has been little previous research that has described the ways that students' engage in *mathematics* lectures.

Initial Framework

Although one of the goals of this research is to construct a framework for analyzing data, we have adopted an initial theoretical lens to help us identify aspects of the lecture or of students' beliefs and habits that might influence the ways the make sense of the lecture.

Sense-Making

The terms "sense" and "sense-making" are widely used in mathematics education, appearing in research literature (e.g., Schoenfeld, 1992), the Common Core Standards (National

Governor's Association, 2010), and an NCTM series (e.g. NCTM, 2011). Although these terms do not have standard definitions, they are typically used in conjunction with describing the act of constructing a personal understanding of a mathematical fact, procedure, concept, or theory.

Lecture Components

Lectures contain numerous components that students must attend to and interpret including proofs, definitions, statements of theorems and algorithms. These components can be broadly distinguished by their mode of presentation: written, spoken, and gestural. Beyond this, we divide lecture components into communicational aspects and mathematical aspects. *Communicational aspects* of a lecture include immediacy, gesture, temporal-spatial components, and organizational cues; these cues can organize transitions from moment-to-moment, from day-to-day, and from unit-to-unit or class-to-class. The *mathematical aspects* of a lecture consist of facts, procedures/algorithms, and processes (including problem-solving, communication, justification, and representation).

Meta-Components of Lectures

A lecture may contain—or be influenced by—aspects that appear as components yet are part of a broader idea. We identify two such aspects: broad mathematical concepts and motivation, and pedagogical actions and motivation. *Mathematical concepts and motivation* are the broad mathematical ideas that a lecture addresses (often indirectly) and the reasons for those ideas being seen as important. For example, while the components of a lecture may include specific examples of equivalence relations, understanding the general concept, its significance in mathematics, and the reason for including it in the lecture, may affect the students' sense-making. In addition to the actions that an instructor performs during the lecture, the instructor's *pedagogical motivation* is their motivation for choosing and ordering specific ways of presenting their ideas. For example, a lecturer might present specific examples before stating a general rule in order to illustrate that rule. Understanding this pedagogical motivation may affect the way a student makes sense of the components of the lecture.

Data Collection and Analytical Methods

We have collected and are in the process of analyzing data in a pilot study. The participants are six mathematics majors who were enrolled in a standard abstract algebra class. The class was videotaped 6 times over the course of the semester. The video of the observations was transcribed to describe the written, spoken, and gestural components. In this way, the class observations were designed to capture as much of the "text" of the lecture as possible. After each observed class period, we collected the participants' notes. We identified places in the students' notes that didn't match what the instructor wrote or said.

We interviewed the students after each recorded lecture. We showed the students video clips of the lecture along with their notes, and asked questions designed to explore how they approached the class, how they took notes, how they understood the purposes of notes, and questions about the mathematical content of the class. The interview data was analyzed using grounded theory (Strauss & Corbin, 1994) through an iterative process of identifying themes, creating codes, applying the codes to the interviews, and comparing our results with each other.

Results and Analysis

We are in the preliminary stages of using this coding manual to fully describe students' sense-making. However, we have so far identified some common themes (supporting excerpts will be shown during the full presentation).

Of the six students we interviewed, five indicated that they try to copy down what the instructor writes on the board. The claim of "copy what's on the board" seems to imply that students *aren't* trying to understand the lecture. However, as they looked back at their notes and re-watched sections of the lecture, the students described numerous acts of sense-making. In addition, students' described their note-taking habits as mediated by their goals, including using their notes for later reference and completing homework and exams.

Although most of the students tended to copy what the lecturer wrote on the board, there were places in every student's notes that differed from what was on the board. Many of these discrepancies occurred at the same place where the student was able to clearly and accurately describe both the lecture component and various meta-components. For example, several students initially struggled to describe the connection that the lecturer described between ordered pairs and rational numbers. Although they could articulate an understanding of a specific component (e.g., the equivalence of two fractions), they couldn't articulate either the meta-component of the role of equivalence relations or the instructor's pedagogical motivation for bringing up the specific numerical example. However, at a later point in the lecture one student added to her notes: "Each [ordered pair] should be thought of as, in this case, rational numbers." When she got to this point while re-watching the lecture during the interview, she was able to articulate the connection.

From the interviews, it appeared that students were trying to make sense of the lecture as they observed it. However, their sense making occurred on multiple levels. Some students were able to make sense of individual mathematical components of the lecture—such as the equivalence of two fractions. Other students were able to make sense of the communicational aspects of the lecture and use the instructor's organization (e.g. of the board space) to develop an understanding of how various ideas are related. Still other students attempted to make sense of the meta-components of the lecture, using this understanding of the significance of various mathematical concepts (e.g., equivalence relations) or the instructor's motivation (e.g., presenting several examples before a general case) to make connections between ideas. In particular, students who *didn't* make sense of these additional components struggled to describe the mathematical ideas in the lecture.

The students all described various barriers to making sense of the lecture. Some students had difficulty translating the instructor's speech into written notes. Other students had difficulty encoding the layout of the notes on the board, and others described their difficulty following the connections the instructor made between the current lecture and ideas that the class had previously encountered. In addition to struggling to encode non-written lecture components in their notes, students described having difficulty figuring out what aspects of the lecture were most important.

Discussion

Despite claiming that they are simply duplicating what the instructor writes on the board, it appears that students are constantly trying to make sense of the lecture. Although they often try to copy the instructor's writing into their notes, places where their notes deviate from what is on the board seem to signify instances where students were successful in making personal sense of the lecture components and, at times, meta-components as well.

This sense-making process is mediated by the students' goals for taking notes and their perceptions of how they will be using the notes. The students face numerous challenges in determining what aspects of the lecture are most important; they also struggle to make sense of aspects of the lecture that are not written on the board, such as the instructor's speech, the board layout, or concepts that are connected across multiple lectures.

More importantly, students can make sense of aspects of the lecture on multiple levels. A student can make sense of a particular mathematical lecture component, an organizational cue, or a meta-component. However, if they only make sense of one of these (such as understanding that ¾=9/12, but not why this example is pedagogically relevant), then they might fail to understand some of the most important aspects of the lecture.

Discussion Questions

1. What are important aspects of students' sense-making during lectures?
2. How might various aspects of lectures affect students' sense-making habits?
3. In what ways can the initial framework described here be enhanced to better capture these important aspects?

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HOW PRE-SERVICE TEACHERS IN CONTENT COURSES REVISE THEIR MATHEMATICAL COMMUNICATION

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Math content courses aim to develop mathematical reasoning and communication skills in future teachers. Instructors often assign problems requiring in-depth written explanations to develop these skills. However, when a student's conception is incorrect, does written feedback from the instructor create the cognitive dissonance necessary to effect realignment of the student's understanding? These conceptions may be mathematical ("what is a fraction?") or meta-mathematical ("what constitutes a justification?"). Assigning problem revisions theoretically creates space for cognitive dissonance by having students rethink their solutions. I investigate a revision assignment in a course for future teachers to understand the nature of students' revisions and the possible impetuses for these revisions. In particular, I find preliminary evidence that students' revisions demonstrate changes in their language, mathematics, and use of examples and representations. Further, students' adoption of new representations in their solutions are largely due to observing peers' presentations rather than to instructor feedback.

Key words: **Pre-service elementary teachers, mathematical communication, mathematical justification, revision, representations, inquiry-based learning**

Research Questions

One goal of math content courses for future teachers is to develop mathematical reasoning and communication skills. However, unlike courses for math majors, future teachers need to be able to reason with conceptual and visual models and not just the axiomatic reasoning of mathematicians. As in most other math courses, instructors often assign problem sets requiring in-depth written explanations to develop and assess these skills. However, when a student communicates a concept incorrectly, does the standard written feedback from the instructor create the cognitive dissonance to effect a realignment of the student's understanding? The conceptions I refer to may be mathematical ("what is a fraction?") or meta-mathematical ("what constitutes a justification?"). Assigning problem set revisions could theoretically create a space for cognitive dissonance, if not the dissonance itself, by asking students to rethink their solutions. I investigate a revision assignment in a course for future teachers to better understand the nature of students' revisions and the possible impetuses for these revisions.

I had several questions about this revision process:

- Q1 What kinds of revisions do students make when asked to revise their solutions to problem sets?
- Q2 What differences can be detected in the influence of peers' in-class presentations and instructor feedback in the revision process?

These questions are important for several reasons. In particular, they address specific practices of teaching mathematical justification and communication. The first question assesses the usefulness of the revision activity as a method of improving students' mathematical justification and communication. The second question illuminates the potential importance of multiple forms of feedback in math content courses for future teachers.

Literature

It is widely acknowledged that teachers need strong mathematical reasoning and justification skills to successfully build mathematical concepts in the classroom (Ball & Bass, 2003; Ball, 1993). Further, there is evidence to suggest many preservice teachers are weak in these skills (Ball, 1990; Morris, 2007). Unfortunately, there is a lack of literature on how preservice teachers can gain these skills (Hiebert & Morris, 2012).

The tools teachers should use in justification are more than the axiomatic systems of mathematicians. They should use conceptual models (Lamon, 1997) and visual representations of those conceptual models (NCTM, 2000). Examples of these include: the number line, the array model of multiplication, and the chip model of integer arithmetic. Given the needs of their future teaching practice, mathematical solutions written by future teachers should utilize these models and representations.

Another area this project addresses is revision of mathematical writing. While there is an ample body of literature on the research methodology for revision of written composition, their heavily linguistic methods do not lend themselves to analyzing mathematical revisions (Fitzgerald, 1987). In a mathematical revision, it is not just the linguistic changes we care about, but the changes in validity of mathematical arguments, clarity of mathematical ideas, and use of mathematical tools (e.g. algebraic models, diagrams, examples). Part of this study seeks to fill this hole by using empirical examples of student revisions to create a systematic framework to describe revisions to written mathematical problem sets (see Q1).

Theoretical Framework

I initially looked to Balacheff's model of a mathematical *conception* as a useful framework for thinking about the pedagogical purpose of a revision activity and the role of feedback in the revision process. In Balacheff's model, a conception is a provisional state of equilibrium of an action-feedback loop between a *milieu* (i.e. learning environment, such as a classroom), and a *subject* (e.g. a student) under proscriptive constraints (Balacheff & Gaudin, 2009). In this model, an instructor's role is to create a milieu that perturbs the equilibrium and the student's job is to select features of the environment to use as feedback. Using this language, after the students create a first draft of their problem sets, the instructor introduced new sources of feedback into the milieu (e.g. student presentations and instructor feedback) and created the explicit need for students to choose from sources of feedback multiple times (e.g. multiple drafts of assignments). This theory is meant to model students' understanding surrounding a mathematical concept or problem, but is problematic when their written output is a proxy for their understanding. A written assignment is a reflection of a student's verbal communication skills, not just reasoning skills. One way to remedy this is to analyze a student's conception of what makes a good written solution, rather than his or her conception of a mathematical topic. This better represents the entwined nature of understanding and communication.

Data

In this math content sequence for pre-service teachers, students completed weekly problem sets comprising challenging problems that require reasoning, justification, and explanation. The course was run using inquiry-based learning (IBL) pedagogy and students presented solutions to problems every week in class. Often, several students present different solutions to the same problem while the class as a whole contributes clarifications, corrections, and modifications to presented solutions. During this discussion students annotated their problem sets with colored pens before turning them in to the instructor for initial feedback. I call these annotations *students' marker comments*. At three points in the semester, students selected problems to revise and resubmit.

The data collected includes the following: first drafts of problem, including students' marker comments; initial instructor feedback on first drafts; final drafts of problem sets; and written student reflections their revision process. The students' marker comments serve as a proxy to understand the effect of peer presentations and class discussion on the revision process. Figure 1 represents the revision process. The items in red are the artifacts I have access to.

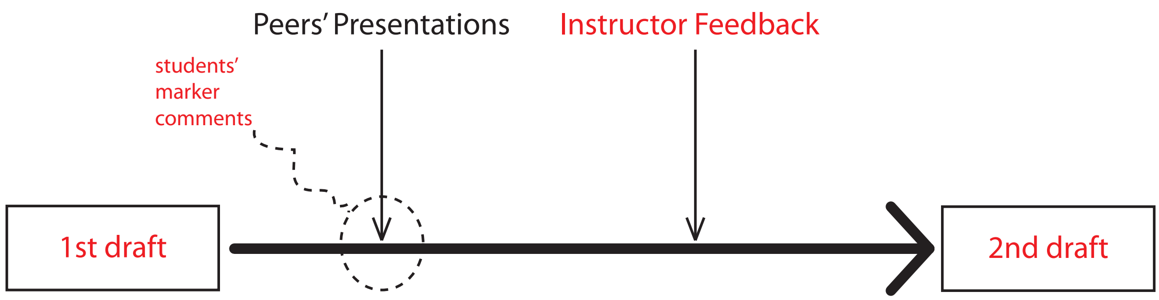

Figure 1: Model of student revision process. The items in red represent artifacts I can see.

I will use the term *one student-worth of data* to describe all of those pieces of data for one student's revision of one problem.

Methods

To understand students' revisions as well as sources for their revisions, I wanted a framework that would both describe the *changes* students made during the revision process as well as *potential changes*—students' marker revisions made during peers' presentations and instructor feedback. To develop this framework I started by analyzing 10 students-worth of data. I wrote memos describing (1) changes I detected between drafts, (2) students' marker comments on their first drafts, and (3) instructor feedback. Figure 2 shows how these memo-ed objects fit into the revision process.

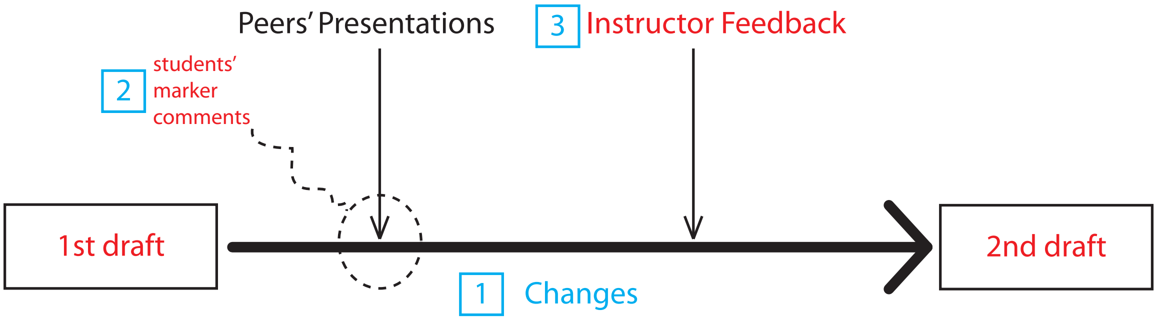

Figure 2: I wrote memos describing [1], [2], and [3] seen in this figure.

I identified themes in the memos and drafted a set of codes and their definitions. I next used this preliminary coding scheme to code the remaining 20-students worth of data from that particular problem set and revised the code definitions when ambiguities arose. Next, I continued that revision process with a new problem set, another 30 students-worth of data, until the definitions stopped changing. This second set of data is what I present below.

One subtlety of the coding is that there is a challenge of systematically identifying every single change between first and second drafts. So rather than identify every single change and count the codes with multiplicity, I used the list of codes as a checklist and recorded only if a certain code occurred or not. This proved to be more systematic and replicable when re-coding for reliability. Further, I used written student revision reflections as member checking; students had written reflections describing the changes they made and I was able to cross-check the changes I identified using their reflections.

Results

The revised codes and definitions arising from this process can be found in Table 1. The codes fall into four larger families: Language (Expo, Expl, Lang), Mathematics (Lar, Sma, Jus, Not, Prec, Def), Examples, and Representations.

Table 1: Revision Framework

Code	Name	Definition
Expo	*Adds Exposition*	This refers to *adding* prose describing the problem premise and goal before steps are carried out. Many students do not include this in early drafts, but it appears in later drafts. This is a specialized kind of explanation, but one so common it gets its own code. This does not refer to editing the exposition, only adding it where there was none before.
Expl	*Adds Explanation*	This includes explaining a diagram, explaining thinking or attempts tried, or making steps in a process explicit. This is different then justification, as it does not explain *why* a process works; it is a *description* of a process or phenomenon. This refers to *adding* explanation to a place in a solution where there was none before, not refining it.

Code	Name	Definition
Lang	*Language*	This describes changes (not content additions) to language. Examples are changes to clarity, flow, word-use, grammar, full sentences, or conciseness. This refers to essentially the same content reorganized or reworded. It does not refer to new content such as an expanded explanation or justification.
Lar	*Large Correction*	This refers to correcting large errors in the original not covered by other codes. For example, this would not refer to filling in gaps in justification or including needed definitions. Instead this would include rectifying missing solutions or large numerical errors.
Sma	*Small Correction*	This refers to correcting small, usually numerical, errors in the original solution.
Jus	*Adds Justification*	This describes added attempts at answering “why?” or filling in logical gaps.
Not	*Introduces New Notation*	This describes a change in notation. One common example is using algebraic notation when previous description was verbal.
Prec	*Adds Precision*	This includes adding constraints on variables, using more precise quantifiers, referencing hypotheses of the problem, or correcting use of the equals sign. Adopting algebraic notation only counts as **Prec** if the previous notation was imprecise.
Def	*Definition*	Adds references to definitions, refines references to definitions, or makes them more explicit.
Ex+	*Add Example*	This refers to adding new examples in the course of an exposition, explanation, or justification.
Ex-	*Delete Example*	This refers to deleting examples from the exposition, explanation, or justification.
Rep+	*Add Representation*	This refers to adding visual/physical models of numbers and/or operations. This could be adding a number line, discrete models of numbers, area models of multiplication, etc. This does not include tables or other record-keeping and problem-solving systems. It also does not cover inclusion of an algebraic representation of a situation; that falls under **Not.**
RepM	*Modify Representation*	This refers to changes made to an existing representation, such as adding lines, circles, or arrows to an existing array or number line model.
Rep-	*Delete Representations*	This refers to deleting a representation from an earlier draft.

To give examples and non-examples of every code here is beyond the scope of this preliminary report, but I will give some flavor of the student work and the codes by providing some examples of the Justification code. Consider a problem students were given: “Show that if the difference between two integers is odd, then their sum is also odd.” A marker comment that received the code **Jus** was the following: “missed how to explain why it’s always true that odd-even or even-odd is odd.” It receive the **Jus** code because a student is noting a gap in logic that must be filled in on a later draft. An instructor comment that received the code **Jus** is the following: “Why will the two numbers always be different? (one odd and one even) Can you explain that more?”. It receive the **Jus** code because it is asking for an explanation of *why* something is true. To see an example of **Jus** coding a change, consider the following paragraph in a first draft: “If the difference between two numbers is odd, that means that one number in the pair is odd and the other is even (as seen in the examples above).” The same student in a second draft attempts (still imperfectly) to more generally address this phenomenon with the following change to her justification: “If we look at all these examples where the difference between two numbers is odd, we find out that the two numbers we are taking the difference of have to be one odd number and one even number. We know this is true for all numbers because of the rules, odd-even=odd, even-odd = odd. These two rules are the only rules that can produce an odd number as its difference.” This examples points to another subtlety in the coding—I used a code if there was a *change* in that dimension, not necessarily ultimate perfection.

When I used these codes on the revision data from one problem set (30 students worth of data), I found the results seen in Figure 3, Figure 4, Figure 5. In all three figures, a given student can receive any given code at most once, so the maximum for all codes is 30.

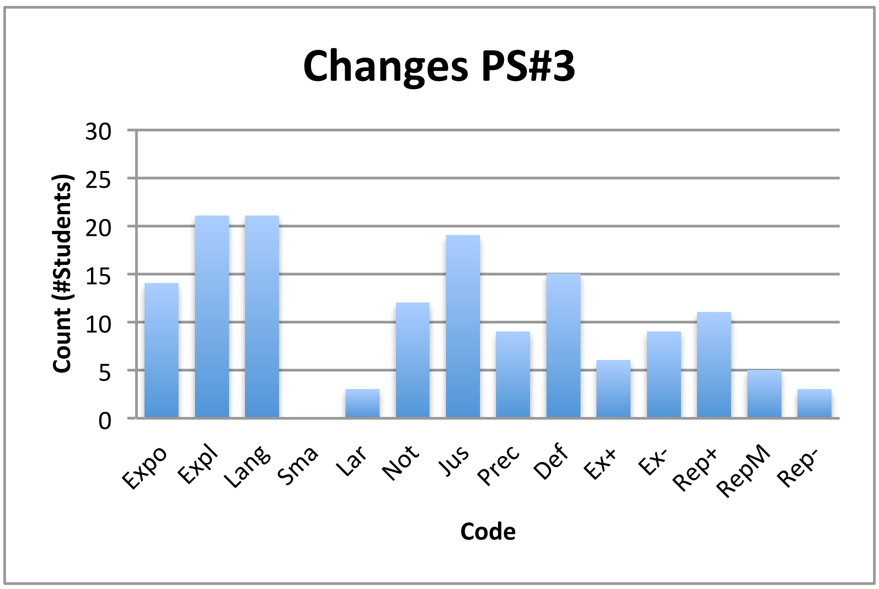

Figure 3: Changes seen between drafts in 30-students worth of data.

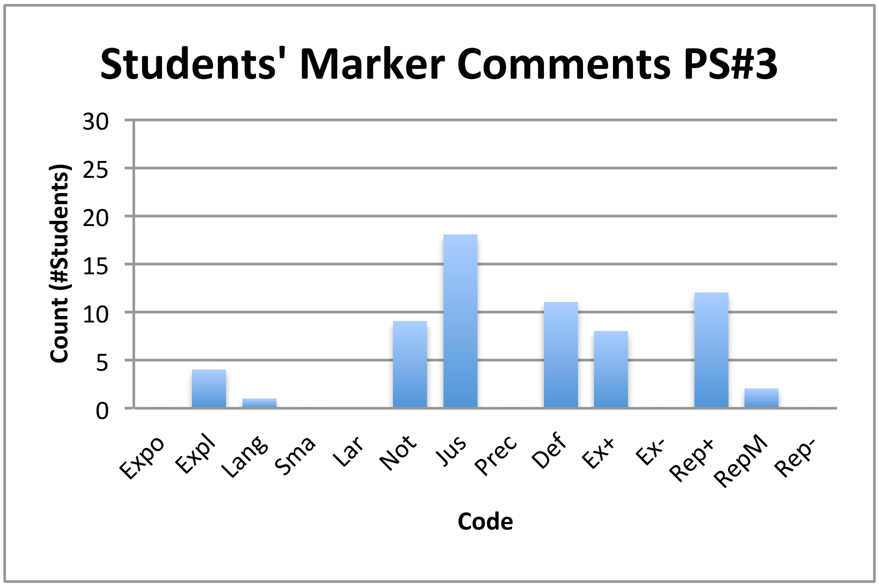

Figure 4: Potential changes seen in 30-students worth of market comments.

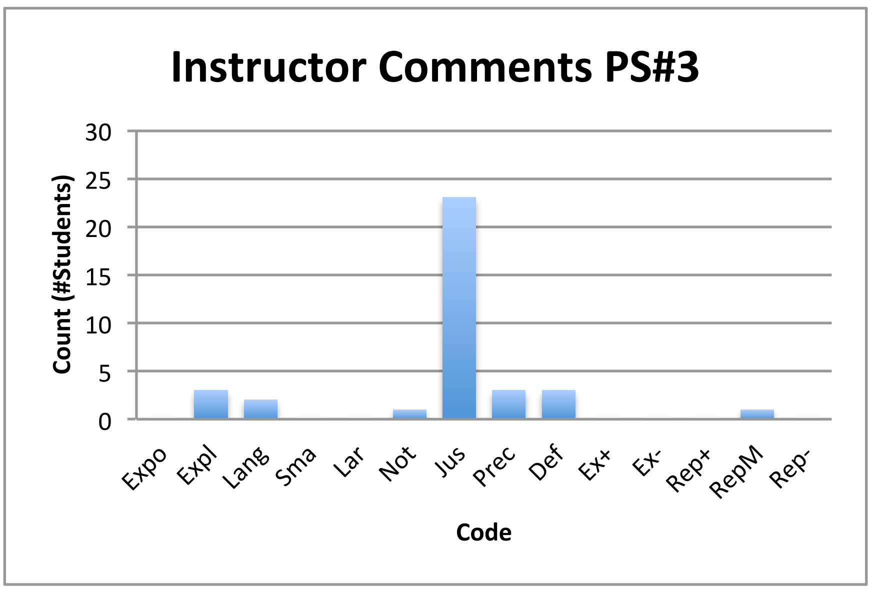

Figure 5: Potential changes seen in 30-students worth of instructor comments.

Discussion

The first observation (answer to Q1) is that students' revisions are rich mathematical changes. They make changes to their expositions, their language, their justifications, their use of definitions, their notation, their precision, their use of examples, and their use of representations. This would indicate that the revision activity is a positive pedagogical exercise.

A more nuanced observation is that peer presentations appear to affect student adoptions of examples and representations more than instructor feedback. This is in part perhaps due to the fact that (as evident in Figure 5), the instructors of this course (both mathematicians by training) fixate on formal mathematical justification in their feedback. However, despite the limited scope of this instructor feedback, over one third of the class incorporated new representations into their solutions, and almost one third of the class added or deleted examples (see Figure 3). I hypothesize that the impetus for these changes can be found in the peers' presentations, as evidenced by the students' marker comments to themselves (see Figure 4).

In addition to changes in justification, changes to examples, and changes to representations, there were changes in language (corresponding to the codes Expo, Expl, and Lang). I want to explore further what these may be due to. I hypothesize that they are both a normal part of the revision process and also highly related to the changes students made to their to justification.

I think these preliminary results have potentially interesting ramifications for using IBL methods with future teachers. We see evidence students utilize peers' presentations differently from instructor feedback, in ways that are important to their future teaching practice.

Questions to the Audience

(1) What other methodologies could I use to analyze this large collection of data?
(2) What theoretical perspectives could help me look at this data in different ways?
(3) What are some alternative frameworks for looking at (mathematical) revisions?
(4) What are some frameworks for teasing apart the code *Justification* in more detail?
(5) What is the role of the specific content area (number and operations) in this analysis? In particular, what observations could generalize to the second and third semesters of the course, whose content areas are geometry and algebra?
(6) What about these results is special to content courses for pre-service teachers? Could there be similar analyses done on student revision in upper division math courses?

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OPPORTUNITY TO LEARN FROM MATHEMATICS LECTURES

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Many mathematics students experience proof-based classes primarily through lectures, although there is little research describing what students actually learn from such classroom experiences. Here we outline a framework, drawing on the idea of the implied observer, to describe lecture content; and apply the framework to a portion of a lecture in an abstract algebra class. Student notes and interviews are used to investigate the implications of this description on students' opportunities to learn from proof-based lectures. Our preliminary findings detail the behaviors, codes, and competencies that an algebra lecture requires. We then compare those with how students behave in response to the same lecture with respect to sense-making and note-taking, and thereby how they approach opportunities to learn.

Keywords: opportunity to learn, implied observer, note-taking, abstract algebra

Many mathematics students experience proof-based classes primarily through lectures (Mills, 2012). Preliminary research indicates that the demands of learning from a live presentation of material, through the instructor's writing, speech, and gestures, are significantly more taxing than learning from a textbook (Fukawa-Connelly, Weinberg, Wiesner, Berube, & Gray, 2012). The goal of our research is to develop and use a framework to describe the opportunities to learn that are available to students in a proof-based mathematics lecture. Focusing on note-taking as one of the primary activities that students engage in during a lecture, our research questions are to identify:

1. What discrepancies between student notes and lecture content are common and under what circumstances do they occur?
2. How and to what extent do such discrepancies represent a missed opportunity to learn?

Lecture Content

Understanding what a student might learn from a lecture requires a description of the lecture itself. Here we briefly present a framework for describing the lecture in terms of its mathematical content and the way it is presented. We also use the idea of the "implied observer" to analyze the demands that the lecture places on the student.

Lectures contain numerous *mathematical components* that students must attend to and interpret including proofs, definitions, statements of theorems and algorithms, examples, and exposition. In addition to their mathematical aspect, these components can also be distinguished by their mode of presentation: written, spoken, and gestural. Lectures also have a temporal-spatial aspect: for example, a lecture is given only once, with limited time and space for writing; instructors may write notes in a non-linear fashion; or they may structure their board-work to make distinctions or connections between ideas.

The *implied observer* provides a description of what is required of an observer of a lecture in order to respond to the lecture in a way that is both meaningful and accurate (Fukawa-Connelly, Weinberg, Wiesner, Berube, & Gray, 2012; Weinberg & Wiesner, 2012); it is created by the lecture itself, as opposed to the intentions of the instructor or the actual observer (i.e. the student's own behaviors, codes, and competencies). The implied observer of a lecture can be characterized by a set of codes, behaviors, and competencies. A *code* is the implied observer's method of ascribing meaning to particular lecture content. The *competencies* of the implied observer are the knowledge, skills, and understandings that are required to understand the lecture. Finally, *behaviors* are actions—often mental actions—that the implied observer takes.

Opportunities for Learning

The National Research Council has defined the opportunity to learn as "circumstances that allow students to engage in and spend time on academic tasks…" (p. 333). Describing the lecture content and the associated implied observer is a necessary precondition to understanding the opportunities a student may have for learning from a lecture. In particular, we view students' opportunity to learn mathematics from a lecture as the interface between the implied observer and the actual observer. In this study, we explore how students react to different points in the lecture and analyze their reaction through the lens of the implied observer.

Data Collection and Analytical Methods

We have collected and are in the process of analyzing data in a pilot study. The participants are six mathematics majors who were enrolled in a standard abstract algebra class. The data corpus consists of videotaped classroom observations, videotaped interviews with students, and the students' notes. The class was videotaped 6 times over the course of the semester. The video was transcribed to include the written, spoken, and gestural components. In this way, the class observations were designed to capture as much of the "text" of the lecture as possible and were used to create a description of the implied observer.

After each observed class period, we collected the participants' notes and identified discrepancies between the students' notes and the "text" of the lecture. Interviews with students after each recorded lecture included showing video clips of the lectures and asking questions about their decision-making. We used the interviews, along with the notes, to help identify the behaviors, codes and competencies of the actual observers. To analyze the data we first read the interview transcripts, making comments indicating the behaviors, codes and competencies that students showed (Strauss & Corbin, 1994). In this proposal we focus on a "chunk" of one class meeting because of its role in motivating a subsequent proof and because multiple students omitted portions of the lecture content from their notes. We are currently attempting to describe to what extent these discrepancies represent a missed opportunity to learn and how they may be explained by differences between the implied and actual observers.

Results and Analysis

Summary of the class. For analysis, we consider a segment in class in which Dr. P asked the question, "What is Q?" meaning, the rational numbers. He noted how they had traditionally described the rational numbers as 'a over b' where a and b are integers and b is non-zero, but said that it was insufficient. Dr. P explained that because they had two elements, it made was reasonable to write the rational number as an ordered pair, or cartesian product. He then stated that a/b would be written as (a,b). Dr. P then noted that ¾ = 9/12 but (3,4) =/ (9,12) although they, meaning the class, would want it to be. Dr. P continued: "Some of these ordered pairs should be related to each other and others not. Now, on what basis do we say this is the same?" He claimed that they should have a way to test when ordered pairs are the same, and noted that cross-multiplication was perhaps the easiest way to test, although reducing to lowest common terms could also work, just that it would require "fiddling." He then defined the relationship a/b = c/d when ad=bc and proceeded to prove that it was an equivalence relation that gave the desired results [Dr. P's board work is included in Appendix 1].

This "chunk" is dense with codes, competencies, and behaviors. For example, there are a variety of symbolic codes embedded in the board work, including the set notation used to defined Q, the notation ZxZ-{0} to represent ordered pairs, and the double arrow indicating equivalent statements. The implied observer has competencies encompassing knowledge of equivalent fractions and an understanding of what makes a well-defined equivalence in mathematics. Behaviors include responding to Dr. P's rhetorical questions by thinking about what difficulties are posed by the familiar definition of the rationals and how the example 3*12=4*9 could be generalized.

We also note that much of Dr. P's presentation was spoken and not written. This potentially widens the gap between the implied and actual observers, as it may be more difficult for students to call up necessary codes and competencies or to enact required behaviors.

Students' note-taking decisions. Students' notes on this segment of class contain a variety of omissions of the written lecture content [See Appendix 1], generally focused on the specific example (3,4) ≠ (9,12). Only 1 student's notes, Jocelyn's, contained additional writing related to the spoken lecture content. These discrepancies between the lecture content and students' notes reflect a variety of student choices.

1) S*ome students omit while also making appropriate mathematical interpretations.* Ted's general note-taking strategy was to write only definitions. In keeping with this, he did not record any part of this segment in his notes. However, in an interview he was able to articulate the dilemma that Dr. P was indicating and how it would be resolved.

2) *Students omit when they think they understand but may not have a complete understanding.* During her interview Petra indicated that she had recorded that (3,4)=/(9,12), and claimed the goal of the lecture was to have them be equal. However, she did not write down the step, "3*12 = 4*9" because, "It was easy... I know, I know why those are equal." This appears to reflect a lack of understanding of the purpose of the example, as she states, "I don't know why [the ordered pairs] have to be related."

3) *Students omit when they don't understand and think they may be confused later.* Meredith's notes do not reference the example. In an interview, Meredith said, "I remember him writing that and saying like, yeah that should be true, and him saying it wasn't. And then I understood that the points weren't the same, but then I didn't understand, I guess the bigger

concept of why the whole thing didn't, so I didn't write it. Because I think it would just confuse me looking back at it."

Discussion

Students' stated reasons for omitting lecture content suggest a variety of gaps between the implied and actual observers. Moreover, these omissions reflect not only a lack of understanding during class but may also limit their opportunity to learn from their notes outside of class. While we believe that our method will produce interesting results, we are cautious in our assertions given that our analysis is ongoing. We believe that the major contribution will be to articulate a method to describe, on a minute-by-minute basis, what a particular student has the opportunity to learn from an undergraduate lecture in a proof-based mathematics class.

Questions we intend to discuss

1) How effective is the implied reader framework at capturing students' opportunity to learn? What aspects of opportunity to learn does this fail to address?
2) To what extent are students' notes an effective tool for investigating opportunity to learn?

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Appendix 1

What Dr. P's board work and students' notes.

$\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ $\frac{3}{4}=\frac{9}{12}$ $(3,4)\neq(9,12)$ **Dr. P:** $3\cdot 12=4\cdot 9$	Jocelyn: $\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-[0])$ $\frac{3}{4}=\frac{9}{12}$ $3\cdot 12=4\cdot 9$
Ted: did not record the chunk in his notes	Kazimir: What is $\mathbb{Q}$? $\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ $\star$ $\frac{3}{4}=\frac{9}{12}$ but $(3,4)\neq(9,12)$
Petra: $\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ $\frac{3}{4}=\frac{9}{12}$ $(3,4)\neq(9,12)$	$\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$ Landon: $\frac{3}{4}=\frac{9}{12}$, but $(3,4)\neq(9,12)$

$\mathbb{Q}=\{\frac{a}{b} \mid a,b\in\mathbb{Z}, b\neq 0\}$ $\frac{a}{b} \longleftrightarrow (a,b)\in\mathbb{Z}\times(\mathbb{Z}-\{0\})$

Meredith:

PARADOXES OF INFINITY – THE CASE OF KEN

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Previous studies have shown that the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation are difficult to understand even for learners with advanced mathematical background such as doctoral students in mathematics. This study examines whether this difficulty is due to the way they are set in everyday life experiences. Some variations of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation and their abstract versions set in the set theoretic language without any reference to everyday life experiences were given to a doctoral student in mathematics. Data collected suggest that the abstract versions can help learners see beyond the metaphorical language of the paradoxes. The main contribution of this study is revealing the possible negative effect of the metaphorical language of the paradoxes of infinity on the understanding of the learner.

Keywords – Infinity, Paradoxes, Cognitive conflict

Introduction and Theoretical Perspectives

Paradoxes involving infinity have been used as a lens in mathematics education research for identifying students' difficulties in understanding infinity. One study was conducted by Mamolo and Zazkis (2008) who used the paradoxes Hilbert's Grand Hotel and the Pin-Pong Ball Conundrum. This study is a part of PhD thesis research of Mamolo (2009) which also included the Pin-Pong Ball Variation. In Mamolo (2009) the author reports that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation. This study examines whether this difficulty in understanding the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation is due to the way these paradoxes are set in everyday life experiences.

What kind of thinking involves in understanding paradoxes like the Pin-Pong Ball Conundrum? Barbara, Dubinsky and McDonald (2005) suggest that it is advanced mathematical thinking. They define Advanced Mathematical Thinking as thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses. They say that comparing $|N|$ with $|2N|$ may require Advanced Mathematical Thinking and the ability to understand that there is a one-to-one relationship between N and 2N is probably not available through experience in the physical world.

We consider two other theoretical frameworks to analyze the data. One is APOS analysis of conceptions of infinity by Dubinsky, Weller, McDonald, and Brown (2005). They suggested that interiorizing infinity to a process corresponds to an understanding of potential infinity - infinity is imagined as performing an endless action. The ability to conceive of the process as a totality occurs as a consequence of encapsulation of the process to an object, and corresponds to a conception of actual infinity.

The other theoretical framework is reducing abstraction by Hazzan (1999). According to this perspective abstractness of mathematical concepts can be reduced by connecting them to real-life situations and establishing a right relationship (in the sense of Wilensky) between the learner and the mathematical concept.

Research Method

Ken is a PhD student in mathematics at a big ten university in the Midwest in the USA. He was sent the following questions in a questionnaire and later interviewed after getting his answers to the questionnaire.

1. A large barrel has Pin-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 2 is removed. Again, in half the remaining time the ball number 3 is removed, and so on. At the end of the minute, how many Pin-Pong balls remain in the barrel?

2. A large barrel has Pin-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 11 is removed. Again, in half the remaining time the ball number 21 is removed, and so on. At the end of the minute, how many Pin-Pong balls remain in the barrel?

3. Let $A_n = A_{n-1} - \{n\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

4. Let $A_n = A_{n-1} - \{10(n-1)+1\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

Questions 1 and 2 are variations of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation, given in the appendix with their normative solutions, respectively with the same end result but different processes. For example in Question 1 the number of balls in the barrel decreases and is always infinite as time approaches the end of one minute but in the Pin-Pong Ball Conundrum the number of balls increases and is finite as time approaches the end of one minute. Questions 3 and 4 are the abstract versions of 1 & 2. Abstract versions don't have a sense of time.

Results

He answered all the questions correctly. From what he wrote at the end of the questionnaire it is clear that he saw that Questions 3 & 4 formalize the processes described in Questions 1 & 2 respectively. And Question 2 helped him in Question 4. So he clearly saw the connection between the concrete versions and the abstract versions. He also reduced the abstraction in Question 4 by going back to Question 2. Even though, arguably Ken is capable of Advanced Mathematical Thinking, he had trouble understanding the processes in Questions 1 and 2. The abstract versions helped him to see that the process can be continued.

Researcher: what if you did not get number 3 and 4 and you got only 1 and 2?

Ken: yeah then … I would still probably I need to take more time I will probably end up assuming that I have to think that this process can be done and I would still give the same answer but after I mean it take bit more time to kind of assume that to take that.

So without Questions 3 and 4 he thinks he would have answered Questions 1 and 2 the same way but it would have taken him more time. Ken never questioned the plausibility of the Questions 3 and 4. As an advanced graduate student in pure mathematics he knows the mathematical language well. He can work in the mathematical realm. So he did not have any trouble with Questions 3 and 4. Though he interiorized the action of removing the ball number n in Question 1 as a process he could not encapsulate this process to an object:

Ken: I started from the first question but I didn't write down answers because at some point I was little bit confused about problem 1 because since it was kind of a practical procedure although it was clear what was going on I mean specially answering the last part

APOS analysis can be applied to Questions 3 and 4 as well. Apparently Ken did not have any trouble with encapsulating the intersection of infinitely many sets to an object – he got little help from Question 1 and 2 in describing this object.

Discussion & Conclusions

Paradoxes involving infinity can provide a window to infinity. The cognitive conflict elicited by a paradox is difficult for a learner to resolve. Resolving this cognitive conflict requires the learner to make a cognitive leap from the intuitive to the formal or from the real world to the mathematical realm. But some of the paradoxes make this cognitive leap difficult as they are too far away from the reality but yet set in the everyday life experiences. If we compare Zeno's paradox of Achilles and Tortoise and the Pin-Pong Ball Conundrum, we can see that Achilles and Tortoise is about a real life situation and the Pin-Pong Ball Conundrum is not a real life situation though it involves real life objects. Even in the mathematical realm the concept of infinity is a difficult concept to grasp. Bolzano and Galileo could not grasp infinity though they considered abstract mathematical entities like intervals and sets of numbers. So when the concept of infinity is presented through everyday life experiences with an infinite process in a finite time interval far away from reality it adds to the difficulty of grasping infinity. We can see it from Ken. Our findings agree with Mamolo (2009) who found that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation. There is further evidence in Mamolo and Zazkis (2008): "Based on the results of our research, and specifically acknowledging the similarity in responses of students with different mathematical sophistication, we suggest that a formal mathematical view of infinity implied in conventional resolutions of the paradoxes may not be reconcilable with intuition and 'real life' experience."

The concept of infinity in mathematics is very mathematical and counter intuitive. This study reveals that the metaphorical language of the paradoxes could have a negative effect on the understanding of the learner.

Questions for the Audience

1. Tasks in Questions 1 & 2 are the same in some sense: Always in half the remaining time a ball is removed. But at the end of the minute the outcomes in them are very different. In the first task the barrel is empty and in the other it has infinitely many balls. Can this happen in the physical world?
2. Can we imagine an infinite process where each step takes some time in a finite time?
3. Is it effective to teach mathematical concepts that are counter intuitive and do not relate much to the experience in the physical world using a real life context?

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Appendix

The Pin-Pong Ball Conundrum

An infinite set of numbered Pin-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the minute, how many Pin-Pong balls remain in the barrel?

Solution

In this thought experiment there is an infinite sequence of time intervals of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Since in the time interval of length $\frac{1}{2^n}$ the ball number *n* is removed and there is a one to one correspondence between the sets $\{\frac{1}{2^n} / n \in N\}$ and *N*, the set of positive integers, at the end of the minute the barrel is empty.

The Pin-Pong Ball Variation

An infinite set of numbered Pin-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 11 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 21 is removed, and so on. After the experiment is over, at the end of the minute, how many Pin-Pong balls remain in the barrel?

Solution

In this variation the same infinite sequence of time intervals of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is there. But in the time interval of length $\frac{1}{2^n}$ the ball number $10(n-1)+1$ is removed. And there is a one to one correspondence between the sets $\{\frac{1}{2^n} / n \in N\}$ and $\{10(n-1)+1 / n \in N\}$. So at the end of one minute the barrel has the balls numbered 2, 3, … , 9, 10, 12, 13, … , 20, 22, … - this corresponds to the set $N - \{10(n-1)+1 / n \in N\}$.

VERIFYING TRIGONOMETRIC IDENTITIES: PROOF AND STUDENTS' PERCEPTIONS OF EQUALITY

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This preliminary study explores how students' perceptions of the equality of trigonometric expressions evolve during the process of verifying trigonometric identities (VTI). If students already view the purported equality as being true, VTI may not offer much in the way of learning experiences for students. Using a semiotic perspective to analyze student work, this study attempts to describe the evolution of students' perceptions of identities upon application of VTI, focusing on the components of the student's VTI process that contribute to the evolution. Initial analyses of interviews conducted while students proved identities indicate that students are not fully convinced that the identities are initially true. However, successful VTI, signaled, for example, by the use of an idiosyncratic equality construction, endows equality on not only the purported identity but on ancillary equality statements generated as part of the VTI process.

Key words: Trigonometric identities, Proof, Equal sign, Semiotics

Verifying trigonometric identities (VTI) involves many domains of mathematics, such as the use of algebraic skills to manipulate expressions and the equality concept embedded in verification and identities. Additionally, while verification is proof-making, students engage in problem solving in order to complete the task. Yet, trigonometry, especially VTI, has been under-studied. Some studies focused on the medium through which students interacted with trigonometry while learning (e.g., Choi-Koh, 2003; Weber, 2005a). Other research described the understanding of trigonometric concepts and objects developed by students (e.g., Brown, 2005; Moore, 2010).

Few studies have explored the verification of trigonometric identities by students. For example, Delice (2002) investigated students simplifying trigonometric expressions. He found that students followed certain patterns of simplifying. However, the study assumed that students constructed no new knowledge during the simplification task, and thus students experienced no changes in their perceptions of the mathematical objects with which they operated.

Background

Historically, VTI has been a part of the trigonometry curriculum. However, the National Council of Teachers of Mathematics (NCTM) recommended scaling back trigonometric identity verification in the school-level curriculum (NCTM, 1989). The deemphasizing of VTI seemingly ignored NCTM's own statement that VTI "improves [students'] understanding of trigonometric properties and provides a setting for deductive proof" (NCTM, 1989, p. 165). In a critique of the trigonometry standard and the standards overall, Wu (n.d.) wondered how average, non-college bound students were to understand identities such as $\sec^2 x = 1 + \tan^2 x$ if not given the opportunity to prove them. Additionally, the more recent Common Core State Standards for Mathematics do not explicitly suggest the verification of general identities; strands F-TF 8 and 9 address proving the Pythagorean identity and the addition and subtraction formulas and then subsequently using them to solve problems (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

VTI is an individualized process that a person utilizes to demonstrate, to himself and to an external audience, that one trigonometric expression is equivalent to another trigonometric expression. Harel and Sowder (1998) defined *proving* as a "process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). Thus, VTI is a proving process. Within VTI, the theorem to be proved is the purported equality of the two trigonometric expressions. Henceforth, this purported equality will be referred to as the *theorem identity*. The theorems used by a person to prove the theorem identity are equivalent expressions or identities; hence, the theorems of VTI are statements of equality.

Viewing VTI as proof construction suggests exploration of student notions of the theorem identity and equality. As noted by Weber (2001), studies have shown that inadequate understanding of the mathematical content of a theorem affects correct use of the theorem. Additionally, as the theorems in VTI are equality statements, student understanding of the theorems includes understanding of equality and the *equals* symbol. Studies have found that students of all ages, even within the college ranks, may view the equal symbol as operational rather than relational (Kieran, 1981; Weinberg, 2010). Furthermore, weak notions of the equal symbol may affect performance in algebra (Alibali, Knuth, McNeil, & Stephens, 2006). These weaknesses may in turn affect a student's ability to use the identities during identity verification.

Finally, CadwalladerOlsker (2011), quoting de Villiers, stated that one role proof serves is that of the verification of a theorem. Thus, when a student verifies the theorem identity to be true, he or she has proven that the purported equality is indeed true, and now that equality may be treated as a theorem. However, "novice proof writers often complain that it is pointless to prove theorems that 'everybody knows,' or that have already been proven in the past" (CadwalladerOlsker, 2011, p. 40). If students engaged in VTI have this attitude of the theorem identity and of VTI, then the question exists of whether or not students can make conceptual gains through VTI. If students view the theorem identity as already true, does VTI really offer a learning experience for them? If no need exists for students to verify a truth, will they experience a change in their perceptions of the objects with which they work?

Purpose

Tall (2002) suggested that the process of proving a statement encapsulates the theorem as a concept. Once encapsulated, the mathematical object may be used to build new knowledge and prove more theorems. This encapsulation process may occur in VTI as students verify the equality statements, creating new identities. In other words, students should experience a shift in their perceptions of the theorem identities and enhance their understanding of identities.

The purpose of this present study is to describe the evolution of students' perceptions of the theorem identity upon application of VTI, focusing on the components of the student's VTI process that contribute to the evolution.

Theoretical Framework

As suggested by Weinberg (2010) in his study of students' conceptions of equality, in this present study, a semiotic perspective is used to analyze students' VTI solutions. Semiotics places the emphasis on the connections between the representation of ideas and how these representations influence the activity of solving the problem. In other words, as a student writes his VTI solution, how he writes the solution is intertwined with his notions at play in the process of VTI; the representations and the notions affect each other.

Methodology

This study uses a pragmatic approach to research, placing primary importance on the research questions and using methods of data collection and analysis which best address the question (Creswell & Plano Clark, 2011). For this reason, qualitative methods were used to explore students' conceptions as they solve problems.

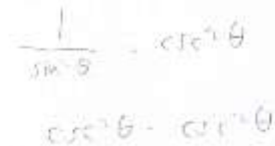
During the spring 2012 semester, 33 students enrolled in a trigonometry course at a large, Midwestern university participated in a study of their problem solving behavior while verifying trigonometric identities. About two weeks after the conclusion of the unit on VTI, and after a semester exam that included VTI items, 8 students agreed to be interviewed as they verified trigonometric identities. While solving the problems, the students were encouraged to think aloud, providing their motivations for their decisions and actions. During the interviews, students' conceptions of identities, equality, and VTI were explored through follow-up questions. The interviews were recorded using a digital voice recorder and were subsequently transcribed. The transcripts of the interviews were analyzed using a grounded, open-coding approach (Strauss & Corbin, 1998).

Preliminary Results

Certain features of VTI appeared to transform students' perceptions regarding the equality of trigonometric expressions. Prior to engagement in VTI, some students were skeptical of the equality purported by the theorem identity. According to one student, Amber, "There's a lot of skepticism in the beginning. … If I don't see those steps in my head, or if they don't look the same in the beginning, then I'm basically trying to prove it wrong." Other students viewed the expressions as equal, but the equality was attenuated by viewing it as tentative or assumed; they did not hold strong convictions concerning the equality. Another student, Bella, stated, "I haven't shown that in my own head that they're equal. I'm trusting that they're equal. … It's just I'm assuming they're equal until I say, until I have shown that they're not. But you don't know for absolute positively sureness that they are equal." Upon application of VTI, skepticism, doubt, or tentativeness was removed for the students, and the students believed they could unequivocally assert that the expressions were equal.

The transformation of the theorem identity's status occurred in both a public and private fashion and depended upon how the students structured their verifications. Some students mentioned accepting the truthfulness of the theorem identity due to an unraveling process in their minds. For others, the shifting of their perceptions occurred when they wrote a reflexive statement, such as $B = B$, for their final VTI step. As Amber put it, "In my mind, it's verified, once they look, I mean, once they are the exact same on either side, then for me, it's verified." This *reflexive step* was an important and necessary step for some students to successfully conclude VTI. It signaled the beginning of the transformation, although for some students, the reflexive step seemed somewhat ritualistic. Nevertheless, through these experiences, students ascertained for themselves the truth of the theorem identity.

For their written solutions, most students employed a columns method, manipulating one expression and cascading the manipulated expressions down to form a column. A few students rewrote the unchanged target expression and connected this expression with an equal sign to each of the manipulated expressions in the column, creating a series of equality statements. While seemingly stating that the expressions were equal, students tended to believe the expressions were not equal until VTI was completed, for example, with the writing of the reflexive step. At this point, in addition to the theorem identity, all of the intermediate equalities that had been written could be retroactively declared as true. For example, consider the work of Alan in verifying the identity $1/(1-\cos^2\theta) = \csc^2\theta$:



In describing his use of the reflexive step, Alan commented, "To me that's just verifying that I was able to get the correct answer and to show whoever is looking at this that if you eventually get to the steps, you will get what equal the cosecant squared equals cosecant squared. It's pretty much to me just to show that I'm done, … to kind of put any doubt out of peoples' minds." In referring to the initial "equality" written, Alan clarifies, "Well, that is true because I was eventually able to work it down. … I got, was able to get cosecant squared on the other side, that, they ended up being true." Thus, reaching the reflexive step not only indicated that the theorem identity was true; Alan was able to consider the first "equality" written a true equality as well.

Audience Questions

1. During VTI, equality appears to have a time-dependent nature. What theories or frameworks exist to describe this behavior?
2. Although this study views VTI as proof-making, doing so seems somewhat naïve. What are some other frameworks that could adequately capture student behavior in VTI?

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FOSTERING STUDENTS' UNDERSTANDING OF THE CONNECTION BETWEEN FUNCTION AND DERIVATIVE: A DYNAMIC GEOMETRY APPROACH

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Students' difficulties with relating the graphs of functions to the graphs of their derivatives have been well documented in the literature. Here I present a Geometer's Sketchpad based applet, which was used as part of a technologically enriched Calculus I course. Individual interviews with students conducted after this in-class activity show evidence of varied and powerful student problem solving strategies that emerged after participation in the activity.

Key words: [Graphing, Technology, Derivative, Calculus, Geometer's Sketchpad]

Students' difficulties with graphing derivatives of functions have been a recurring theme in the calculus education literature. Given the recurrence of this theme it is unfortunate that only a few instructional interventions aimed at bridging this gap have been studied. Here I present a Geometer's Sketchpad (GSP) based applet that is designed to address a reoccurring reason for students' difficulty with graphing derivatives. Individual interviews conducted with students after this in-class activity revealed a diverse range of student-invented problem solving approaches for tackling novel graphing tasks.

Background Literature

A growing body of literature supports the assertion that a coordination of both visual-graphical and analytic reasoning is essential for students to form a rich understanding of mathematics (Aspinwall & Shaw, 2002; Zazkis, Dubinsky, & Dautermann, 1996). Zimmerman (1991) went so far as to state that, "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136). Students' difficulties with graphing derivatives have been studied in the education research literature for at least thirty years (Orton, 1983; Nemirovsky & Rubin, 1992; Aspinwall, Shaw and Presmeg, 1997; Haciomeroglu, Aspinwall, & Presmeg, 2010). Some researchers have attributed these difficulties to pre-tertiary mathematics education that commonly deemphasizes the importance of graphing and graphical intuition in favor of more symbolic and algorithmic approaches to mathematics (Vinner, 1989). However, this places the blame elsewhere and makes students' difficulties with graphing a foregone conclusion, rather than an issue that can be addressed. Another series of explanations revolves around the coordination of different types of quantities. In order to take the Cartesian graph of a function (for which the function is not given) and sketch the graph of its derivative one needs to coordinate two very different types of quantities—the function's instantaneous rate of change, which is a gradient measure, and the height of the derivative function relative to the x-axis, which is a linear measure. Coordination of varying quantities is in general difficult for students (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Coordination of the gradient and linear quantities required for graphing derivatives can be viewed as a special case of this difficulty.

This outlook provides an explanation for the most common error that occurs when sketching the graph of a derivative, which is simply to redraw the original graph (Kung & Speer, in press; Nemirovsky & Rubin, 1992). Studies that have examined students working on these graphing tasks in clinical interview environments have revealed that students attended to the desired attribute of the original function, its slope, but coordinated it with the slope of the derivative function, leading to a redrawing of the original function, rather than the drawing of the desired graph. In essence, students coordinated a quantity with itself rather than coordinating two different quantities. A visual link that coordinates the correct two quantities is built into the design of the applet discussed in the following section.

The Applet and Associated Activity

The applet described in this section aims to provide a manipulatable link between gradients and linear measures, one that can be used both in association with functions and independently of functions. This is done through the use of slope-widgets. Each widget consists of two vertically aligned manipulatable points one of which has a line through it. The slope of this line is controlled by the height of the other point (see Figure 1 and Figure 2). When the slope point is placed on a function, the second point is manipulated so that the line approximates a tangent line. The widgets provide a tangible visual link between the slope of the function and the value of its derivative at that point. When several slope widgets are used in unison they plot out the path of the derivative function using this tangible link.

Students were given access to the slope widget applet during class time and asked to work in small groups to graph the derivatives of several functions with the aid of this applet.

Participants

One group of three students (two male and one female) was video recorded during group work throughout the semester. The group was chosen based on their scores on the Calculus Concept Readiness test (Carlson, Madison & West, Submitted) to be fairly representative of the class as a whole. Each member of the group also participated in a series of three individual problem solving interviews, conducted by the author of this paper, which included graphing tasks as well as other calculus tasks. These interviews were conducted in a technology free environment in which students had access to only pencil and paper. The data in the following section comes from the first set of interviews.

Data

The participants demonstrated a diverse range of strategies for approaching graphing tasks, particularly considering they worked with each other four days a week in class. One particularly interesting task, which illustrates some of this diversity, was given during the first set of interviews. Each of the students was given the graph of a function without being given its formula and told that the given function was the derivative of another function (Figure 3). They were asked to sketch the graph of the original function. This task was given early in the semester, when students had not yet encountered the concept of anti-derivative (or integral). The only instruction they had received which targeted graphing derivatives was the slope widget task described in the previous section. When the task was presented the students had only seen 'graph the derivative'-type tasks, but not its 'inverse'. I was interested in studying the problem solving strategies that students implement in approaching this novel-for-them task.

All three students, which will be referred to as Allison, Brad and Carson, where able to successfully complete the task fairly quickly (under three minutes) and then spent several minutes convincing themselves of the legitimacy of their solution. Each demonstrated a unique problem solving strategy.

Brad was the only student whose reasoning centred around the gradient-linear measure relationship.

> Brad [00:33:58]: [reading] Below is the graph of the derivative of a function sketch the graph of the original function. [begins sketching function] So from here on the original the slope is increasing until it crosses zero and then it begins decreasing and then it hits zero and after it hits zero it goes back up until it hits zero again.

Alison, in contrast, used a hybrid strategy starting with algebraic reasoning and adjusting her algebraic images based on graphical reasoning. She started with familiar graphical shapes by drawing a $y=x^2$ graph and a $y=x^3$ graph stating that one was the derivative of the other. Then she noted that unlike the $y=x^2$ shaped graph the given graph had a portion that was under the x-axis and that this portion of the graph corresponded to a portion of the original graph which had a negative slope. She then used this to adjust the shape of the x^3 graph appropriately.

Carson, demonstrated a different type of hybrid strategy. He first began by reasoning that the zeros of the given function corresponded to maxima, minima or saddle points. However, he had not yet encountered the appropriate terminology for these types of points and instead drew small sections of functions to describe each of these phenomena. He then expanded his reasoning to try to figure out which of the three situations he was dealing with. He started by reasoning that the derivative of a downward facing parabola is a line with a negative slope which crosses the x-axis at the same x-value as the vertex of the parabola. This was used to deal with the first zero of the given function. Then he reasoned that the derivative of an upward facing parabola is a positively sloped line that crosses the x-axis at the same x-value as the vertex of the parabola. Connecting these two shapes provided the desired cubic function shape.

Discussion

What is particularly interesting is that on the anti-derivative task two of the three students used strategies that incorporated both analytic reasoning and graphical reasoning. If I expand to look at other questions asked during the interviews, there is evidence of such thinking from all three of the studies participants. This is surprising given the limited experience these students had before the course with graphical modes of thinking. Students' difficulties with both subscribing to graphical modes of reasoning (Vinner, 1989) and translating between these and analytic modes (Apsinwall & Shaw, 2002; Haciomeroglu et al., 2010) has been documented in the literature. Such preferences for one mode of thinking over another are common enough that some authors draw fairly rigid distinctions between students who prefer either graphical or analytic modes of reasoning (e.g. Apsinwall et al., 1997; Haciomeroglu et al., 2010). The students in this study, partially due to their exposure to applets such as the one described above, moved between these representational forms relatively fluidly. In other parts of the interview data students were able to spot their own mistakes when one mode's results seemed to contradict another's. It is important to mention that these were not especially talented mathematics students. They were chosen to be fairly representative of the class as a whole. In fact Brad, in spite of putting considerable effort into his schooling, was unable to finish the course with a passing grade. The flexible thinking that these students demonstrated can be attributed to the types of powerful reasoning that the applet was able to foster. Further inquiry into the specifics of how students interacted with the applet in class will hopefully help me better develop a theory regarding how the applet affected these students' thinking about graphing of derivatives.

Questions

1) What theoretical framework can be used to analyze these data, and the rest of the interview data from my study? Will coordinating several frameworks be preferable?

2) Have you ever experienced phenomena with your students that are not in accord with the research literature? Was technology involved? How did you make sense of it?

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Appendix

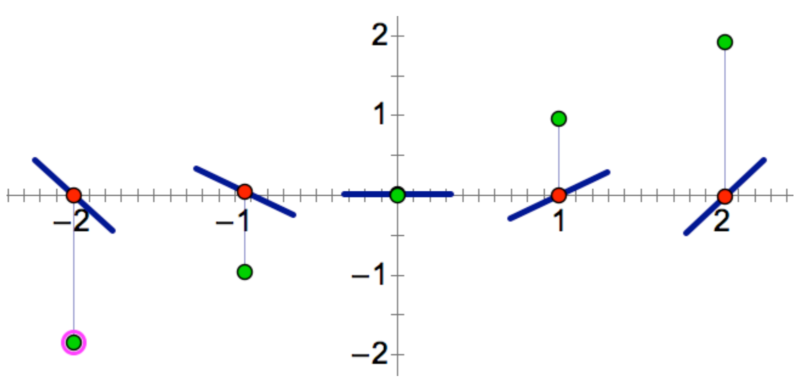

Figure 1: Slope-widgets

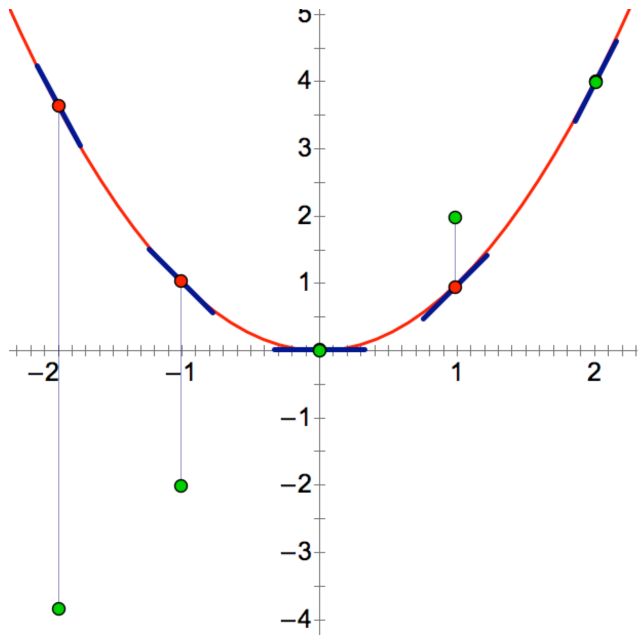

Figure 2: Slope-widgets on a function

Below is the graph of the derivative of a function. Sketch the graph of the original function.

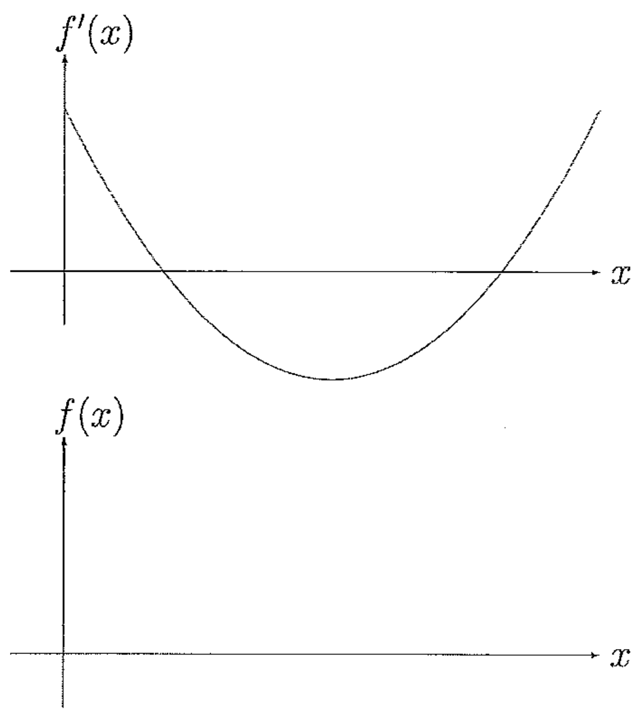

Figure 3: The anti-derivative task

AN ANNOTATION TOOL DESIGNED TO INTERFACE WITH WEBWORK: INTERPRETTING STUDENTS' WRITTEN WORK

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Gulden Karakok
University of Northern Colorado

Aaron Wangberg
Winona State University

We present how we are using tablets with an open-source online homework system to collect students' written work to calculus problems. The new whiteboard feature captures all student written work in real time. An annotation tool has also been incorporated into the system. Through this tool, we are examining how students solve chain rule problems and what actions they take to correct their mistakes. At this poster, we will allow users to try out the annotation tool and provide results of how we have used it to date.

Key words: calculus, chain rule, online homework, collaborative methods

The collection of student work through online assessment systems is popular for many courses in mathematics. Such systems provide opportunities for students to practice problems and receive immediate feedback on correctness of their answers. However, with enhancements to such systems, both students and teachers could receive feedback on students' understanding of a concept through investigation of their solutions as well as the final answers to the problems. We will demonstrate the use of such enhancements to the open-source system WeBWorK. The enhancements included the creation of a whiteboard area in which students can provide their solutions to the problems (Figure 1) and an annotation system for researchers/instructors.

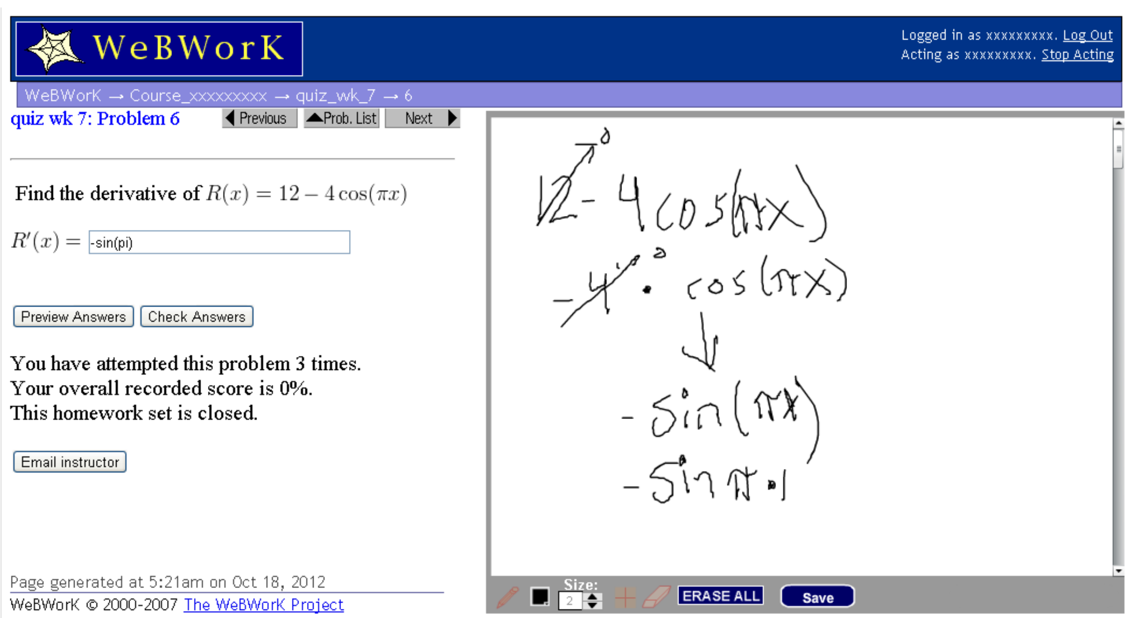


Figure 1: Example of student whiteboard area in WeBWorK

Success in calculus can be linked to students' understanding of the concept of function (Carlson, 1998; Carlson, Oehrtman, & Engelke, 2010; Engelke, 2007). While there is fairly extensive literature about student understanding of function, there is little that focuses explicitly on student understanding of function composition. Being able to work with functions in different contexts is particularly problematic for students, especially when dealing with function composition (Engelke, Oehrtman, & Carlson, 2005). One of the critical units where this concept is foundational to calculus is the chain rule and its applications. As such, we chose to study students' understanding of calculus concepts and problems which are closely related to function composition such as the chain rule, related rates, and optimization. We are seeking evidence of how and what students transfer from classroom experience to online activities.

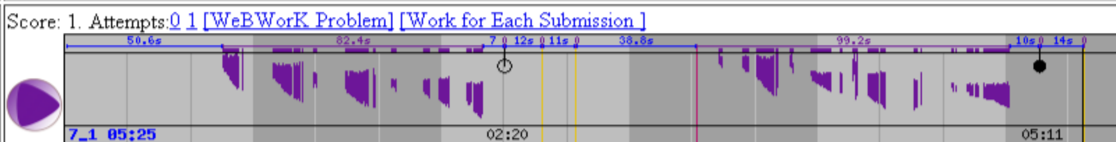
Student work on calculus quizzes was collected via the whiteboard interface in WeBWorK. This data was analyzed using the newly created coding feature of the WeBWorK system which allows one to create keys and descriptions of student work. (See Figure 2) This feature was designed to allow researchers to apply open and axial coding procedures as described by Strauss and Corbin (1990).

Score: 0. Attempts:0 0 0 [WeBWorK Problem] [Work for Each Submission]

Click to manually add Annotation

Time	From	Key	Comment
01:17	ADW	chain_rule_failure_recognition?, deriv_const_correct?, deriv_c_func_failure?	Ignored needing to take the derivative of the inside. Incorrectly made the constant go to zero when differentiating a constant time a function.
02:22	ADW	chain_rule_failure_partial_recognition?, chain_rule_incorrect_implementation?, d_oversimplification?, meta_transfer_failure?, chain_rule_failure_inside?	Student recognized the need to use another rule (maybe chain rule).
02:59	ADW	deriv_trig_correct?	Recognized that derivative of cosine should be negative sine.

Score: 1. Attempts:0 1 [WeBWorK Problem] [Work for Each Submission]



Click to manually add Annotation

Time	From	Key	Comment
02:13	ADW	deriv_const_correct?, chain_rule_failure_partial_recognition?, chain_rule_incorrect_implementation?, marching_d_evidence?, meta_transfer_partial_success?, representation_issue_operations?, parentheses_attention?, chain_rule_failure_inside?	Student wrote what appears to be subtraction but used multiplication in the submission. Student made careful use of parentheses when using the Marching D, but not for multiplication.
05:03	ADW	chain_rule_fix_inside?, new_strategy_product_rule?, chain_rule_correct_implementation?	Student more carefully applied Marching D in conjuction with use of product rule. Product rule was not necessary. Student persisted with no parentheses when doing algebraic operations.

Figure 2: Student work map and associated codes

We met via the internet to explore how the coding tool facilitated our research goals. In the first session the researchers spent 71 minutes examining 3 minutes of student work. The first student was chosen because they had used all three attempts for entering a solution, but had not successfully solved the problem. Each line of the student's written work was discussed and analyzed resulting in multiple keys being created. The following week, the researchers met again and analyzed the work of a second student. For this meeting, a student was chosen who had only used two attempts and successfully solved the problem. This student was chosen because they had made errors (some similar to the first student), but had successfully corrected the errors. We anticipated some overlap of the codes we created for the errors but also the creation of new codes to identify actions taken to correct those errors. Our second coding session lasted about 50 minutes covering again about 3 minutes of student problem solving time.

We will share how the system collects data, facilitates coding, our data, and our coding. Examples from 16 codes will be shared to highlight what we learned in this process. Participants are invited to try the coding system. We are curious to know how others might envision using these tools.

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REASONING ABILITIES THAT SUPPORT STUDENTS IN DEVELOPING MEANINGFUL FORMULAS TO RELATE QUANTITIES IN AN APPLIED PROBLEM CONTEXT

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This poster illustrates and describes student thinking when responding to applied problems to relate two quantities that cannot be directly related by a single formula. Students who understood the meaning of the directive to define one quantity in terms of another, and who also conceptualized variables as representing varying values that a quantity can assume were successful in constructing a meaningful formula to relate the values of two quantities that cannot be directly related by a single formula. Students who failed to construct meaningful formulas during their solution process either held the view that a variable is an unknown value to be solved for or could not meaningfully interpret the directive of the problem statement.

Key words: Function, Quantity, Variable

Introduction And Theoretical Framework

It has been widely reported that a process view of function is critical for understanding and using ideas of function composition and function inverse (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Students who do not conceptualize a function as relating a continuum of values in a function's domain to a continuum of values in a function's range are unable to imagine linking two function processes together for the purpose of relating two quantities that cannot be related by a single formula. The purpose of this study was to reveal how students' conceptions of quantity, variable, and function impact their reasoning when responding to a problem that required the use of function composition to relate two quantities in an applied problem.

Methods

We administered two applied problems (Figure 1) to 123 precalculus level students and used open coding (Strauss & Corbin, 1990) to analyze student written responses. We then conducted clinical interviews with 5 of these students to characterize their thinking relative to their conceptions of quantity, variable, and function and how these conceptions impacted their solution approach. The clinical interviews followed the methodology described by Goldin's (2000) principles of structured, task-based interviews, including metacognitive questions to reveal the reasoning processes students used to determine their answers.

Task 1:	The perimeter of a rectangle is 40 feet and the length of one side of the rectangle is 8 feet. Determine the area of the rectangle.
Task 2:	The perimeter of a rectangle is 40 feet and the length of one side of the rectangle is w feet. Express the area of the rectangle A in terms of the rectangle's width w.

Figure 1. Assessment and clinical interview tasks.

Results and Findings

Of the 123 students, 107 provided a correct answer to the first task, and of these 107 students, only 46 provided a correct answer to the second task. Of the 61 students who did not answer the second question correctly, only 4 advanced their solution to the point of expressing the length of the rectangle in terms of the width (e.g., $l = 20 - w$), suggesting that this construction is key to students' advancing their solution towards expressing the area of the rectangle in terms of the width of the rectangle. Analysis of interview data revealed that in order for students to necessitate relating length and width in a single formula, they needed to have a process view of function. Additionally, the interview data revealed that students did not transfer the algorithm to determine a specific value of A in terms of a specific value of w, Task 1, to the general case of expressing A in terms of w, Task 2. Analysis of the interview data further revealed that students who did not construct the formula, $l = 20 - w$ viewed the variables w and l as unknowns to be solved for rather than varying values that the width and length can assume. A variation view of variable, as discussed by Trigueros and Jacobs (2008), appears to be critical for viewing the problem goal as that of defining a function process to express the area of a rectangle with a perimeter of 40 feet in terms of the rectangle's width w. Analysis of the clinical interview data further revealed that an inability to understand the directive to express the area of a rectangle A in terms of its width w as a request to write "A = <some expression containing a w>" led to students not constructing a clear goal to guide their solution attempt. This study identified critical steps that students must engage with in order to successfully complete this task. Further research is necessary to determine how these ideas of variable, quantity, and function can be developed in curriculum as well as teaching practices so that more students are able to successfully complete and understand these function compositions in applied problems.

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COACHING THE COACHES: SUPPORTING UNIVERSITY SUPERVISORS IN THE SUPERVISION OF ELEMENTARY MATHEMATICS INSTRUCTION

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University of Alabama

This study evaluated program changes that required university supervisors to attend professional development and to use an observation protocol that reflected high quality mathematics pedagogy. The study used both qualitative and quantitative data to analyze the impact of the professional development. Qualitative data consisted of background information, observations, and interviews. Quantitative data included teacher candidates' performance on the Reformed Observation Teaching Protocol (RTOP) and belief scores from the Mathematics Beliefs Instrument (MBI). This study was designed to fill in a gap in the literature to investigate the role university supervisors play in changing teacher candidates' beliefs about mathematics and their instructional practice. By examining the effects of professional development, this study provided research about the type of support university supervisors need to challenge teacher candidates' beliefs about mathematics.

The theoretical framework for this study is a combination of Fuller's (1969)'s concern theory and social constructivism (Cobb, Yackle, & Wood, 1992; Meehan, Holmes, & Tangney, 2001).

This study was designed to investigate two research questions: 1.What are the effects of training university supervisors in mathematics pedagogy and coaching practices on their supervision practices in observing mathematics lessons of elementary teacher candidates? 2. What are the effects of training university supervisors in mathematics education and coaching practices on elementary teacher candidates' beliefs and their instruction in mathematics?

A mixed-method design was necessary to fully capture the relationship between the university supervisor and the support provided to teacher candidates. The quantitative data included pre-post data analyzed using descriptive statistics & graphs using ANCOVA and Paired t-tests. Qualitative data included reflective analysis, interviews, and post-conferences. Descriptive, explanatory and interpretive coding was used to analyze the qualitative data. The sample included 11university supervisors and 78 teacher candidates.

Analysis of the data revealed that the supervision practice of the university supervisors changed as a result of the professional development. University supervisors added paraphrasing and mediating questions to their coaching practice. They fostered reflection by allowing the teacher candidates to problem solve. Teacher candidates also experienced changes in their beliefs about mathematics and their instructional practice.

This study revealed that professional development for university supervisors does make a difference. By focusing on the university supervisor as part of the education of teacher candidates, the cohesiveness of the teacher preparation program is strengthened.

Key Words: Coaching, Mathematics Methods, Professional Development, Supervision, Teacher Education, University Supervisors

RUME “non-blind” submission

THE FLIPPED CLASSROOM MODEL FOR COLLEGE ALGEBRA: EFFECTS ON STUDENT ACHIEVEMENT

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The past few years have seen a substantial rise in the use and interest in a teaching and learning paradigm most commonly known as the flipped classroom. It is called the flipped class model because the whole classroom/homework paradigm is "flipped". In its simplest terms, what used to be classwork (the lecture) is done at home via teacher-created videos and what used to be homework (assigned problems) is now done in class.

This quantitative research compares 5 sections (N=144) of college algebra using the flipped classroom methods with 6 sections (N=181) of traditional college algebra and its effect on student achievement as measured through a common final exam. In the traditional sections, students spent class time receiving lecture and reviewing homework and exams. In the flipped sections, students viewed short video lectures and submitted basic homework solutions online. Students then completed their homework assignments in class with the instructor either in small-groups or active whole-class discussions. All sections took a common final exam and a pre/post algebra readiness exam.

Because the learning of mathematics is built upon a foundation of a student’s prior knowledge, it is imperative that students understand the foundations before progressing in the subject. In a hierarchically-organized subject, such as mathematics, failure to learn prerequisite skills is likely to interfere with students' learning of later skills. The traditional framework of most college algebra classes includes lectures provided by the instructor and homework completed by the student. Because the flipped model does not alter the student-teacher interaction times and maintains an institutions course scheduling, a change to a flipped model is practical and reasonable. The flipped model allows constructivist learning and allows instructors to discover new ways to learn about students, provide instant feedback, adapt instruction, and provide students with anytime lectures. Screencasting of lectures in a flipped model creates a permanent archive for students to pause, rewind and review lectures.

The data were analyzed using multiple regressions and mixed ANOVA with interactions of the intervention measured with gender and ACT mathematics scores. The main independent variable is learning environment (flipped vs. traditional) with secondary independent variables of gender and ACT mathematics scores. The dependent variable is score on common final exam and differences in a pre/post algebra inventory. Regression and ANOVA are appropriate because this tells us how these independent variables interact with each other and what effects these interactions have on the dependent variables.

This research of the effects of a flipped classroom using teacher created online videos on student achievement in college algebra may provide valuable insight into the best-practices of technology in mathematics education.

THAT'S NICE…BUT IS IT WORTH SHARING?

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Key words: Calculus, Group-work, Equity, Peer-assessment

Introduction

Productive groupwork requires "group-worthy" problems (cf. Complex Instruction; Featherstone et al., 2011), which are nontrivial, require multiple competencies, and often have multiple solution paths (see also "problem aesthetic"; Schoenfeld, 1991). In contrast, groupwork with standard tasks often degenerates into one student "teaching" the other, because the tasks do not support collaborative learning. Short of full-blown collaborative groupwork, partner work is often a productive practice. In this poster, I discuss a particular type of partner work (peer-assessment) and the affordances required of tasks to make the activity productive. I dub these problems "peer-worthy." Peer-worthy problems should satisfy a number of the following criteria; they: (1) are nontrivial, (2) have multiple solution paths, (3) require students to generate examples, and (4) involve explanation. In short, peer-worthy problems require students to generate mathematics in problem situations that allow for many different productive pathways, allowing students to deepen their understanding by making connections.

Method

Students engaged in a peer-assessment activity as a part of their undergraduate calculus course (Reinholz, 2013). Students: (1) solved a weekly homework problem, (2) self-assessed their understanding, (3) traded their work with a partner and performed a peer-assessment, and (4) revised their work based on peer-feedback. I performed microgenetic analyses of student interactions (cf. Schoenfeld, Smith, & Arcavi, 1991), and for this poster I contrast student interactions working on two different problems – one peer-worthy problem and one "unworthy" problem. The peer-worthy problem required students to determine whether a number of statements were always, sometimes, or never true, and provide an explanation or appropriate examples. The comparison problem was a standard related rates problem.

Results, Analysis, and Conclusions

Students discussed the peer-worthy problem by comparing and contrasting various examples, attempting to determine if they actually met the criteria required by the problem. By engaging in the peer-assessment activity, students actually revised and reconsidered their understandings of the problem. In contrast, conversations of the related-rates problem were mostly focused on how to properly differentiate the functions involved, and had little connection to problem solving or deep mathematics. While students did find errors in differentiation, the interactions did little to alter students' understandings beyond a procedural level.

Partner work provides students with a number of opportunities for productive engagement and learning. Peer-worthy tasks require students to bring their various mathematical perspectives to the table, which allows them to make connections between these various viewpoints and develop deeper understandings. This is in contrast to many standard tasks, which are better suited for supporting individual, rather than collaborative, learning.

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